Average distortion embeddings, nonlinear spectral gaps, and a metric John theorem
(after Assaf Naor)

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Norms and convex sets

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The main objects of study in this talk are finite-dimensional normed spaces $X = (\mathbb{R}^d, \|\cdot\|)$, where $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}_+$ satisfies:

- $\|x\| = 0 \;\Leftrightarrow\; x = 0$,
- $\|\lambda x\| = |\lambda| \|x\|$ and
- $\|x + y\| \leq \|x\| + \|y\|$,

where $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$.

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where $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$.

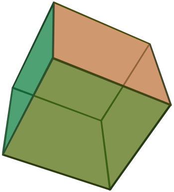
Finite-dimensional normed spaces $X = (\mathbb{R}^d, \|\cdot\|)$ are in 1-1 correspondance with symmetric compact convex sets K in $\mathbb{R}^d$ with non-empty interior via the bijection

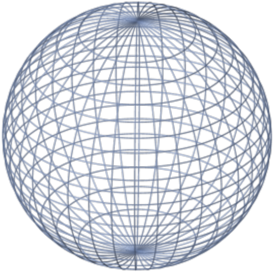
$$X \mapsto \mathbf{B}_X \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : \ \|x\| \leq 1\},$$

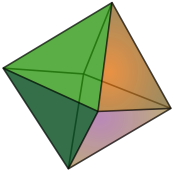
whose inverse is

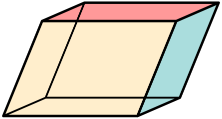
$$K \mapsto \|x\|_K \stackrel{\text{def}}{=} \min\{t \geq 0 : \ x \in tK\}.$$

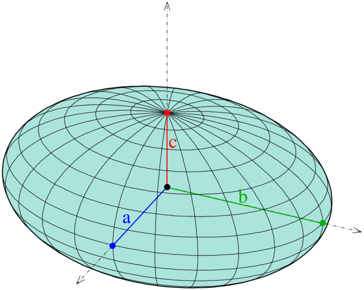
Norms and convex sets

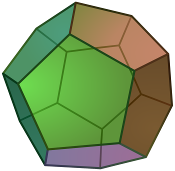












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Theorem (F. John, 1948)

For any normed space $X = (\mathbb{R}^d, \|\cdot\|)$*, there exists a linear operator* $T : X \to \ell_2^d = (\mathbb{R}^d, \|\cdot\|_{\ell_2^d})$ *with* $\|T\|_{\mathrm{op}}\|T^{-1}\|_{\mathrm{op}} \leq \sqrt{d}$*, i.e.*

$$\forall\, x \in \mathbb{R}^d, \qquad \|x\| \leq \|Tx\|_{\ell_2^d} \leq \sqrt{d}\|x\|.$$

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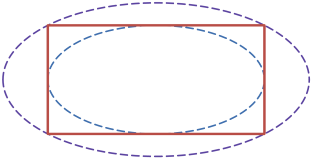

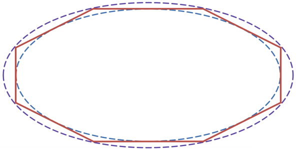

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Equivalently, for any compact symmetric convex set K in $\mathbb{R}^d$, there exists an ellipsoid $\mathcal{E}$ satisfying $\mathcal{E} \subseteq K \subseteq \sqrt{d}\mathcal{E}$.

Moreover, the factor $\sqrt{d}$ is optimal (e.g. for the cube $[-1,1]^d$ which corresponds to the supremum norm on $\mathbb{R}^d$).

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By classical differentiation principles, this is an **equivalent** reformulation of John's theorem.

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• A metric space $(\mathcal{M}, d_{\mathcal{M}})$ embeds into a normed space $(Y, \|\cdot\|_Y)$ with q-**average distortion** at most $D \geq 1$ if for every Borel probability measure μ on $\mathcal{M}$, there exists a D-Lipschitz mapping $f = f_\mu : \mathcal{M} \to Y$ such that

$$\iint_{\mathcal{M}\times\mathcal{M}} \|f(x)-f(y)\|_Y^q \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \geq \iint_{\mathcal{M}\times\mathcal{M}} d_{\mathcal{M}}(x,y)^q \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

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• If $\theta \in (0,1]$, the θ-**snowflake** of a metric space $(\mathcal{M}, d_{\mathcal{M}})$ is the metric space $(\mathcal{M}, d_{\mathcal{M}}^\theta)$.

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- $\frac{1}{2}$ is the least amount of snowflaking for which the resulting distortion depends subpolynomially on d.

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- If average distortion is replaced by bi-Lipschitz then the resulting distortion may grow polynomially in d.
- $\frac{1}{2}$ is the least amount of snowflaking for which the resulting distortion depends subpolynomially on d.
- The bound $O(\sqrt{\log d})$ is optimal for the quadratic average distortion of a d-dimensional normed space.

Application: expanders

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Let $G = (V, E)$ be an r-regular graph and let $\mathrm{A}(G)$ be its normalized adjacency matrix, whose entries are given by

$$\forall\ u, v \in V, \qquad \mathrm{A}(G)_{u,v} = \frac{\mathbf{1}_{\{u,v\}\in E}}{r}.$$

$\mathrm{A}(G)$ is a symmetric stochastic matrix with real eigenvalues

$$1 = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{|V|}(G) \geq -1.$$

Denote by $\gamma(G) = \frac{1}{1-\lambda_2(G)}$ the **reciprocal spectral gap** of G.

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A sequence of r-regular graphs $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ such that $|V_n| \to \infty$ as $n \to \infty$ is an **expander graph sequence** if there exists $C \in (0, \infty)$ such that $\gamma(G_n) \leq C$ for all $n \in N$.

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Suppose that $G = (V, E)$ is an n-vertex connected 4-regular expander which admits a bi-Lipschitz embedding into a d-dimensional normed space $X = (\mathbb{R}^d, \|\cdot\|)$ with distortion at most D. Then $d \geq n^{c/D}$ for some universal constant $c \in (0, \infty)$.

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The sharpness of the JLS theorem had previously been established in important work of **J. Matoušek** (1996) via an ingenious construction of random metric spaces which relied on input from real algebraic geometry.

Proof using the average John theorem

Let $G = (V, E)$ be an n-vertex 4-regular connected graph and denote $\gamma(G) = \gamma$. Suppose that there exists a d-dimensional normed space $X = (\mathbb{R}^d, \|\cdot\|)$ and a map $f : V \to X$ satisfying

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By the average John theorem for the measure $\frac{1}{n}\sum_{u\in V}\delta_{f(u)}$, there exists a $O(\sqrt{\log d})$-Lipschitz map $g:(X,\|\cdot\|^{1/2})\to\ell_2$ with

$$\frac{1}{n^2}\sum_{u,v\in V}\|g(f(u))-g(f(v))\|_{\ell_2}^2\ge\frac{1}{n^2}\sum_{u,v\in V}\|f(u)-f(v)\|.$$

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$$\frac{1}{n^2}\sum_{u,v \in V} \|g(f(u)) - g(f(v))\|_{\ell_2}^2 \geq \frac{1}{n^2}\sum_{u,v \in V} \|f(u) - f(v)\|.$$

Remark. γ is the best constant such that any $h : V \to \ell_2$ satisfies

$$\frac{1}{n^2}\sum_{u,v \in V} \|h(u) - h(v)\|_{\ell_2}^2 \leq \frac{\gamma}{|E|}\sum_{\{a,b\} \in E} \|h(a) - h(b)\|_{\ell_2}^2.$$

Proof using the average John theorem

Applying this to $g \circ f$ along with the upper and lower bounds,

$$\frac{1}{n^2}\sum_{u,v\in V} d_G(u,v) \le \frac{1}{n^2}\sum_{u,v\in V} \|g(f(u)) - g(f(v))\|_{\ell_2}^2$$
$$\le \frac{\gamma}{|E|}\sum_{\{a,b\}\in E} \|g(f(a)) - g(f(b))\|_{\ell_2}^2 \lesssim \gamma D \log d.$$

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Finally, as the graph G is 4-regular, for any fixed $u \in V$, at least $\log_4(\lfloor n/2 \rfloor)$ satisfy $d_G(u,v) \ge \frac{n}{2}$. Therefore

$$\frac{\log n}{100} \le \frac{1}{n^2}\sum_{u,v\in V} d_G(u,v) \lesssim \gamma D \log d,$$

which completes the proof. □

Nonlinear spectral gap inequalities

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Let $\pi = (\pi_1, \ldots, \pi_n)$ be a probability measure on $\{1, \ldots, n\}$. A stochastic matrix $A \in M_n(\mathbb{R})$ is π-**reversible** if $\pi_i a_{ij} = \pi_j a_{ji}$. We think of A as the transition matrix of a Markov chain $\{X_t\}_{t \geq 0}$, i.e.

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Moreover, A defines a self-adjoint operator on $L_2(\pi)$ whose norm is

$$\|x\|_{L_2(\pi)} \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n} \pi_i x_i^2 \right)^{\frac{1}{2}}$$

and therefore has real eigenvalues

$$1 = \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \geq -1.$$

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As in the case of regular graphs, $\gamma(A) \stackrel{\mathrm{def}}{=} \frac{1}{1-\lambda_2(A)}$ is the least constant such that for any $x_1, \ldots, x_n \in \ell_2$, we have

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Definition. Let $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space and $p \in (0, \infty)$. If A is a π-reversible stochastic matrix, denote by $\gamma(A, d_{\mathcal{M}}^p)$ the least constant such that for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{M}$, we have

$$\sum_{i,j=1}^{n} \pi_i \pi_j d_{\mathcal{M}}(x_i, x_j)^p \le \gamma(A, d_{\mathcal{M}}^p) \sum_{i,j=1}^{n} \pi_i a_{ij} d_{\mathcal{M}}(x_i, x_j)^p.$$

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Major open problem. For which metric spaces $\mathcal{M}$ do we have

$$\gamma(A; d_{\mathcal{M}}^p) \le \Psi\left(\frac{1}{1-\lambda_2(A)}\right)$$

for some function Ψ and all reversible stochastic matrices A?

Extrapolation

Extrapolation

We shall need the following vector-valued version of Matoušek's extrapolation principle for Poincaré inequalities (1997) due to T. de Laat and M. de la Salle (2017).

Proposition

For every normed space $(X, \|\cdot\|)$, every π-reversible matrix B and every $1 \leq p \leq q$,

$$\gamma(B, \|\cdot\|^q)^{\frac{p}{q}} \lesssim_{p,q} \gamma(B, \|\cdot\|^p) \lesssim_{p,q} \gamma(B, \|\cdot\|^q).$$

Nonlinear spectral gaps and average distortion

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Suppose that $(\mathcal{M}, d_{\mathcal{M}})$ embeds into $(Y, \|\cdot\|_Y)$ with q-average distortion D and let A be a π-reversible stochastic matrix. Then, given $x_1, \ldots, x_n \in \mathcal{M}$ there exist $y_1, \ldots, y_n \in Y$ satisfying $\|y_i - y_j\|_Y \le D d_{\mathcal{M}}(x_i, x_j)$ for all $i, j \in \{1, \ldots, n\}$ and

$$\sum_{i,j=1}^{n} \pi_i \pi_j \|y_i - y_j\|^q \ge \sum_{i,j=1}^{n} \pi_i \pi_j d_{\mathcal{M}}(x_i, x_j)^q.$$

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Therefore,

$$\sum_{i,j=1}^{n} \pi_i \pi_j d_{\mathcal{M}}(x_i, x_j)^q \leq \gamma(A, \|\cdot\|_Y^q) \sum_{i,j=1}^{n} \pi_i a_{ij} \|y_i - y_j\|_Y^q$$

$$\leq D^q \gamma(A, \|\cdot\|_Y^q) \sum_{i,j=1}^{n} \pi_i a_{ij} d_{\mathcal{M}}(x_i, x_j)^q,$$

which implies that $\gamma(A, d_{\mathcal{M}}^q) \leq D^q \gamma(A, \|\cdot\|_Y^q)$.

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Theorem (A. Naor, 2014)

Suppose that $q, D \in [1, \infty)$. Let $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space and $(Y, \|\cdot\|_Y)$ be a Banach space such that for every $n \in \mathbb{N}$, every reversible stochastic matrix $A \in M_n(\mathbb{R})$ satisfies

$$\gamma(A, d_{\mathcal{M}}^q) \leq D^q \gamma(A, \|\cdot\|_Y^q).$$

Then, for any $\varepsilon > 0$, $\mathcal{M}$ embeds into some ultrapower of $\ell_q(Y)$ with q-average distortion at most $D + \varepsilon$.

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Theorem (A. Naor, 2014)

Suppose that $q, D \in [1, \infty)$. Let $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space and $(Y, \|\cdot\|_Y)$ be a Banach space such that for every $n \in \mathbb{N}$, every reversible stochastic matrix $A \in M_n(\mathbb{R})$ satisfies

$$\gamma(A, d_{\mathcal{M}}^q) \leq D^q \gamma(A, \|\cdot\|_Y^q).$$

Then, for any $\varepsilon > 0$, $\mathcal{M}$ embeds into some ultrapower of $\ell_q(Y)$ with q-average distortion at most $D + \varepsilon$.

The proof is a clever Hahn–Banach separation argument.

Average John as a spectral gap estimate

In view of the duality principle, the average John theorem is **equivalent** to the following statement.

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Theorem

Let $(X, \|\cdot\|)$ be a finite-dimensional normed space. Then, for every $n \in \mathbb{N}$, every π-reversible stochastic matrix $A \in M_n(\mathbb{R})$ satisfies

$$\gamma(A, \|\cdot\|_X) \leq \frac{C \log(\dim(X) + 1)}{1 - \lambda_2(A)},$$

where $C \in (0, \infty)$ is a universal constant.

Thank you!