Séminaire BOURBAKI Avril 2023
75e année, 2022–2023, n° 1206

EXPONENTIAL GROWTH RATES IN HYPERBOLIC GROUPS [after Koji Fujiwara and Zlil Sela]

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A classical result of Jørgensen and Thurston shows that the set of volumes of finite volume complete hyperbolic 3-manifolds is a well-ordered subset of the real numbers of order type ω^ω; moreover, each volume can only be attained by finitely many isometry types of hyperbolic 3-manifolds.

Fujiwara and Sela (2020) established a group-theoretic companion of this result: If Γ is a non-elementary hyperbolic group, then the set of exponential growth rates of Γ is well-ordered, the order type is at least ω^ω, and each growth rate can only be attained by finitely many finite generating sets (up to automorphisms).

In this talk, we outline this work of Fujiwara and Sela and discuss related results.

Contents

1. MAIN RESULTS

Geometric group theory provides a rich interaction between the Riemannian geometry of manifolds and the large-scale geometry of finitely generated groups. This bond is particularly strong in the presence of negative curvature and explains a variety of rigidity phenomena. The group-theoretic analogues of closed hyperbolic manifolds are hyperbolic groups; more generally, the group-theoretic analogues of finite volume complete hyperbolic manifolds are relatively hyperbolic groups.

The volume growth behaviour of Riemannian balls in the universal covering of a compact Riemannian manifold is the same as the growth behaviour of balls in Cayley graphs of the fundamental group. By definition, the exponential growth rates of finitely generated groups measure the exponential expansion rate of balls in Cayley graphs and thus are entropy-like invariants. While there is no direct connection between the volume of a hyperbolic manifold M and the exponential growth rates of $\pi_1(M)$, the results of Fujiwara and Sela (2020) show that certain sets of such values share fundamental structural similarities.

To state these results, for a finitely generated group Γ, we write $\operatorname{Exp}(\Gamma) \subset \mathbb{R}$ for the (countable) set of all exponential growth rates $\mathrm{e}(\Gamma, S)$ with respect to finite generating sets S of Γ. The automorphism group $\operatorname{Aut}(\Gamma)$ acts on the set $\mathrm{FG}(\Gamma)$ of all finite generating sets of Γ and $\mathrm{e}(\Gamma, f(S)) = \mathrm{e}(\Gamma, S)$ holds for all $S \in \mathrm{FG}(\Gamma)$ and all $f \in \operatorname{Aut}(\Gamma)$. More details on terminology and notation can be found in Appendix A.

Theorem 1.1 (well-orderedness; Fujiwara and Sela, 2020, Theorem 2.2)
If Γ is a hyperbolic group, then $\operatorname{Exp}(\Gamma)$ is well-ordered (with respect to the standard order on $\mathbb{R}$).

Theorem 1.2 (finite ambiguity; Fujiwara and Sela, 2020, Theorem 3.1)
The set $\{S \in \mathrm{FG}(\Gamma) \mid \mathrm{e}(\Gamma, S) = r\}/\operatorname{Aut}(\Gamma)$ is finite for every non-elementary hyperbolic group Γ and every $r \in \mathbb{R}$.

Theorem 1.3 (growth ordinals; Fujiwara and Sela, 2020, Proposition 4.3)
Let Γ be a non-elementary hyperbolic group. Then the ordinal number $\operatorname{ord}_{\mathrm{Exp}}(\Gamma)$ associated with $\operatorname{Exp}(\Gamma)$ satisfies $\operatorname{ord}_{\mathrm{Exp}}(\Gamma) \geq \omega^\omega$.

Moreover, Fujiwara and Sela (2020, Proposition 4.3) show that $\operatorname{ord}_{\mathrm{Exp}}(\Gamma) = \omega^\omega$ if epi-limit groups over Γ have a Krull dimension. In analogy with the case of hyperbolic 3-manifolds, they conjecture that $\operatorname{ord}_{\mathrm{Exp}}(\Gamma) = \omega^\omega$ holds for all non-elementary hyperbolic groups Γ (Fujiwara and Sela, 2020, Section 4) .

Example 1.4. — If F is a finitely generated free group of rank at least 2, then limit groups over F have a Krull dimension (Louder, 2012). Hence, Theorems 1.1–1.3 show that $\operatorname{ord}_{\mathrm{Exp}}(F) = \omega^\omega$ and each value in $\operatorname{Exp}(F)$ is realised by only finitely many generating sets (up to automorphisms of F).

The key idea for the proofs of Theorems 1.1–1.3 is inspired by the proofs by Thurston and Jørgensen for the set of volumes of hyperbolic 3-manifolds and model theory: One passes from sequences of generating sets (of bounded size) of the given hyperbolic group Γ to a limit group over Γ with an associated finite generating set; i.e., limit groups play the role of cusped manifolds. The main challenge is then to compute the exponential growth rate of this limiting object in terms of the exponential growth rates appearing in the original sequence.

Overview

Basics on hyperbolic groups, exponential growth rates, and well-ordered countable sets are recalled in Appendix A. We briefly explain the manifold context of the results above in Section 2, with a focus on hyperbolic and simplicial volume. Section 3 gives a proof outline of the main results. Finally, in Section 4, we mention applications and extensions of the main results.

2. CONTEXT: VOLUMES OF MANIFOLDS AND HYPERBOLICITY

The results of Fujiwara and Sela (2020) are analogues of the behaviour of volumes of finite volume complete hyperbolic 3-manifolds. We recall this background in Section 2.1. The situation for simplicial volume is discussed in Section 2.2. In addition, we mention right-computability as a further structural property of "volume" sets (Section 2.3).

2.1. Hyperbolic volume

The structure and volumes of hyperbolic 3-manifolds was analysed in the breakthrough work of Jørgensen and Thurston.

THEOREM 2.1 (volumes of hyperbolic 3-manifolds; Thurston, 1979, Chapter 6)
The set

$$\{\mathrm{vol}(M) \mid M \textit{ is a finite volume complete hyperbolic } 3\textit{-manifold}\}$$

is well-ordered (with respect to the standard order on $\mathbb{R}$) and the associated ordinal is ω^ω. Moreover, every value arises only from finitely many isometry classes of finite volume hyperbolic 3-manifolds.

We briefly summarise the main steps of the proof (Gromov, 1981); the key is to study the convergence of sequences of hyperbolic manifolds and to understand the role of hyperbolic manifolds with cusps as limits of such sequences:

1. Every sequence $(M_n)_{n\in\mathbb{N}}$ of complete hyperbolic 3-manifolds with uniformly bounded volume contains a subsequence that converges in a strong geometric sense to a finite volume complete hyperbolic 3-manifold M and $\lim_{n\to\infty} \mathrm{vol}(M_n) = \mathrm{vol}(M)$. Furthermore, for "non-trivial" such sequences, one can show that $\mathrm{vol}(M) > \mathrm{vol}(M_n)$ holds for all members M_n of the subsequence.

   This can be used to show that the set of hyperbolic volumes is well-ordered and that every value can only be obtained in finitely many ways.
2. Every finite volume complete hyperbolic 3-manifold with $k \in \mathbb{N}$ cusps can be obtained for each $p \in \{0, \dots, k\}$ as the limit of a sequence of finite volume complete hyperbolic 3-manifolds with exactly p cusps.

   This can be used to show that the volume ordinal is at least ω^k. Constructing hyperbolic 3-manifolds with arbitrarily large numbers of cusps thus shows that the volume ordinal is at least ω^ω. In combination with the first part, one can derive that the volume ordinal equals ω^ω.

In contrast, in higher dimensions, the set of volumes of finite volume complete hyperbolic manifolds leads to the ordinal ω. This follows from Wang's finiteness theorem and the unboundedness of hyperbolic volumes.

THEOREM 2.2 (Wang's finiteness theorem; Wang, 1972). — *Let $n \in \mathbb{N}_{\geq 4}$ and $v \in \mathbb{R}_{\geq 0}$. Then there exist only finitely many isometry classes of finite volume complete hyperbolic n-manifolds M with $\mathrm{vol}(M) \leq v$.*

2.2. Simplicial volume

Simplicial volume is a homotopy invariant of closed manifolds. For several geometrically relevant classes of Riemannian manifolds, the simplicial volume encodes topological rigidity properties of the Riemannian volume.

DEFINITION 2.3 (simplicial volume; Gromov, 1982). — *The* simplicial volume *of an oriented closed connected manifold M is the ℓ^1-semi-norm of its (singular) $\mathbb{R}$-fundamental class:*

$$\|M\| := \|[M]_{\mathbb{R}}\|_1 := \inf\Big\{\sum_{j=1}^{k} |a_j| \;\Big|\; \sum_{j=1}^{k} a_j \cdot \sigma_j \text{ is a singular } \mathbb{R}\text{-fundamental cycle of } M\Big\}$$

For genuine hyperbolic manifolds, the simplicial volume leads to the same ordering and finiteness behaviour as the hyperbolic volume (Section 2.1):

Example 2.4 (hyperbolic manifolds). — If M is an oriented closed connected hyperbolic manifold of dimension n, then

$$\|M\| = \frac{\mathrm{vol}(M)}{v_n},$$

where $v_n \in \mathbb{R}_{>0}$ is the hyperbolic volume of ideal regular geodesic n-simplices in hyperbolic n-space (Thurston, 1979; Benedetti and Petronio, 1992). A similar relationship also holds in the complete finite volume case (Thurston, 1979; Fujiwara and Manning, 2011, Appendix A). In particular, this proportionality can be used to prove mapping degree estimates in terms of the hyperbolic volume for continuous maps between hyperbolic manifolds (Gromov, 1982).

Passing to the setting of fixed hyperbolic fundamental groups, we obtain:

Example 2.5 (hyperbolic fundamental group). — Let Γ be a finitely presented group and let $n \in \mathbb{N}$. Then the set

$$\mathrm{SV}_\Gamma(n) := \{\|M\| \mid M \text{ is an oriented closed connected } n\text{-manifold with } \pi_1(M) \cong \Gamma\}$$

is a subset of $\{\|\alpha\|_1 \mid \alpha \in H_n(\Gamma;\mathbb{R}) \text{ is integral}\}$, where a class in $H_n(\Gamma;\mathbb{R})$ is *integral* if it lies in the image of the change of coefficients map $H_n(\Gamma;\mathbb{Z}) \to H_n(\Gamma;\mathbb{R})$ (Löh, 2023, Section 3.1).

If Γ is hyperbolic and $n \geq 2$, then $\|\cdot\|_1$ is a norm on $H_n(\Gamma;\mathbb{R})$ (by the results of Mineyev (2001) on bounded cohomology and the duality principle). In particular: The set $\mathrm{SV}_\Gamma(n) \subset \mathbb{R}$ is well-ordered and the ordinal associated with $\mathrm{SV}_\Gamma(n)$ is

- either 0 (if $H_n(\Gamma;\mathbb{R}) \cong 0$);
- or ω (if $H_n(\Gamma;\mathbb{R}) \not\cong 0$): In this case, normed Thom realisation shows that indeed infinitely many different values are realised (Löh, 2023, Section 3.1).

For $n \geq 4$, finite ambiguity breaks down in this general topological setting: If M is an oriented closed connected n-manifold, then for each $k \in \mathbb{N}$, the manifold M and the iterated connected sums $M_k := M \# (S^2 \times S^{n-2})^{\# k}$ have the same simplicial volume (Gromov, 1982) and isomorphic fundamental groups. However, the manifolds $M_0, M_1, \ldots$ all have different homotopy types (as can be seen from the homology in degree 2).

2.3. Right-computability

In the previous discussion, we focussed on the order structure of volumes and exponential growth rates. Many real-valued invariants in geometric group theory and geometric topology also carry another, complementary, structure: They tend to have an intrinsic limit on their computational complexity. In particular, such a limit gives additional constraints on the possible sets of values.

DEFINITION 2.6 (right-computable). — *A real number α is* right-computable *if the set $\{x \in \mathbb{Q} \mid x > \alpha\}$ is recursively enumerable.*

For example, simplicial volumes of oriented closed connected manifolds are right-computable real numbers (Heuer and Löh, 2023). On the group-theoretic side, right-computability naturally arises for stable commutator length of recursively presented groups (Heuer, 2019) or L^2-Betti numbers of groups with controlled word problem (Löh and Uschold, 2022). Concerning exponential growth rates, we have the follwing:

PROPOSITION 2.7 (right-computability of exponential growth rates)
There exists a Turing machine that

– *given a finite presentation $\langle S \mid R \rangle$ and a finite set S' of words over $S \sqcup S^{-1}$,*
– *does*
  - not *terminate if S' does not represent a generating set of the group Γ described by $\langle S \mid R \rangle$;*
  - *terminate and return an enumeration of $\{x \in \mathbb{Q} \mid x > \mathrm{e}(\Gamma, S')\}$ if S' represents a generating set of Γ.*

COROLLARY 2.8. — *Let Γ be a finitely presented group.*

1. *For every $S \in \mathrm{FG}(\Gamma)$, the real number $\mathrm{e}(\Gamma, S)$ is right-computable.*
2. *For every $r \in \mathbb{Q}$, the truncated set $\{S \in \mathrm{FG}(\Gamma) \mid \mathrm{e}(\Gamma, S) < r\}$ is recursively enumerable.*

Proofs of these observations are provided in Appendix B. In particular, such results could be used to give a crude a priori upper bound for $\mathrm{ord}_{\mathrm{Exp}}(\Gamma)$ by a "large" countable ordinal for all finitely presented groups Γ with well-ordered set $\mathrm{Exp}(\Gamma)$.

3. PROOF TECHNIQUE

We outline the proofs of Theorem 1.1–1.3 by Fujiwara and Sela (2020). These proofs roughly follow the blueprint of the case of hyperbolic 3-manifolds (Section 2.1), where limit groups will play the role of cusped manifolds:

- A compactness phenomenon turns convergence of exponential growth rates into convergence of actions (of subsequences) to the action of a limit group.
- The main challenge is then to compute the exponential growth rates of these limit groups as the limit of the given exponential growth rates.

Before going into these arguments, we first recall basic notions on limit groups.

3.1. Limit groups

Limit groups are groups that arise as “limits” – in various senses – of groups. These groups are convenient tools in the model theory of groups and in geometric group theory (Kharlampovich and Myasnikov, 1998a,b; Sela, 2006; Groves and Wilton, 2018). In analogy with 3-manifolds, limit groups over hyperbolic groups admit a JSJ-decomposition (Sela, 2009; Weidmann and Reinfeldt, 2019, Section 4). Limit groups over a given group Γ are the finitely generated subgroups of non-principal ultraproducts of Γ. More explicitly:

Definition 3.1 (limit group). — *Let Γ be a group.*

- *A* stable homomorphism *from a group Λ to Γ is a sequence $(f_n\colon \Lambda \to \Gamma)_{n\in\mathbb{N}}$ of homomorphisms with the property*

$$\forall_{x\in\Lambda} \quad \exists_{N\in\mathbb{N}} \quad (\forall_{n\in\mathbb{N}_{\geq N}} \quad f_n(x) = 1) \vee (\forall_{n\in\mathbb{N}_{\geq N}} \quad f_n(x) \neq 1).$$

  The stable kernel *of a stable homomorphism $f_*\colon \Lambda \to \Gamma$ is defined as*

$$\ker f_* := \Big\{ x \in \Lambda \;\Big|\; \exists_{N\in\mathbb{N}} \quad \forall_{n\in\mathbb{N}_{\geq N}} \quad f_n(x) = 1 \Big\} \subset \Lambda.$$

- *A* limit group over Γ *is a group of the form $\Lambda/\ker f_*$, where Λ is a finitely generated group and $f_*\colon \Lambda \to \Gamma$ is a stable homomorphism. The canonical projection $f\colon \Lambda \to \Lambda/\ker f_*$ is the* limit homomorphism *and we say that f_** converges to f*. A limit group over Γ is an* epi-limit group over Γ *if the f_n can be chosen to be epimorphisms.*
- *A* limit group *is a limit group over a finitely generated free group.*

Example 3.2. — If Γ is a group, then every finitely generated subgroup of Γ can be viewed as a limit group over Γ (via the inclusion homomorphisms). In particular, Γ is an epi-limit group over Γ if Γ is finitely generated.

3.2. Limits and their exponential growth rate

THEOREM 3.3 (compactness; Fujiwara and Sela, 2020, proof of Theorem 2.2)

Let Γ be a hyperbolic group, let (X, d) be a Cayley graph of Γ, and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite generating sets of Γ such that the sequence $(\mathrm{e}(\Gamma, S_n))_{n\in\mathbb{N}}$ is bounded. Then there exists a subsequence (again denoted by $(S_n)_{n\in\mathbb{N}}$) with the following properties:

- *All S_n have the same size. Let S be a set of this cardinality and let F be the free group freely generated by S.*
- *There exist epimorphisms $f_n \colon F \to \Gamma$ for all $n \in \mathbb{N}$ such that $f_n(S)$ is conjugate to S_n. Moreover, f_* is a stable homomorphism $F \to \Gamma$. Let L denote the associated limit group.*
- *The sequence*
$$\Big(F \curvearrowright_{f_n} \Big(X, \frac{1}{\max_{s\in S} d(1, f_n(s))}\Big)\Big)_{n\in\mathbb{N}}$$
of actions (induced by $f_n \colon F \to \Gamma$ and the translation action of Γ on X) converges in the F-Gromov–Hausdorff distance to a faithful action of L on a real tree.

Proof. — The boundedness of the exponential growth rates allows us to fix the size of the generating set because the exponential growth rate grows at least linearly in the size of the generating set by an estimate of Arzhantseva and Lysenok (2006).

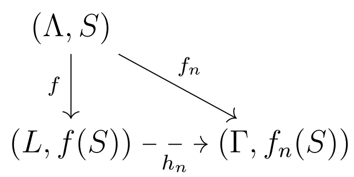
One can then apply the Bestvina–Paulin method after conjugating and rescaling appropriately (Fujiwara and Sela, 2020, proof of Theorem 2.2; Weidmann and Reinfeldt, 2019, Section 2). □

THEOREM 3.4 (limits of exponential growth rates; Fujiwara and Sela, 2020, Proposition 2.3)

Let Γ be a hyperbolic group, let Λ be a finitely generated group with finite generating set S, and let $(f_n \colon \Lambda \to \Gamma)_{n\in\mathbb{N}}$ be a stable homomorphism consisting of epimorphisms that converges to a limit group $f \colon \Lambda \to L$ over Γ with a faithful action on a real tree. Then $\mathrm{e}(\Gamma, f_n(S)) \leq \mathrm{e}(L, f(S))$ holds for all large enough $n \in \mathbb{N}$ and
$$\lim_{n\to\infty} \mathrm{e}(\Gamma, f_n(S)) = \mathrm{e}(L, f(S))$$

Sketch of proof. — By precomposition, without loss of generality we may assume that Λ is free and that S a free generating set.

We first explain why "$\leq$" holds (provided the limit exists): Because Γ is hyperbolic, for all large enough $n \in \mathbb{N}$, there exists a homomorphism $h_n \colon L \to \Gamma$ with $f_n = h_n \circ f$ (Weidmann and Reinfeldt, 2019, Lemma 6.5, Corollary 7.13):



Therefore, monotonicity of the exponential growth rates (Remark A.5) implies that $\mathrm{e}(\Gamma, f_n(S)) \leq \mathrm{e}(L, f(S))$ for all large enough $n \in \mathbb{N}$.

The hard work lies in proving convergence and "$\geq$": Given $\varepsilon \in \mathbb{R}_{>0}$, the goal is to show that for all large enough $n \in \mathbb{N}$, we have

$$\log \mathrm{e}(\Gamma, f_n(S)) \geq \log \mathrm{e}(L, f(S)) - \varepsilon.$$

Matters would be simple if, given $N \in \mathbb{N}$, the multiplication-projection map

$$\begin{aligned} B_N(L, f(S))^q &\to B_{q \cdot N}(\Gamma, f_n(S)) \\ (w_1, \dots, w_q) &\mapsto h_n(w_1 \cdot \dots \cdot w_q) \end{aligned}$$

were injective for all large enough $n \in \mathbb{N}$ and all $q \in \mathbb{N}$. However, this will not happen in general. Using the faithful limit action of L on a real tree, Fujiwara and Sela (2020, proof of Proposition 2.3) find enough freeness inside L to show through delicate estimates that there exists a $b \in \mathbb{N}$, a four-element subset $U \subset B_b(L, f(S))$, and a constant $C \in \mathbb{R}_{>0}$ with:

> For all $q \in \mathbb{N}$, there is a map $\varphi_q \colon L^q \to L$ of the form
>
> $$(w_1, \dots, w_q) \mapsto w_1 \cdot u_1 \cdot \dots \cdot w_q \cdot u_q,$$
>
> where the "separators" $u_1, \dots, u_q \in U$ may depend on $w_1, \dots, w_q$ and satisfy a "small cancellation condition" that ensures the following: Given $N \in \mathbb{N}$, for all large enough $n \in \mathbb{N}$ and all $q \in \mathbb{N}$, the map $h_n \circ \varphi_q \colon L^q \to \Gamma$ is injective on at least a subset $A_{N,n,q}$ of size $(1/C \cdot \beta_N(L, f(S)))^q$ of $B_N(L, f(S))^q$. In particular,
>
> $$\beta_{q \cdot (N+b)}(\Gamma, f_n(S)) \geq \# A_{N,n,q} \geq \Big(\frac{1}{C} \cdot \beta_N(L, f(S))\Big)^q.$$
>
> More specifically, this works for all $n \in \mathbb{N}$ that are large enough so that h_n is injective on $B_{2 \cdot N}(L, f(S))$; such n exist in view of the convergence of actions.

Given $\varepsilon \in \mathbb{R}_{>0}$, we choose $N \in \mathbb{N}$ large enough to have

$$\frac{1}{N+b} \cdot (\log \beta_N(L, f(S)) - \log C) \geq \frac{1}{N} \cdot \log \beta_N(L, f(S)) - \varepsilon.$$

Then, we obtain for all large enough $n \in \mathbb{N}$ that

$$\begin{aligned} \log \mathrm{e}(\Gamma, f_n(S)) &= \lim_{q \to \infty} \frac{1}{q \cdot (N+b)} \cdot \log \beta_{q \cdot (N+b)}(\Gamma, f_n(S)) \\ &\geq \frac{1}{N+b} \cdot \log \frac{\beta_N(L, f(S))}{C} \\ &\geq \frac{1}{N} \cdot \log \beta_N(L, f(S)) - \varepsilon \\ &\geq \log \mathrm{e}(L, f(S)) - \varepsilon, \end{aligned}$$

as desired. □

Remark 3.5. — The proof of Theorem 3.4 is mainly based on properties of the limit action on the real tree. In fact, the theorem also holds under the following weaker assumptions on Γ (Fujiwara, 2021, Proposition 3.2): The group Γ is finitely generated, equationally Noetherian, and admits a non-elementary isometric action on a hyperbolic graph X that satisfies a uniform weak proper discontinuity condition (Fujiwara, 2021, Definition 2.1) and that admits a constant N such that for every $S \in \mathrm{FG}(\Gamma)$, the set S^N contains an element that acts hyperbolically on X.

3.3. Well-orderedness

Sketch of proof of Theorem 1.1. — If the given hyperbolic group Γ is virtually cyclic, then $\mathrm{Exp}(\Gamma) = \{1\}$, which clearly is well-ordered.

In the following, we consider the case when Γ is non-elementary hyperbolic. We assume for a contradiction that there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of finite generating sets of Γ such that $(\mathrm{e}(\Gamma, S_n))_{n\in\mathbb{N}}$ is strictly monotonically decreasing. In particular, the sequence $(\mathrm{e}(\Gamma, S_n))_{n\in\mathbb{N}}$ is bounded. In view of the compactness theorem (Theorem 3.3) and the invariance of the exponential growth rates under conjugation, we may assume without loss of generality that there exists a free group F with free generating set S and epimorphisms $(f_n\colon F \to \Gamma)_{n\in\mathbb{N}}$ with $f_n(S) = S_n$ and such that f_* converges to a limit group $f\colon F \to L$ over Γ with a faithful action on a real tree.

We therefore obtain from Theorem 3.4 that

$$\begin{aligned}\lim_{n\to\infty} \mathrm{e}(\Gamma, S_n) &= \lim_{n\to\infty} \mathrm{e}(\Gamma, f_n(S)) \\ &= \mathrm{e}(L, f(S)) && \text{(Theorem 3.4)} \\ &\geq \mathrm{e}(\Gamma, S_N) > \mathrm{e}(\Gamma, S_{N+1}) > \dots, && \text{(for all } N \gg 0\text{; Theorem 3.4)}\end{aligned}$$

which is impossible. This contradiction shows that no such strictly decreasing sequence exists and hence $\mathrm{Exp}(\Gamma)$ is well-ordered. □

3.4. Finite ambiguity

Sketch of proof of Theorem 1.2. — Let $r \in \mathbb{R}_{>1}$ and let us assume for a contradiction that there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of finite generating sets that all represent different $\mathrm{Aut}(\Gamma)$-orbits and that satisfy $\mathrm{e}(\Gamma, S_n) = r$ for all $n \in \mathbb{N}$.

Proceeding as before, by the compactness theorem (Theorem 3.3), we may assume without loss of generality that there exists a free group F with free generating set S and epimorphism $(f_n\colon F \to \Gamma)_{n\in\mathbb{N}}$ with $f_n(S) = S_n$ and such that f_* converges to a limit group $f\colon F \to L$ over Γ with a faithful action on a real tree. Hence, Theorem 3.4 shows that

$$\mathrm{e}(L, f(S)) = \lim_{n\to\infty} \mathrm{e}(\Gamma, f_n(S)) = r$$

On the other hand, by passing to a subsequence, we may furthermore assume that for all $n \in \mathbb{N}$, there exists a homomorphism $h_n\colon L \to \Gamma$ with $f_n = h_n \circ f$ (as in the proof of Theorem 3.4), that at most one of the epimorphisms h_n is an isomorphism (because the S_n lie in different $\mathrm{Aut}(\Gamma)$-orbits), and that the kernels of the h_n contain no torsion. Then a careful refinement of the proof of Theorem 3.4 shows that the strict inequality

$$\mathrm{e}(L, f(S)) > \mathrm{e}(\Gamma, f_n(S)) = r$$

holds for all $n \in \mathbb{N}$ (Fujiwara and Sela, 2020, Proposition 3.2). This contradicts the previous computation that $\mathrm{e}(L, f(S)) = r$. □

3.5. Growth ordinals

Sketch of proof of Theorem 1.3. — It suffices to show that $\mathrm{ord}_{\mathrm{Exp}}(\Gamma) \geq \omega^m$ for every $m \in \mathbb{N}$. Let $m \in \mathbb{N}$. We consider the sequence

$$L_1 := \Gamma * F_m \to L_2 := \Gamma * F_{m-1} \to \cdots \to L_m := \Gamma * \mathbb{Z} \to L_{m+1} := \Gamma$$

of epimorphisms, where F_j is a free group of rank j and where the epimorphisms successively kill free generators and keep the Γ-factor intact. It helps to think of j as the number of cusps.

Let us first focus on a single step: If Λ is a non-elementary hyperbolic group, then there exists a stable homomorphism $(f_n \colon \Lambda * \mathbb{Z} \to \Lambda)_{n \in \mathbb{N}}$ consisting of epimorphisms that converges to $\Lambda * \mathbb{Z}$. Let S be a finite generating set of Λ and let $\widetilde{S} \subset \Lambda * \mathbb{Z}$ be a generating set of $\Lambda * \mathbb{Z}$, e.g., obtained by adding a free generator of $\mathbb{Z}$. By passing to subsequences of f_*, one can achieve the following strict monotonicity:

- The sequence $(\mathrm{e}(\Lambda, f_n(\widetilde{S})))_{n \in \mathbb{N}}$ is increasing and converges to $\mathrm{e}(\Gamma * \mathbb{Z}, \widetilde{S})$; this uses Theorem 3.3 and Theorem 3.4, as before.
- The values in the sequence are all different; this uses a finite ambiguity theorem for finitely generated subgroups of limit groups over Λ (Fujiwara and Sela, 2020, Theorem 5.8).

For notational simplicity, we now restrict to the case $m = 2$. We choose a finite generating set S of Γ and take the extended finite generating set $\widetilde{S}$ of $L_1 = \Gamma * F_2$. Applying the single step to $L_1 \to L_2$ leads to a stable homomorphism f^1_* with strict monotonicity. Let f^2_* be a stable homomorphism for $L_2 \to L_3$. For each $n_1 \in \mathbb{N}$, we apply the single step to $L_2 \to L_3$ and the generating set $f_{n_1}(\widetilde{S})$ to select a subsequence of f^2_* with strict monotonicity. By composing with f_{n_1}, we obtain a sequence $f_{n_1,*}$ from L_1 to $L_3 = \Gamma$ such that $(\mathrm{e}(\Gamma, f_{n_1,n}(\widetilde{S})))_{n \in \mathbb{N}}$ is strictly increasing and converges to $\mathrm{e}(L_2, f_{n_1}(\widetilde{S}))$. By varying n_1, we thus see that $\mathrm{ord}_{\mathrm{Exp}}(\Gamma) \geq \omega^2$.

For higher values of m, one iterates these considerations appropriately. □

To prove $\mathrm{ord}_{\mathrm{Exp}}(\Gamma) \leq \omega^\omega$ under additional hypotheses, Fujiwara and Sela (2020, proof of Theorem 4.2) construct proper epimorphism chains of limit groups over Γ from convergent sequences of convergent sequences of etc... of exponential growth rates of Γ; the Krull dimension property then gives control on the maximal lengths of such chains, whence on the maximal powers of ω that appear below a given threshold.

4. APPLICATIONS AND EXTENSIONS

The well-orderedness of exponential growth rates (Theorem 1.1) in particular contains the fact that all non-elementary hyperbolic groups have uniformly exponential growth.

4.1. Hyperbolic groups are Hopfian

A group Γ is *Hopfian* if every self-epimorphism $\Gamma \to \Gamma$ is an automorphism. This property has applications in the context of degrees of self-maps of closed manifolds. Hyperbolic groups are known to be Hopfian (Sela, 1999; Weidmann and Reinfeldt, 2019). Using that the exponential growth rates of hyperbolic groups are well-ordered, Fujiwara and Sela (2020, Corollary 2.9) complete an approach to proving that hyperbolic groups are Hopfian outlined by de la Harpe (2002); this is not an independent alternative proof because the current proof of Theorem 1.1 uses the very results on limit groups that go into the previous proofs that hyperbolic groups are Hopfian.

Corollary 4.1. — *Every hyperbolic group is Hopfian.*

Proof. — Elementary hyperbolic groups are Hopfian because they are virtually cyclic (whence finitely generated and residually finite).

Let Γ be a non-elementary hyperbolic group and let $f \colon \Gamma \to \Gamma$ be an epimorphism. Because $\mathrm{Exp}(\Gamma)$ is well-ordered (Theorem 1.1), there exists a finite generating set S of Γ with $\mathrm{e}(\Gamma) = \mathrm{e}(\Gamma, S)$. *Assume* for a contradiction that the kernel of f is non-trivial. Then, Arzhantseva and Lysenok (2002) show that there is a *strict* monotonicity

$$\mathrm{e}(\Gamma, S) > \mathrm{e}(\Gamma, f(S)).$$

However, this contradicts the minimality property of S. Thus, f is an automorphism. □

All finitely generated residually finite groups are Hopfian. While fundamental groups of closed hyperbolic manifolds are residually finite and hyperbolic, it is a long-standing open problem whether all hyperbolic groups are residually finite.

4.2. Generalisations

The methods discussed in Section 3 by Fujiwara and Sela (2020) extend to cover also the following generalisations:

- If Γ is a hyperbolic group, then the set

  $$\{\mathrm{e}(H, S) \mid H < \Gamma \text{ finitely generated and non-elementary},\ S \in \mathrm{FG}(H)\}$$

  is well-ordered (Fujiwara and Sela, 2020, Theorem 5.1). This can be viewed as an addition to the Tits alternative for hyperbolic groups.
- Moreover, in this subgroup setting, there is a corresponding finite ambiguity statement for non-elementary hyperbolic groups (Fujiwara and Sela, 2020, Theorem 5.3).

As a consequence, they also obtain analogous results for limit groups over non-elementary hyperbolic groups (Fujiwara and Sela, 2020, Corollary 5.6–5.10). Furthermore, the approach is robust enough to admit an extension to the case of sub-semigroups (Fujiwara and Sela, 2020, Section 6).

Fujiwara (2021) adapted the method to obtain well-orderedness of exponential growth rates sets for other classes of groups, including certain groups acting acylindrically on hyperbolic spaces, rank-1 lattices, fundamental groups of strictly negatively curved Riemannian manifolds, and certain relatively hyperbolic groups. These results can, for instance, be applied to certain subgroups of right-angled Artin groups (Kerr, 2021, Corollary 1.0.11).

APPENDIX A. TERMINOLOGY

For the sake of completeness, we recall the basic terminology appearing in the main results (Section 1).

A.1. Hyperbolic groups

Finitely generated groups are hyperbolic if their Cayley graphs are "negatively curved" in the sense that geodesic triangles in are uniformly slim (Gromov, 1987; Bridson and Haefliger, 1999):

Definition A.1 (hyperbolic group). — *A finitely generated group* Γ *is* hyperbolic *if the Cayley graph of* Γ *with respect to one (whence every (Bridson and Haefliger, 1999, Theorem III.H.1.9)) finite generating set is a hyperbolic metric space. A hyperbolic group is* non-elementary *if it is not virtually cyclic.*

Example A.2 (hyperbolic groups). — Fundamental groups of closed smooth manifolds that admit a Riemannian metric of negative sectional curvature are hyperbolic in view of the Švarc–Milnor lemma and the fact that CAT(-1)-spaces are hyperbolic metric spaces. In particular, this includes the fundamental groups of closed hyperbolic manifolds. Such fundamental groups are virtually cyclic if and only if the dimension is at most 1.

Finitely generated free groups are hyperbolic. The class of hyperbolic groups is closed under quasi-isometries (Bridson and Haefliger, 1999, Theorem III.H.1.9) and under certain amalgamations (Bestvina and Feighn, 1996).

The group $\mathbb{Z}^2$ is *not* hyperbolic. More generally, all finitely generated groups that contain a subgroup isomorphic to $\mathbb{Z}^2$ are *not* hyperbolic (Bridson and Haefliger, 1999, Corollary III.Γ.3.10). In general, subgroups of hyperbolic groups need *not* be hyperbolic.

A.2. Exponential growth rates of groups

The exponential growth rate of groups measures the exponential expansion rate of the size of balls in Cayley graphs (de la Harpe, 2000, Chapter VII.B):

Remark A.3. — Let Γ be a finitely generated group and let $S \subset \Gamma$ be a finite generating set of Γ. We write $\beta_n(\Gamma, S)$ for the number of elements in the n-ball $B_n(\Gamma, S)$ of the Cayley graph of Γ with respect to S. Then $\beta_{n+m}(\Gamma, S) \leq \beta_n(\Gamma, S) \cdot \beta_m(\Gamma, S)$ for all $n, m \in \mathbb{N}$. Therefore, the Fekete lemma shows that the limit of $(\beta_n(\Gamma, S)^{1/n})_{n\in\mathbb{N}}$ exists and that

$$\lim_{n\to\infty} \beta_n(\Gamma, S)^{1/n} = \inf_{n\in\mathbb{N}_{>0}} \beta_n(\Gamma, S)^{1/n}.$$

Definition A.4 (exponential growth rate). — *Let Γ be a finitely generated group.*

- *Let $S \subset \Gamma$ be a finite generating set of Γ. The* exponential growth rate of Γ with respect to S *is defined as*

$$\mathrm{e}(\Gamma, S) := \lim_{n\to\infty} \beta_n(\Gamma, S)^{1/n}.$$

- *We write* $\mathrm{Exp}(\Gamma) := \{\mathrm{e}(\Gamma, S) \mid S \in \mathrm{FG}(\Gamma)\}$ *for the (countable) set of all exponential growth rates of* Γ*, where* $\mathrm{FG}(\Gamma)$ *denotes the set of finite generating sets of* Γ*.*
- *The* exponential growth rate of Γ *is the infimum*

$$\mathrm{e}(\Gamma) := \inf \mathrm{Exp}(\Gamma).$$

- *The group Γ has* exponential growth *if there exists an $S \in \mathrm{FG}(\Gamma)$ with $\mathrm{e}(\Gamma, S) > 1$. The group Γ has* uniform exponential growth *if $\mathrm{e}(\Gamma) > 1$.*

Remark A.5 (monotonicity of exponential growth rates). — Let Γ and Λ be finitely generated groups.

1. If $f : \Gamma \to \Lambda$ is an epimorphism, and $S \in \mathrm{FG}(\Gamma)$, then $\mathrm{e}(\Gamma, S) \geq \mathrm{e}(\Lambda, f(S))$.
2. If Λ is a subgroup of Γ and Λ has exponential growth, then also Γ has exponential growth.

Example A.6 (exponential growth). — Finitely generated free groups have exponential growth if and only if they are of rank at least 2. More precisely (de la Harpe, 2000, Proposition VII.13): If F is a free group and $S \in \mathrm{FG}(F)$, then

$$\mathrm{e}(F, S) \geq 2 \cdot \mathrm{rk}(F) - 1.$$

Because non-elementary hyperbolic groups [uniformly] contain free groups of rank 2, monotonicity shows that they have [uniform] exponential growth (Koubi, 1998).

There exist finitely generated groups that have exponential growth but do *not* have uniform exponential growth (Wilson, 2004). In particular, for such groups Γ, the set $\mathrm{Exp}(\Gamma)$ is *not* well-ordered.

Exponential growth rates seem to be fragile under quasi-isometries: It is an open problem to determine whether uniform exponential growth is stable under quasi-isometries.

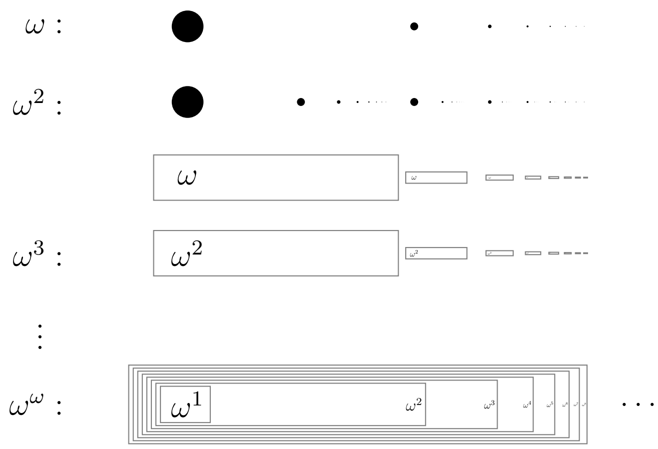


FIGURE 1. The ordinals ω, ω^2, ..., ω^ω, schematically

A.3. Well-ordered countable sets and ordinals

Well-orderings are orderings that allow for induction principles. Moreover, well-orderings admit an arithmetic, the ordinal arithmetic.

DEFINITION A.7 (well-ordered sets, ordinals). — *An ordered set* $(A, <)$ *is* well-ordered *if every non-empty subset of* A *contains a* $<$*-minimal element. An* ordinal *is an isomorphism class of well-ordered ordered sets. An ordinal is* countable *if the underlying set is countable.*

In the context of Section 1, the following ordinals are important (Figure 1):

Example A.8 (ω^ω). — The natural numbers $\mathbb{N}$ are well-ordered with respect to the standard order. The corresponding ordinal is denoted ω. For $k \in \mathbb{N}$, we write ω^k for the ordinal represented by $\mathbb{N}^k$ with the lexicographic order. Equipping the finite support functions $\mathbb{N} \to \mathbb{N}$ with the lexicographic order leads to a well-ordered set; its ordinal number is denoted by ω^ω. The ordinal ω^ω can alternatively also be described as $\sup_{k \in \mathbb{N}} \omega^k$.

Example A.9. — The subsets $A := \{1 - 1/n \mid n \in \mathbb{N}_{>0}\}$ and $B := \bigcup_{n \in \mathbb{N}}(n + A)$ of $\mathbb{R}$ are well-ordered with respect to the standard order on $\mathbb{R}$. The set A represents the ordinal ω and B represents the ordinal ω^2. The subset $\mathbb{Q}_{\geq 0} \subset \mathbb{R}$ is *not* well-ordered.

APPENDIX B. RIGHT-COMPUTABILITY OF EXPONENTIAL GROWTH RATES

We provide proofs for the right-computability claims in Section 2.3.

Proof of Proposition 2.7. — Let $F(S)$ be the set of reduced words over $S \sqcup S^{-1}$. In particular, $F(S)$ is a free group, freely generated by S, with respect to the composition given by concatenation and reduction. It is well known that we can Turing-enumerate all finite subsets of $F(S)$ that represent generating sets of Γ under the canonical projection $F(S) \to \langle S \mid R\rangle = \Gamma$ (by Turing-enumerating the normal closure of R in $F(S)$).

Therefore it suffices to show that there exists a Turing machine that given a finite generating set $S' \subset F(S)$ of Γ enumerates the set $A(S') := \{x \in \mathbb{Q} \mid x > \mathrm{e}(\Gamma, S')\}$. By the Fekete lemma (Remark A.3), for all generating sets S', we have

$$A(S') = \Big\{x \in \mathbb{Q} \;\Big|\; \exists_{n\in\mathbb{N}_{>0}} \quad x^n > \beta_n(\Gamma, S')\Big\}.$$

The numbers $\beta_n(\Gamma, S')$ are not necessarily computable in terms of n and S' (as the word problem might not be solvable in Γ), but recursively enumerating the normal closure of R in $F(S)$ shows that there exists a Turing machine that given a finite generating set $S' \subset F(S)$ of Γ enumerates $\{(n, m) \mid n, m \in \mathbb{N},\ m \geq \beta_n(\Gamma, S')\}$; hence, there is also a Turing machine for $A(\,\cdot\,)$. □

Proof of Corollary 2.8. — The first part is a direct consequence of Proposition 2.7.

For the second part, let $r \in \mathbb{Q}$. For all $S \in \mathrm{FG}(\Gamma)$, we have

$$\mathrm{e}(\Gamma, S) < r \iff \exists_{x\in\mathbb{Q}} \quad (r > x \wedge x > \mathrm{e}(\Gamma, S)).$$

Let $\langle S \mid R\rangle$ be a finite presentation of Γ. Using a Turing machine as provided by Proposition 2.7, we can thus construct a Turing machine that enumerates all finite sets S' of words over $S \sqcup S^{-1}$ such that S' represents a generating set of Γ and such that $\mathrm{e}(\Gamma, S') < r$. □

For simplicity, we restricted the discussion to finitely presented groups. Similar arguments also apply to finitely generated recursively presented groups. Conversely, one might wonder whether every right-computable real number ≥ 1 can be realised as the exponential growth rate of some finite generating set of some finitely/recursively presented group.

Acknowledgements

This work was supported by the CRC 1085 *Higher Invariants* (Universität Regensburg, funded by the DFG).

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