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TWO PROOFS OF THE $P = W$ CONJECTURE
[after Maulik–Shen and Hausel–Mellit–Minets–Schiffmann]

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INTRODUCTION

For a smooth projective complex algebraic curve C, non-abelian Hodge theory of Hitchin (1987b) and Donaldson (1983, 1987), Corlette (1988) and Simpson (1994a, 1988, 1992) provides a real analytic isomorphism

$$\mathcal{M}^{\mathrm{Dol}}_{n,d}(C) \simeq \mathcal{M}^{\mathrm{B}}_{n,d}(C) \tag{1}$$

where $\mathcal{M}^{\mathrm{Dol}}_{n,d}(C)$ is the *Dolbeault* moduli space of rank n degree d semistable *Higgs bundles* $(E, \theta \colon E \to E \otimes \omega_C)$ on C and $\mathcal{M}^{\mathrm{B}}_{n,d}(C)$ is the *Betti* moduli space of d-twisted n-dimensional representations of the fundamental group $\pi_1(C)$.

These moduli spaces are complex algebraic varieties, and are smooth when n and d are coprime; however, the above isomorphism is transcendental and does not preserve the complex structures on these moduli spaces. In particular, the Dolbeault moduli space depends on the complex structure of C, whereas the Betti moduli space only depends on the topology (i.e. the genus) of C. The Betti moduli space is affine via its construction as an affine geometric invariant theory quotient, while the Dolbeault moduli space contains many compact subvarieties that appear as fibres of the *Hitchin map*. The Hitchin map is a proper morphism $h \colon \mathcal{M}^{\mathrm{Dol}}_{n,d}(C) \to \mathcal{A}_n$ to an affine space which records the coefficients of the characteristic polynomial of the Higgs field $\theta \colon E \to E \otimes \omega_C$ viewed as a twisted endomorphism of the vector bundle E. By the spectral correspondence (Beauville, Narasimhan, and Ramanan, 1989; Hitchin, 1987b; Schaub, 1998), the general fibres $h^{-1}(a)$ are abelian varieties isomorphic to the Jacobian of an associated spectral curve $\mathcal{C}_a \subset T^*C := \mathrm{Tot}(\omega_C)$, which is a degree n cover of C.

The homeomorphism (1) induces an isomorphism on cohomology, but this does not preserve mixed Hodge structures. Assume from now on that n and d are coprime. Then the Hodge structure of the Dolbeault moduli space is pure, as there is a $\mathbb{G}_m$-action which scales the Higgs field and provides a deformation retract onto its fixed locus, which is a smooth projective subvariety. The mixed Hodge structure on the Betti moduli space is genuinely mixed and does not see the complex structure of C; in fact, it turns out to be of Hodge–Tate type. Consequently, one can ask if the non-trivial weight filtration on the cohomology of the Betti moduli space has a geometric interpretation on the Dolbeault side via non-abelian Hodge theory.

Hausel and Rodriguez-Villegas (2008) spotted a symmetry in the Hodge numbers of Betti moduli spaces and formulated a Curious Hard Lefschetz conjecture relating opposite graded pieces of the weight filtration; they proved this in rank 2 and Mellit (2019) recently proved this in higher ranks. This lead de Cataldo, Hausel, and Migliorini (2012) to formulate (and prove in rank $n = 2$) the $P = W$ *conjecture*: the *weight filtration* W on the cohomology of the Betti moduli space should correspond to the *perverse Leray filtration* P associated to the Hitchin map on the Dolbeault moduli space. This would imply that the Curious Hard Lefschetz property is explained by the Relative Hard Lefschetz Theorem for the Hitchin map. In 2022, two independent proofs of the $P = W$ conjecture were given by Maulik–Shen (MS) and Hausel–Mellit–Minets–Schiffmann (HMMS).

Although these two proofs are very different, they share a common starting point: the cohomology of the Dolbeault moduli space is generated by certain *tautological classes* obtained from Künneth components of the Chern classes of the universal Higgs bundle $\mathcal{E} \to \mathcal{M}_{n,d}^{\mathrm{Dol}} \times C$ by work of Markman (2002). If these tautological classes are appropriately normalised, their weights are known by work of Shende (2017). Since the weight filtration is multiplicative (i.e. the cup product adds weights), the $P = W$ conjecture can be studied entirely on the Higgs moduli space as a question about the interaction between products of tautological classes and the perverse filtration. One key challenge is that the perverse filtration is not in general multiplicative.

Maulik and Shen reduce the $P = W$ conjecture to a sheaf-theoretic property they call *strong perversity* for cup products with tautological classes in the derived category of constructible sheaves over the Hitchin base. To prove that the Chern classes of the universal bundle have the predicted strong perversity, they work with parabolic moduli spaces appearing in Yun's *global Springer theory* (Yun, 2012, 2011). The parabolic structure is given by a full flag in a fibre and induces certain tautological line bundles which have the desired strong perversity over an open in the Hitchin base by work of Yun. Using the decomposition theorem (Beĭlinson, Bernstein, and Deligne, 1982), this would automatically extend to the whole Hitchin base if the parabolic Hitchin map had full supports. Unfortunately, this is not the case, but previous work of Maulik and Shen (2021) that builds on work of Ngô (2006, 2010) and Chaudouard and Laumon (2016) offers a solution: instead of classical Higgs bundles, consider D-twisted Higgs bundles $(E, \theta\colon E \to E \otimes \omega_C(D))$ for an effective divisor D. Maulik and Shen prove a parabolic support theorem for D of sufficiently high degree and then pass from D-twisted Higgs bundles to classical Higgs bundles via a *vanishing cycles* argument as in Maulik and Shen (2021). Consequently, they deduce the image of the weight filtration under non-abelian Hodge theory is contained in the perverse filtration and conclude they are equal using the Relative and Curious Hard Lefschetz Theorems.

Hausel, Mellit, Minets and Schiffmann approach the $P = W$ conjecture by constructing Lie algebras acting on the cohomology using two natural operations on the cohomology of Higgs moduli spaces (or rather their stacks): cup products with tautological classes and *Hecke correspondences* that modify a vector bundle at single point in C. By the Relative Hard Lefschetz Theorem for the Hitchin fibration, a choice of relatively ample divisor class provides an identification between opposite graded pieces of the perverse filtration and defines a nilpotent operator on the associated graded vector space that can be extended to an $\mathfrak{sl}_2$-triple $\langle \mathbf{e}, \mathbf{f}, \mathbf{h} \rangle$ whose $\mathbf{h}$-graded pieces are the perverse graded pieces. The idea is to find a lifted $\mathfrak{sl}_2$-triple acting on the cohomology (rather than the associated graded object for P) where the first two operators come from tautological and Hecke operators respectively, the third induces the perverse filtration and the tautological classes are $\mathbf{h}$-eigenvectors in order to describe the interaction between tautological classes and the perverse filtration. The desired $\mathfrak{sl}_2$-triple is found by considering an action of a much larger Lie algebra $\mathcal{H}_2$ of polynomial Hamiltonian vector fields on the plane and by using the spectral correspondence to instead work with sheaves on surfaces and their cohomological Hall algebras as in Mellit, Minets, Schiffmann, and Vasserot (2023). In this story, there is again a technical problem to overcome: Hecke correspondences do not preserve semistability and so first a result is shown on the *elliptic* locus, where the spectral curves are integral, and then parabolic bundles are used to pass from the elliptic locus to the whole moduli space. This proof actually shows the perverse and weight filtrations both coincide with the filtration induced by this $\mathfrak{sl}_2$-triple and also gives a new proof of the Curious Hard Lefschetz Theorem.

Recently, a third proof due to Maulik, Shen, and Yin (2023) appeared, which offers a new perspective; we do not include the details due to lack of time and space but merely mention the main results. A key theorem is that the elliptic locus is a *twisted (self)-dualisable abelian fibration satisfying Fourier vanishing*; this builds on Arinkin's work on compactified Jacobians of locally planar integral curves (Arinkin, 2013). For any such fibration, Maulik, Shen, and Yin (2023) prove a (motivic) decomposition and show that the perverse filtration is multiplicative; hence the tautological operators have the expected perversity and the *Chern filtration* (see §2.4.1) is contained in the perverse filtration, which suffices to conclude the equality $P = W$ using the Curious Hard Lefschetz Theorem.

Although the methods of proofs are very different, they involve similar moduli spaces (for example, parabolic Higgs bundles and elliptic loci appear in all three proofs). The third proof (Maulik, Shen, and Yin, 2023) may come closer to relating to the approach in (HMMS) as follows. As a consequence of the semi-classical limit of the geometric Langlands correspondence, the Hitchin fibration is expected to be self-dual, in the sense that there is a derived equivalence which should exchange Hecke operators and certain tautological operators. This expectation is realised over the regular locus by Donagi and Pantev (2012) and over the elliptic locus by Arinkin (2013). Furthermore, Polishchuk (2007) produced actions of $\mathfrak{sl}_2$ (and $\mathcal{H}_2$) on the rational tautological Chow ring of the Jacobian of a curve using the Fourier transform, which suggests a possible deeper relation between Maulik, Shen, and Yin (2023) and (HMMS).

The goal of this report is to give an overview of the main ideas in the proofs of Maulik–Shen (MS) and Hausel–Mellit–Minets–Schiffmann (HMMS), and the relevant background.

Many of the techniques and ideas in the proofs play an important role in the study of moduli spaces more generally: operators constructed from tautological classes and natural correspondences generate actions of interesting algebras in various contexts, such as Hilbert schemes (Nakajima, 1997; Grojnowski, 1996), infinite symmetric powers (Kimura and Vistoli, 1996), Jacobians (Polishchuk, 2007) and CoHAs (Mellit, Minets, Schiffmann, and Vasserot, 2023). Furthermore vanishing cycles, support theorems and duality for abelian varieties are important tools for studying (Higgs) moduli spaces.

0.1. Structure of the paper

In §1, we introduce the moduli spaces and recall non-abelian Hodge theory, then provide some background on mixed Hodge structures and perverse sheaves in order to state the $P = W$ conjecture. In §2, we introduce the tautological classes and state Markman's result on tautological generation and Shende's computation of the weights of the tautological generators; this leads to various reformulations of the $P = W$ conjecture purely in terms of the interaction of the perverse filtration with tautological classes on the Dolbeault side. Finally in §3 and in §4 respectively, we discuss the proofs of Maulik–Shen (MS) and Hausel–Mellit–Minets–Schiffmann (HMMS).

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1. MODULI SPACES AND THE $P = W$ CONJECTURE

Throughout, we let C denote a smooth projective geometrically connected algebraic curve of genus g over a field k, which will often be the complex numbers.

1.1. Moduli spaces and non-abelian Hodge theory

The terminology *non-abelian* Hodge theory is used to indicate a generalisation of classical Hodge theory, which relates various cohomology theories with coefficients in the abelian group $\mathbb{C}$, to a version taking values in the non-abelian coefficient group $\mathrm{GL}_n(\mathbb{C})$ and leads to diffeomorphism of various moduli spaces related to these cohomology theories.

For a smooth complex projective variety X, the de Rham isomorphism and Hodge decomposition give isomorphisms

$$H^k_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{C}) \simeq H^k_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

between the (topological) *Betti* cohomology of singular cochains on the underlying topological space X^{an}, the (differential geometric) *de Rham* cohomology groups and the (holomorphic) *Dolbeault* cohomology groups, which can be expressed as sheaf cohomology groups via the Dolbeault isomorphisms $H^{p,q}(X) \simeq H^q(X, \Omega^p_X)$. The Hodge decomposition involves relating both cohomology theories to harmonic operators; similarly, harmonic metrics also play a central role in the non-abelian version of this story.

Let us focus on the case of a smooth complex projective curve C. We have

$$H^1_{\mathrm{B}}(C^{\mathrm{an}}, \mathbb{C}) \simeq H^1_{\mathrm{dR}}(C^{\mathrm{an}}, \mathbb{C}) \simeq H^{0,1}(C) \oplus H^{1,0}(C) \simeq H^1(C, \mathcal{O}_C) \oplus H^0(C, \omega_C).$$

The idea of non-abelian Hodge theory is to replace the abelian coefficient group $\mathbb{C}$ with the non-abelian coefficient group $\mathrm{GL}_n(\mathbb{C})$. For $n = 1$, on the Betti side we have

$$H^1_B(C, \mathbb{C}^*) \simeq \mathrm{Hom}(\pi_1(C), \mathbb{C}^*) = \mathrm{Rep}(\pi_1(C), \mathbb{C}^*)$$

characters of the fundamental group and on the Dolbeault side we have pairs consisting of a holomorphic line bundle parametrised by $H^1(C, \mathcal{O}^*_C) \simeq \mathrm{Pic}(C)$ and a holomorphic 1-form parametrised by $H^0(C, \omega_C)$. For $n > 1$, the Betti moduli space $\mathcal{M}^{\mathrm{B}}_{\mathrm{GL}_n}(C)$ parametrises representations $\pi_1(C) \to \mathrm{GL}_n(\mathbb{C})$ and the Dolbeault moduli space $\mathcal{M}^{\mathrm{Dol}}_{n,0}(C)$ parametrises certain rank n degree 0 holomorphic vector bundles on C equipped with a Higgs field, which is a bundle endomorphism twisted by the canonical line bundle ω_C. We formally introduce these moduli spaces below. The non-abelian Hodge theorem gives a real analytic isomorphism between the Dolbeault and Betti moduli spaces, which is not compatible with their complex structures. The proof uses the existence of certain harmonic metrics and a third de Rham moduli space of holomorphic bundles with connections appears; however, we will not introduce the de Rham side, as the $P = W$ conjecture relates only the Betti and Dolbeault moduli spaces.

Non-abelian Hodge theory can be viewed as a complexification of the Narasimhan–Seshadri Theorem, which relates irreducible unitary representations of the fundamental group with stable vector bundles. The notion of stability appearing here was introduced by Mumford to construct moduli spaces of (semi)stable vector bundles as geometric invariant theory quotients. We begin by introducing moduli spaces of vector bundles and the Narasimhan–Seshadri theorem, before proceeding to non-abelian Hodge theory.

1.1.1. *Moduli of semistable vector bundles.* — In this section, we can work over an arbitrary field k. The *slope* of a vector bundle E on C is $\mu(E) := \deg(E)/\operatorname{rk}(E)$ and we say E is *semistable* (respectively stable) if for every proper subbundle $E' \subsetneq E$, we have $\mu(E') \leq \mu(E)$ (respectively a strict inequality). Any semistable vector bundle E has a (non-unique) *Jordan–Hölder filtration* $0 = E_0 \subsetneq E_1 \subsetneq \cdots E_l = E$ by subbundles whose successive quotients are stable of the same slope as E; the associated graded object $\mathrm{gr}(E) = \oplus_{i=1}^{l} E_i/E_{i-1}$ is unique and is *polystable* (that is, it is a direct sum of stable vector bundles of the same slope). Two semistable bundles are *S-equivalent* if their Jordan–Holder graded bundles are isomorphic. Every vector bundle has a unique maximally destabilising subbundle, which is a maximal subbundle of maximal slope; this inductively gives rise to a unique *Harder–Narasimhan* filtration whose successive quotients are semistable with decreasing slopes. If k is not algebraically closed, then stability is not an open property, but by also asking for the base change to an algebraic closure to be stable, we obtain a notion of *geometrically stable*, which is open in families; we will mostly suppress this subtlety as we are primarily concerned with the case when $\mathrm{k} = \mathbb{C}$. We will also focus on vector bundles of coprime rank and degree, where semistability is equivalent to (geometric) stability.

Consider the moduli functor $\underline{\mathcal{N}}^s_{n,d}(C)\colon \mathrm{Sch}_{\mathrm{k}} \to \mathrm{Sets}$ that sends a k-scheme T to the set of vector bundles on $C \times T$ which are fibrewise geometrically stable of rank n and degree d modulo the relation $\mathcal{E} \cong \mathcal{F} \otimes \pi_T^* \mathcal{L}$ for a line bundle $\mathcal{L}$ on T (we allow tensoring by pullbacks of line bundles on T to make this a Zariski sheaf). This moduli functor is coarsely represented by a scheme $\mathcal{N}^s_{n,d} = \mathcal{N}^s_{n,d}(C)$, which we refer to as the moduli space of stable vector bundles. By deformation theory, $T_{[E]}\mathcal{N}^s_{n,d} \simeq \mathrm{Ext}^1(E,E)$ and the obstructions to smoothness (of the stack) lie in Ext^2 and so vanish on a curve. Since (geometrically) stable bundles E only have scalar automorphisms, we have $\chi(E,E) := \sum_{i\geq 0}(-1)^i \dim \mathrm{Ext}^i(E,E) = 1 - \dim \mathrm{Ext}^1(E,E)$. Hence, $\mathcal{N}^s_{n,d}$ is smooth and, for $g > 1$, we have by Riemann–Roch

$$\dim(\mathcal{N}^s_{n,d}) = -\chi(E,E) + 1 = n^2(g-1) + 1.$$

Over $\overline{\mathrm{k}}$, stable bundles exist for all n and d provided $g > 1$; thus $\mathcal{N}^s_{n,d}$ is non-empty.

The stable moduli space has a compactification, the moduli space $\mathcal{N}_{n,d} = \mathcal{N}_{n,d}(C)$ of (S-equivalence classes of) semistable vector bundles, which is constructed via geometric invariant theory as a projective variety (Mumford, Fogarty, and Kirwan, 1994; Newstead, 1978; Seshadri, 1967). Its geometric points parametrise polystable vector bundles of rank n and degree d on C. Over the complex numbers, $\mathcal{N}_{n,d}$ can also be constructed as a (real) symplectic reduction of a gauge group acting on the space of holomorphic structures on a fixed smooth complex vector bundle (Atiyah and Bott, 1983).

If n and d are coprime, semistability and stability coincide and $\mathcal{N}^s_{n,d} = \mathcal{N}_{n,d}$ is a smooth projective variety. If also $C(\mathrm{k}) \neq \emptyset$, then $\mathcal{N}_{n,d}$ is a fine moduli space, as there is a universal family $\mathcal{U} \to \mathcal{N}^s_{n,d} \times C$ obtained via pullback from the stack of stable rank n and degree d bundles, which is a neutral $\mathbb{G}_m$-gerbe over $\mathcal{N}_{n,d}$ (see Heinloth, 2010, Corollary 3.12). Conversely, if n and d are non-coprime, $\mathcal{N}^s_{n,d}$ does not admit a universal family (Ramanan, 1973).

1.1.2. *The Narasimhan–Seshardi correspondence.* — Assume now that $\mathrm{k} = \mathbb{C}$. Given a unitary representation $\rho\colon \pi_1(C) \to \mathrm{U}(n)$ (or equivalently a unitary local system), one can construct an associated rank n degree 0 vector bundle

$$E_\rho = \widetilde{C} \times^\rho \mathbb{C}^n$$

from the representation ρ and the $\pi_1(C)$-bundle given by the universal cover $\widetilde{C} \to C$.

Theorem 1.1 (The Narasimhan–Seshadri Theorem (Narasimhan and Seshadri, 1965))
A degree 0 *holomorphic vector bundle on a smooth projective curve* C *of genus* $g \geq 2$ *is stable if and only if it arises from an irreducible unitary representation.*

Although this statement concerns an individual vector bundle, its proof uses moduli spaces: the (harder) forward direction involves showing that the natural map from the moduli space of irreducible unitary representations to the moduli space of stable vector bundles is a surjective real-analytic submersion by a deformation theory argument.

Narasimhan and Seshadri moreover proved for arbitrary d that $\mathcal{N}_{n,d}(C)$ is real analytically isomorphic to a moduli space of twisted unitary representations of the fundamental group, which can be viewed as unitary representations of the fundamental group of a punctured curve with prescribed monodromy at the puncture depending on d. The analogue of these moduli space of twisted unitary representations obtained by replacing $\mathrm{U}(n)$ with $\mathrm{GL}_n(\mathbb{C})$ are the twisted Betti moduli spaces (see §1.1.3) appearing in the non-abelian Hodge correspondence.

Let us outline a gauge-theoretic proof of the Narasimhan–Seshadri Theorem due to Donaldson (1983), as similar ideas are used in the non-abelian Hodge correspondence. Donaldson proved that a vector bundle is polystable if and only if it admits a Hermitian metric whose associated unitary Chern connection is projectively flat (such connections are called Hermitian–Einstein). For a vector bundle with such a Hermitian–Einstein connection, one can use the Chern–Weil description of the degree to prove the inequality needed for (poly)stability. Conversely, to find such a Hermitian–Einstein connection on a (poly)stable vector bundle, Donaldson considered the Yang–Mills flow and used Uhlenbeck compactness and stability to show the limit lies in the same gauge orbit.

1.1.3. *The Betti moduli spaces of representations.* — By replacing the unitary group $\mathrm{U}(n)$ with its complexification $\mathrm{GL}_n(\mathbb{C})$, we can take an algebro-geometric quotient of the conjugation action on the space of representations $\pi_1(C) \to \mathrm{GL}_n(\mathbb{C})$ using geometric invariant theory (Mumford, Fogarty, and Kirwan, 1994). The resulting quotient is an affine complex variety and is known as the Betti moduli space or character variety. Similarly to Betti cohomology, it forgets the complex structure on C and only depends on the underlying topological space.

We can define the Betti moduli space over an arbitrary field k and may replace GL_n with any reductive Lie group G over k (and one can also replace C with a higher dimensional variety). For the conjugation action of G on the affine variety

$$\mathrm{Rep}(\pi_1(C), G) := \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} : \prod_{i=1}^{n} [A_i, B_i] = I\},$$

there is an associated affine GIT quotient $\mathcal{M}_G^{\mathrm{B}} = \mathcal{M}_G^{\mathrm{B}}(C) := \mathrm{Rep}(\pi_1(C), G)/\!/G$ given by taking the spectrum of the k-algebra of G-invariant functions on $\mathrm{Rep}(\pi_1(C), G)$, which is finitely generated as G is reductive. The notation for the GIT quotient indicates that this is not an orbit space and orbits are identified if their closures meet. Consequently the Betti moduli space parametrises closed orbits or equivalently semisimple G-representations. To avoid these GIT identifications, we can instead consider the Betti stack of G-representations which is the quotient stack of representations modulo conjugation

$$\mathfrak{M}^{\mathrm{B}}_{G} = \mathfrak{M}^{\mathrm{B}}_{G}(C) := [\mathrm{Rep}(\pi_1(C), G)/G].$$

The Betti moduli stack classifies G-local systems on C, or equivalently G-bundles with a flat connection by the Riemann–Hilbert correspondence. For $G = \mathrm{GL}_n$, the existence of a flat connection implies that the corresponding vector bundle has degree 0.

To obtain twisted Betti moduli spaces in non-zero degrees, one considers the space of representations of a punctured curve $C \setminus \{x\}$ with prescribed diagonal monodromy at x as follows. For $G = \mathrm{GL}_n$ or SL_n, the degree d twisted Betti moduli space is the quotient $\mathcal{M}^{\mathrm{B}}_{G,d}(C) = \mathrm{Rep}_d(\pi_1(C), G)/\!/G$, where

$$\mathrm{Rep}_d(\pi_1(C), G) = \{(A_1, \dots, A_g, B_1, \dots, B_g) \in G^{2g} : \prod_{i=1}^{n} [A_i, B_i] = e^{2\pi i d/n} I_n\}.$$

We note that this moduli space only depends on the value of d modulo n. If n and d are coprime, then $\mathcal{M}^{\mathrm{B}}_{\mathrm{GL}_n,d}$ is smooth by Hausel and Rodriguez-Villegas (2008, Theorem 2.2.5), as $\mathrm{Rep}_d(\pi_1(C), \mathrm{GL}_n)$ is smooth and the GL_n-quotient coincides with the free quotient by PGL_n.

If we now replace GL_n with PGL_n, then the prescribed monodromy becomes trivial and so we get an honest representation $\rho \colon \pi_1(C) \to \mathrm{PGL}_n$ of the fundamental group. The obstruction to lifting ρ to GL_n is given by a Schur multiplier $\delta(\rho)$ arising from the exact sequence

$$\to H^1(\pi_1(C), \mathbb{G}_m) \to H^1(\pi_1(C), \mathrm{GL}_n) \to H^1(\pi_1(C), \mathrm{PGL}_n) \xrightarrow{\delta} H^2(\pi_1(C), \mathbb{G}_m).$$

The PGL_n-representations whose image under δ vanishes lift to GL_n and otherwise they lift to a twisted GL_n-representation. Consequently, we can stratify the PGL_n-moduli space using Schur multipliers

$$\mathcal{M}^{\mathrm{B}}_{\mathrm{PGL}_n} = \bigsqcup_{d \in \mathbb{Z}/n\mathbb{Z}} \mathcal{M}^{\mathrm{B}}_{\mathrm{PGL}_n,d}$$

where

$$\mathcal{M}^{\mathrm{B}}_{\mathrm{PGL}_n,d} \simeq \mathcal{M}^{\mathrm{B}}_{\mathrm{GL}_n,d}/(\mathbb{C}^*)^{2g} \simeq \mathcal{M}^{\mathrm{B}}_{\mathrm{SL}_n,d}/\mu_n^{2g}.$$

For $G = \mathrm{GL}_n$, we will simply write $\mathcal{M}^{\mathrm{B}}_{n,d} = \mathcal{M}^{\mathrm{B}}_{\mathrm{GL}_n,d}$ for the Betti moduli space and

$$\mathfrak{M}^{\mathrm{B}}_{n,d} = [\mathrm{Rep}_d(\pi_1(C), \mathrm{GL}_n)/\mathrm{GL}_n]$$

for the Betti moduli stack.

1.1.4. *The Dolbeault moduli space.* — In non-abelian Hodge theory, the complexified version of the moduli space of (semi)stable vector bundles appearing in the Narasimhan–Seshadri Theorem is the Dolbeault moduli space $\mathcal{M}_{n,d} := \mathcal{M}^{\mathrm{Dol}}_{n,d}(C)$ of (semi)stable Higgs bundles introduced by Hitchin (1987b).

We can also define the Dolbeault moduli space over an arbitrary field k (and also in much greater generality over a higher-dimensional base or in fact in families). A *Higgs bundle* on C is pair $(E, \theta : E \to E \otimes \omega_C)$ consisting of a vector bundle E on C and a *Higgs field* θ, which is an endomorphism of E twisted by the canonical bundle. Semistability of Higgs bundles is given by verifying an inequality of slopes for all Higgs subbundles (that is, subbundles which are invariant under the Higgs field). There is a moduli space $\mathcal{M}_{n,d} = \mathcal{M}^{\mathrm{Dol}}_{n,d}(C)$ of (S-equivalence classes of) semistable Higgs bundles which can be constructed as a quasi-projective variety using geometric invariant theory (Nitsure, 1991; Simpson, 1994a) (see also Remark 1.3 below) or as a hyperkähler reduction via gauge theory (Hitchin, 1987b). The moduli space $\mathcal{M}^s_{n,d}$ of (geometrically) stable Higgs bundles is a smooth open subvariety. If n and d are coprime, then $\mathcal{M}^s_{n,d} = \mathcal{M}_{n,d}$ is smooth and is a fine moduli space which admits a universal family. If additionally $C(\mathrm{k}) \neq \emptyset$, the stack of stable rank n degree d Higgs bundles is a neutral $\mathbb{G}_m$-gerbe over $\mathcal{M}^s_{n,d}$.

From an algebraic perspective, $\mathcal{M}^s_{n,d}$ naturally arises as an algebraic symplectic analogue of $\mathcal{N}^s_{n,d}$, as it contains $T^*\mathcal{N}^s_{n,d}$ as a dense open set: by deformation theory and Serre duality we have

$$T^*_{[E]}\mathcal{N}^s_{n,d} = \mathrm{Ext}^1(E, E)^* \cong \mathrm{Hom}(E, E \otimes \omega_C)$$

which is the space of Higgs fields on E. The algebraic symplectic form on this cotangent bundle extends to $\mathcal{M}^s_{n,d}$ and moreover it admits a Lagrangian fibration to an affine space given by the Hitchin fibration (see §1.2 below). In particular, for $g \geq 2$, we have

$$\dim(\mathcal{M}_{n,d}) = 2\dim(\mathcal{N}_{n,d}) = 2(n^2(g-1)+1).$$

Tensoring a Higgs bundle by a line bundle $L \in \mathrm{Pic}^e(C)$ preserves (semi)stability and determines an isomorphism $\mathcal{M}_{n,d} \to \mathcal{M}_{n,d+ne}$; hence, over $\overline{\mathrm{k}}$, the Higgs moduli space up to isomorphism only depends on the value of d modulo n. Moreover, there is a χ-independence phenomena: various cohomological invariants of $\mathcal{M}_{n,d}$ are independent of d. Non-abelian Hodge theory gives an isomorphism on cohomology for different d's coprime to n, and in this coprime setting, Kinjo and Koseki (2021) and de Cataldo, Maulik, Shen, and Zhang (2021) prove the isomorphism on cohomology preserves the perverse filtration and the cup product respectively, whereas Groechenig, Wyss, and Ziegler (2020) show the point counts of $\mathcal{M}_{n,d}$ over finite fields are independent of d.

Similarly to the Betti moduli spaces, for any complex reductive group G, one can define moduli spaces of semistable G-Higgs bundles $\mathcal{M}_G(C)$ which are given by a principal G-bundle $P \to C$ with a section $\theta \in H^0(C, \mathrm{ad}(P) \otimes \omega_C)$ of the associated adjoint bundle of P twisted by the canonical bundle that satisfy a natural notion of semistability.

For $G = \mathrm{SL}_n$, we can realise $\mathcal{M}_{\mathrm{SL}_n} = \mathcal{M}_{\mathrm{SL}_n}(C)$ as the closed subvariety of $\mathcal{M}_{n,0}$ consisting of GL_n-Higgs bundles (E, θ) such that $\det(E) \simeq \mathcal{O}_C$ and $\mathrm{tr}(\theta) = 0$. Since $\mathcal{M}_{n,0}$ and $\mathcal{M}_{\mathrm{SL}_n}$ are in general singular, due to the existence of strictly semistable Higgs bundles for non-coprime invariants, it is convenient to replace the SL_n-Higgs moduli space by twisted versions in non-zero degrees as follows. For a line bundle $L \in \mathrm{Pic}^d(C)$, consider the closed subvariety

$$\mathcal{M}_{\mathrm{SL}_n,L} \subset \mathcal{M}_{n,d}$$

consisting of GL_n-Higgs bundles (E, θ) such that $\det(E) \simeq L$ and $\mathrm{tr}(\theta) = 0$. We shall refer to this as a (twisted) SL_n-Higgs moduli space, which is smooth if n and d are coprime. If k is algebraically closed, different choices of degree d line bundles gives isomorphic twisted SL_n-moduli spaces, as we can multiply by a root of their difference.

The PGL_n-Higgs moduli spaces can be described as the following quotients

$$\mathcal{M}_{\mathrm{PGL}_n,d} \cong \mathcal{M}_{\mathrm{SL}_n,L}/\operatorname{Jac}^0(C)[n] \cong \mathcal{M}_{n,d}/T^* \operatorname{Jac}(C),$$

where $\operatorname{Jac}^0(C)[n]$ denotes the group of n-torsion points acting by tensorisation. Even for n and d coprime, this moduli space may acquire singularities from quotienting by the non-free action of the finite group $\operatorname{Jac}^0(C)[n]$. However, for n and d coprime, it is smooth when viewed as a Deligne–Mumford stack:

$$\mathfrak{M}_{\mathrm{PGL}_n,d} \cong [\mathcal{M}_{\mathrm{SL}_n,L}/\operatorname{Jac}^0(C)[n]\,] \cong [\mathcal{M}_{n,d}/T^* \operatorname{Jac}(C)].$$

Let us briefly introduce stacks of Higgs bundles, as we will sometimes need them later on. The stack of rank n degree d Higgs bundles $\mathfrak{Higgs}_{n,d}$ is a singular Artin stack which contains the stack $\mathfrak{M}_{n,d}$ of semistable Higgs bundles as a finite type open substack. The substack $\mathfrak{M}^s_{n,d}$ of (geometrically) stable Higgs bundles is smooth and is a $\mathbb{G}_m$-gerbe over $\mathcal{M}^s_{n,d}$, as every endomorphism of a (geometrically) stable Higgs bundle is given by a scalar multiple of the identity. There is a natural map $\mathfrak{M}_{n,d} \to \mathcal{M}_{n,d}$ which is a good moduli space (Alper, 2013) if the characteristic of k is zero (and an adequate moduli space in positive characteristic).

1.1.5. *Non-abelian Hodge theory.* — Over $\mathrm{k} = \mathbb{C}$, the Dolbeault moduli space $\mathcal{M}^{\mathrm{Dol}}_{n,d}(C)$ of semistable Higgs bundles on C and the Betti moduli space $\mathcal{M}^{\mathrm{B}}_{n,d}(C)$ of twisted representations of the fundamental group of C are both complex algebraic varieties, which are smooth when n and d are coprime.

Theorem 1.2 (Non-abelian Hodge Theory (Hitchin, 1987b; Donaldson, 1987; Corlette, 1988; Simpson, 1988))

There is a real analytic isomorphism

$$\mathcal{M}^{\mathrm{Dol}}_{n,d}(C) \simeq \mathcal{M}^{\mathrm{B}}_{n,d}(C).$$

This isomorphism is transcendental and is proved by relating both moduli problems to moduli of harmonic metrics. In the degree $d = 0$ case, the Betti moduli space classifies complex vector bundles with flat connections and by work of Donaldson (1987) and Corlette (1988) such a flat connection admits a harmonic metric if and only if it is completely reducible. This harmonic metric splits the connection into a skew-Hermitian and Hermitian part, which respectively give a holomorphic structure on the bundle and an endomorphism valued 1-form. The existence of such a harmonic metric on a Higgs bundle can be interpreted in terms of polystability by work of Hitchin (1987b) and Simpson (1988), analogously to Donaldson's proof of the Narasimhan–Seshadri Theorem. For further details, we refer to Simpson (1997, 1994b) and the Bourbaki seminar of Le Potier (1991).

The complex structures on the Betti moduli space and Dolbeault moduli space are very different; for example, only the Dolbeault moduli space sees the complex structure on C. In fact, these two different complex structures appear as part of the hyperkähler structure on the smooth locus coming from the construction of moduli space of solutions to Hitchin's equations as a hyperkähler reduction (Hitchin, 1987b).

In rank $n = 1$, the Betti moduli space

$$\mathcal{M}^{\mathrm{B}}_{1,d}(C) \cong (\mathbb{C}^*)^{2g}$$

of a genus g curve C is diffeomorphic (but not biholomorphic) to

$$\mathcal{M}^{\mathrm{Dol}}_{1,d}(C) \cong \mathrm{Pic}^d(C) \times H^0(C, \omega_C) \cong \mathbb{C}^g/\Lambda \times \mathbb{C}^g,$$

as via polar coordinates we have a (non-holomorphic) diffeomorphism $\mathbb{C}^* \simeq S^1 \times \mathbb{R}$. The diffeomorphism given by non-abelian Hodge theory depends on the complex structure of C; for an explicit coordinate description, see (Goldman and Xia, 2008, §7.4).

The above statement admits many variants and generalisations. Simpson (1994b) proved a non-abelian Hodge correspondence for any smooth complex projective variety. For punctured non-compact curves, Simpson (1990) also worked out a corresponding non-abelian Hodge correspondence via harmonic bundles which uses filtered (or parabolic) Higgs bundles with certain tameness or regularity conditions at the boundary. In higher dimensions, this has been extended to quasi-projective varieties with a smooth completion with smooth boundary by Biquard (1997) and with normal crossings boundary by Mochizuki (2009). The case of wild singularities was studied by Biquard and Boalch (2004) and Mochizuki (2011). Simpson (1992) showed one can replace the general linear group with another complex reductive group G (or even a real reductive group) to obtain non-abelian Hodge theory with values in G. Non-abelian Hodge theory has been developed in positive characteristic by Ogus and Vologodsky (2007) and over p-adic fields by Faltings (2005) and Lan, Sheng, and Zuo (2019), among others.

The Dolbeault and Betti moduli spaces have alternative generalisations which have no analogue under non-abelian Hodge theory. On the Dolbeault side, one can consider D-twisted Higgs bundles (also known as meromorphic Higgs bundles), which are pairs $(E, \theta\colon E \to E \otimes \omega_C(D))$ for an effective divisor D on C. The moduli space $\mathcal{M}^D_{n,d}$ of D-twisted Higgs bundles only admits a Poisson structure rather than an algebraic symplectic structure. These D-twisted Higgs moduli spaces play a key role in Maulik and Shen's proof of the $P = W$ conjecture and have played a prominent role in the study of Higgs bundles since their use by Ngô (2006). There is no Betti analogue of these

D-twisted Higgs moduli spaces. On the Betti side, one can consider character varieties where the fundamental group is replaced by any finitely generated group (Lawton and Sikora, 2017; Lubotzky and Magid, 1985).

1.2. The Hitchin map and the spectral correspondence

A regular semisimple endomorphism of a finite dimensional complex vector space is uniquely determined by its eigenvalues and their (1-dimensional) eigenspaces. The Hitchin map on the moduli space $\mathcal{M}_{n,d}$ of Higgs bundles on C records the eigenvalues (or rather the coefficients of the characteristic polynomial) of the Higgs field and the spectral correspondence describes the fibres in terms of families of 1-dimensional eigenspaces (or more precisely rank 1 torsion-free sheaves on certain spectral covers of C).

1.2.1. *The Hitchin map.* — Viewing a Higgs field $\theta\colon E \to E \otimes \omega_C$ as an ω_C-twisted endomorphism of a rank n vector bundle E, we can consider its characteristic polynomial whose coefficients are given by $a_i(\theta) = (-1)^i \operatorname{tr}(\wedge^i\theta) \in H^0(C, \omega_C^{\otimes i})$. The assignment $(E,\theta) \mapsto a(\theta) = (a_1(\theta), \dots, a_n(\theta))$ defines the *Hitchin map*

$$h\colon\ \mathcal{M}_{n,d} \to \mathcal{A}_n := \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}).$$

Alternatively, the Hitchin map can be described as the affinisation morphism for the Higgs moduli space. Since $T^*\mathcal{N}^s_{n,d} \subset \mathcal{M}_{n,d}$ is a dense open subset with complement of codimension at least 2 (Hitchin, 1987a, Proposition 6.1 iv) and the Higgs moduli space is normal (Simpson, 1992, Corollary 11.7), the rings of regular functions on $T^*\mathcal{N}^s_{n,d}$ and $\mathcal{M}_{n,d}$ coincide. The Hitchin map coincides with the affinisation morphism

$$h\colon \mathcal{M}_{n,d} \to \operatorname{Spec} \mathcal{O}(\mathcal{M}_{n,d}) = \operatorname{Spec} \mathcal{O}(T^*\mathcal{N}^s_{n,d}) = \mathcal{A}_n.$$

One of the most intriguing and intensively studied fibres is the fibre $h^{-1}(0)$ which is called the *nilpotent cone*. We will see in Proposition 1.7 that there is a $\mathbb{G}_m$-action on $\mathcal{M}_{n,d}$ which induces a deformation retract onto the nilpotent cone.

1.2.2. *Spectral curves.* — Let $\pi\colon T^*C \to C$ denote the projection; then there is a tautological section $\lambda \in H^0(T^*C, \pi^*\omega_C)$. For $a = (a_1, \dots, a_n) \in \mathcal{A} = \mathcal{A}_n$, consider

$$\lambda^n + \pi^*a_1\lambda^{n-1} + \cdots + \pi^*a_n \in H^0(T^*C, \pi^*\omega_C^{\otimes n})$$

whose vanishing locus $\mathcal{C}_a \subset T^*C$ is the *spectral curve* over $a \in \mathcal{A}$. The projection π restricts to a degree n cover $\pi_a\colon \mathcal{C}_a \to C$. The spectral curve $\mathcal{C}_a$ may singular, although the singularities are locally planar as $\mathcal{C}_a \subset T^*C$, and moreover may be non-reduced (e.g. for $a = 0$). We obtain a family of spectral curves $\mathcal{C}/\mathcal{A}$ with $\mathcal{C} \hookrightarrow T^*C \times \mathcal{A} \to C \times \mathcal{A}$.

Since the projection from the cotangent bundle

$$\pi\colon T^*C = \underline{\operatorname{Spec}}_C(\operatorname{Sym}^\bullet \mathcal{T}_C) \to C$$

is affine, we have $\pi_*\mathcal{O}_{T^*C} = \operatorname{Sym}^\bullet \mathcal{T}_C$ and π_* induces an equivalence of categories between quasi-coherent $\mathcal{O}_{T^*C}$-modules F and quasi-coherent $\operatorname{Sym}^\bullet(\mathcal{T}_C)$-modules π_*F such that π_*F is coherent if and only if $\operatorname{supp}(F) \hookrightarrow T^*C \to C$ is finite. Restricting π to a spectral curve $\pi_a\colon \mathcal{C}_a \to C$, the $\mathcal{O}_C$-algebra $(\pi_a)_*\mathcal{O}_{\mathcal{C}_a}$ is isomorphic to the quotient of $\mathrm{Sym}^\bullet(\mathcal{T}_C)$ by the ideal generated by the image of $u_a\colon \mathcal{T}_C^{\otimes n} \to \mathrm{Sym}^\bullet(\mathcal{T}_C)$ given by summing the maps $\mathcal{T}_C^{\otimes n} \xrightarrow{a_i} \mathcal{T}_C^{\otimes n-i} \hookrightarrow \mathrm{Sym}^\bullet(\mathcal{T}_C)$. Thus $(\pi_a)_*\mathcal{O}_{\mathcal{C}_a} = \mathcal{O}_C \oplus \mathcal{T}_C \oplus \cdots \oplus \mathcal{T}_C^{\otimes(n-1)}$ and the arithmetic genus of $\mathcal{C}_a$ is $g(\mathcal{C}_a) = 1 - \chi(\mathcal{C}_a, \mathcal{O}_{\mathcal{C}_a})$. By adjunction, we compute

$$\chi(\mathcal{C}_a, \mathcal{O}_{\mathcal{C}_a}) = \chi(C, (\pi_a)_*\mathcal{O}_{\mathcal{C}_a}) = \sum_{i=0}^{n} \chi(C, \mathcal{T}^{\otimes i}) = (2-2g)\left(\sum_{i=0}^{n-1} i\right) + n(1-g) = n^2(1-g)$$

and so the arithmetic genus is given by $g(\mathcal{C}_a) = 1 + n^2(g-1)$.

1.2.3. *The spectral correspondence.* — Via a type of abelianisation procedure described by Hitchin (1987b) for smooth spectral curves, and by Beauville, Narasimhan, and Ramanan (1989) for integral spectral curves, and by Schaub (1998) in general, Higgs bundles on C can be related to certain rank 1 torsion-free sheaves on spectral curves $\mathcal{C}_a$.

Given a Higgs bundle $(E, \theta\colon E \to E \otimes \omega_C)$, we obtain from θ a homomorphism

$$\mathrm{Sym}^\bullet(\mathcal{T}_C) \otimes E \to E$$

thus there is an $\mathcal{O}_{T^*C}$-module F such that $\pi_*(F) = E$ and the support of F is a finite cover of C. Let $a = h(E, \theta)$; then by the Cayley–Hamilton Theorem, we have $\theta^n + a_1\theta^{n-1} + \cdots + a_n = 0$ and thus $\mathrm{supp}(F) = \mathcal{C}_a$. Furthermore, since E is torsion-free, the restriction of F to its support $\mathcal{C}_a$ is torsion-free (as the sheaf of rational functions on $\mathcal{C}_a$ is obtained from that of C by tensoring over $\mathcal{O}_C$ by $\mathcal{O}_{\mathcal{C}_a}$). Let η_i be generic points on the irreducible components $\mathcal{C}_{a,i}$ of $\mathcal{C}_{a,\mathrm{red}}$ and write $\mathcal{O}_{\eta_i} := \mathcal{O}_{\mathcal{C},\eta_i}$; then

$$\sum_i l_{\mathcal{O}_{\eta_i}}(\mathcal{O}_{\eta_i}) = \deg(\pi_a) = \mathrm{rk}(E) = l_{\mathcal{O}_{C,\eta}}(E_{C,\eta}) = \sum_i l_{\mathcal{O}_{\eta_i}}(F_{\eta_i})$$

and since $l_{\mathcal{O}_{\eta_i}}(F_{\eta_i}) \le l_{\mathcal{O}_{\eta_i}}(\mathcal{O}_{\eta_i})$, this inequality must be an equality on each connected component, which means F has rank 1 (in the sense of Schaub, 1998, Définition 1.2).

Conversely, given a rank 1 torsion-free sheaf F on a spectral curve $\mathcal{C}_a$, its pushforward $E = (\pi_a)_*F$ to C is torsion-free (and thus locally free, as C is smooth) of rank n by the above computation. The $\mathcal{O}_{\mathcal{C}_a}$-structure on F gives a $\mathrm{Sym}^\bullet(\mathcal{T}_C)$-module structure on E, which in particular gives a homomorphism $\mathcal{T}_C \to \mathcal{E}nd(E)$ determining a Higgs field $\theta\colon E \to E \otimes \omega_C$. The degree of F and $E = (\pi_a)_*F$ are related as follows:

$$\deg(F) := \chi(\mathcal{C}_a, F) - \mathrm{rk}(F)\chi(\mathcal{C}_a, \mathcal{O}_{\mathcal{C}_a}) = \chi(C, E) - n^2(1-g) = \deg(E) + (n - n^2)(1-g).$$

If $\mathcal{C}_a$ is smooth, then the torsion-free sheaf F is a line bundle and so there is a bijection between rank n Higgs bundles on C with characteristic polynomial specified by a and line bundles on $\mathcal{C}_a$.

We outlined the idea behind the spectral correspondence above; let us briefly explain its interaction with families and semistability (see Theorem 1.4 below for a concise summary). The above correspondence works functorially in families and respects isomorphisms, so we obtain an isomorphism over $\mathcal{A}$ between the stack of rank n degree d Higgs bundles on C and the stack of torsion-free rank 1 degree $d + (n - n^2)(1-g)$ sheaves on $\mathcal{C}/\mathcal{A}$. By a result of Schaub (1998), the spectral correspondence respects semistability of Higgs bundles (E,θ) and a natural notion of semistability for the corresponding torsion-free rank 1 sheaf $F=(\pi_a)_*E$ on the spectral curve $\mathcal{C}_a$, which involves testing an inequality for all torsion-free rank 1 quotients of F restricted to subcurves of $\mathcal{C}_a$ (see Chaudouard and Laumon, 2016, Remarque 4.2). In particular, if $\mathcal{C}_a$ is integral, all torsion-free sheaves are stable and the corresponding Higgs bundles over $a\in\mathcal{A}$ are automatically stable.

Remark 1.3. — Viewing $F=\pi_*E$ as a pure 1-dimensional sheaf on T^*C or its compactification $\mathbb{P}(\mathcal{O}_C\oplus\mathcal{T}_C)$, Simpson proved that slope semistability of (E,θ) corresponds to Gieseker semistability of F on $\mathbb{P}(\mathcal{O}_C\oplus\mathcal{T}_C)$ with respect to an appropriate polarisation. Consequently, one can construct the moduli space of semistable Higgs bundles on C as the open subscheme of the moduli space of Gieseker semistable 1-dimensional sheaves on $\mathbb{P}(\mathcal{O}_C\oplus\mathcal{T}_C)$ whose support is disjoint from the boundary. Since the moduli space of Gieseker semistable sheaves on $\mathbb{P}(\mathcal{O}_C\oplus\mathcal{T}_C)$ (with fixed Hilbert polynomial) is projective, one immediately deduces that the moduli space of Higgs bundles is quasi-projective.

To succinctly summarise the above discussion, we need to introduce the smooth (or regular) and elliptic loci in the Hitchin base; the latter acquires its name from Ngô's work on the fundamental lemma and endoscopy, where the word elliptic is used in analogy with elliptic elements in reductive groups. Let $\mathcal{A}^{\mathrm{ell}}$ (resp. $\mathcal{A}^{\mathrm{sm}}$) denote the locus in $\mathcal{A}$ such that the spectral curve $\mathcal{C}_a$ is integral (resp. smooth and connected) and let $h^{\mathrm{ell}}\colon \mathcal{M}^{\mathrm{ell}}\to\mathcal{A}^{\mathrm{ell}}$ denote the restriction to the elliptic locus (and similarly $h^{\mathrm{sm}}\colon \mathcal{M}^{\mathrm{sm}}\to\mathcal{A}^{\mathrm{sm}}$). By Beauville, Narasimhan, and Ramanan (1989, Remark 3.5), $\mathcal{A}^{\mathrm{sm}}\subset\mathcal{A}^{\mathrm{ell}}\subset\mathcal{A}$ are non-empty open subsets. For a projective flat family of curves $\mathcal{X}/S$, we let $\mathcal{P}_e(\mathcal{X}/S)$ denote the relative Picard scheme of degree e line bundles on $\mathcal{X}/S$ and, if the fibres of $\mathcal{X}/S$ are integral, we let $\overline{\mathcal{P}}_e(\mathcal{X}/S)$ denote the relative compactified Picard scheme of torsion-free sheaves of rank 1 and degree e. Finally, let $\overline{\mathcal{P}}_e^{ss}(X)$ be the good moduli space representing S-equivalence classes of semistable (in the sense of Schaub) torsion-free sheaves of rank 1 and degree e on a projective curve X.

Theorem 1.4 (Spectral correspondence (Hitchin, 1987b; Beauville, Narasimhan, and Ramanan, 1989; Schaub, 1998))

Let $h\colon \mathcal{M}_{n,d}(C)\to\mathcal{A}:=\oplus_{i=1}^n H^0(C,\omega_C^{\otimes i})$ denote the Hitchin map and $\mathcal{C}/\mathcal{A}$ denote the spectral curve and write $e:=d+(n-n^2)(1-g)$. Then we have the following isomorphisms.

(i) *As stacks over $\mathcal{A}$, we have an isomorphism*

$$\mathfrak{Higgs}_{n,d}(C)\simeq\mathfrak{Coh}_{1,e}^{t.f.}(\mathcal{C}/\mathcal{A}),$$

where the right side is the stack of torsion-free rank 1 degree e sheaves on $\mathcal{C}/\mathcal{A}$.

(ii) *As schemes over $\mathcal{A}^{\mathrm{sm}}$ and $\mathcal{A}^{\mathrm{ell}}$, we have isomorphisms*

$$\mathcal{M}_{n,d}^{\mathrm{sm}}(C)\simeq\mathcal{P}_e(\mathcal{C}^{\mathrm{sm}}/\mathcal{A}^{\mathrm{sm}})\quad\textit{and}\quad \mathcal{M}_{n,d}^{\mathrm{ell}}(C)\simeq\overline{\mathcal{P}}_e(\mathcal{C}^{\mathrm{ell}}/\mathcal{A}^{\mathrm{ell}}).$$

(iii) *For any $a \in \mathcal{A}$, we have an isomorphism*

$$h^{-1}(a) \simeq \overline{\mathcal{P}}_e^{ss}(\mathcal{C}_a).$$

More generally, we can consider pairs $(E, E \to E \otimes L)$ where ω_C is replaced by a line bundle L; these pairs have an analogous Hitchin map and spectral description.

1.2.4. *Properness of the Hitchin map.* — As the properness of the Hitchin map is used to define the perverse filtration, we sketch the proof via the spectral correspondence.

Proposition 1.5. — *The Hitchin map $h\colon \mathcal{M}_{n,d}(C) \to \mathcal{A}$ is proper.*

Proof. — Let R be a DVR with fraction field K. For a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{M}_{n,d}(C) \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathcal{A}, \end{tikzcd}$$

the top morphism corresponds to a Higgs bundle (E_K, θ_K) on C_K. By the spectral correspondence (E_K, θ_K) corresponds to a semistable 1-dimensional sheaf F_K on S_K where $S = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{T}_C)$. By Langton's proof (Langton, 1975) of the valuative criterion for properness of semistable sheaves on S, the semistable sheaf F_K on S_K extends to a semistable sheaf F_R on S_R. To show that F_R corresponds to a family of semistable Higgs bundles over R extending (E_K, θ_K) it remains to show that the support of F_R is disjoint from the boundary of $T^*C \subset S$. However, as the map $\mathrm{Spec}(K) \to \mathcal{M}_{n,d}(C) \to \mathcal{A}$ extends to $\mathrm{Spec}(R)$, we know that the spectral curve $\mathcal{C}_K$ extends to $\mathcal{C}_R$, which is the support of F_R (see §1.2.3), and so is disjoint from the boundary. □

1.3. Hodge theory

The Hodge decomposition for a smooth complex projective variety X equips its ith cohomology groups with a pure Hodge structure of weight i: we have a decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p,q} H^{p,q}$$

satisfying $\overline{H^{p,q}} = H^{q,p}$ and $H^{p,q} = 0$ whenever $p + q \neq i$. We define an associated decreasing Hodge filtration $F^j H^i(X, \mathbb{C}) = \sum_{k \geq j} H^{k,i-k}$ such that

$$H^i(X, \mathbb{C}) = F^j H^i(X, \mathbb{C}) \oplus \overline{F^{i-j+1} H^i(X, \mathbb{C})}$$

for any $0 \leq j \leq i$. We say the Hodge structure on the cohomology of X is of *pure Hodge–Tate type* if $H^{p,q} = 0$ whenever $p \neq q$. In particular, this implies that the odd cohomology groups of X must be trivial. For example, the cohomology of projective spaces is of pure Hodge–Tate type.

Deligne showed that this can be extended to singular and non-compact varieties via the notion of a mixed Hodge structure.

Theorem 1.6 (Deligne). — *The cohomology $H^*(X,\mathbb{Q})$ of a complex algebraic variety X admits a mixed Hodge structure (MHS) consisting of an increasing weight filtration $W_\bullet H^i(X,\mathbb{Q})$ and a decreasing Hodge filtration $F^\bullet H^i(X,\mathbb{C})$ for all i such the Hodge filtration induced by $F^\bullet$ on the graded pieces of the weight filtration $\mathrm{Gr}_l^W H^i(X,\mathbb{Q})$ is a pure Hodge structure of weight l. Furthermore, the following properties hold.*

(i) *Mixed Hodge structures are functorial with respect to algebraic morphisms and compatible with the cup product and Künneth isomorphism.*

(ii) *$H^i(X,\mathbb{Q})$ has weights contained in $[0,2i]$; that is, it stabilises outside of this range*

$$W_{-1} = 0 \subset W_0 \subset \cdots \subset W_{2i}H^i(X,\mathbb{Q}) = H^i(X,\mathbb{Q}).$$

(iii) *If X is proper, $H^i(X,\mathbb{Q})$ has weights in $[0,i]$; that is, $W_iH^i(X,\mathbb{Q}) = H^i(X,\mathbb{Q})$.*

(iv) *If X is smooth, $H^i(X,\mathbb{Q})$ has weights in $[i,2i]$; that is, $W_{i-1}H^i(X,\mathbb{Q}) = 0$.*

(v) *If X is smooth, then $W_iH^i(X,\mathbb{Q}) = \mathrm{Im}(j^*\colon H^i(\overline{X},\mathbb{Q}) \to H^i(X,\mathbb{Q}))$ for any smooth compactification $j\colon X \hookrightarrow \overline{X}$.*

We say a class $\gamma \in H^*(X,\mathbb{Q})$ has *weight* j if $\gamma \in W_jH^*(X,\mathbb{Q})$, but $\gamma \notin W_{j-1}H^*(X,\mathbb{Q})$. We define the Hodge subspace by

$$ {}^k\mathrm{Hod}^i(X) := W_{2k}H^i(X,\mathbb{C}) \cap F^kH^i(X,\mathbb{C}). \tag{2}$$

We say that the MHS on $H^*(X,\mathbb{Q})$ is of *mixed Hodge–Tate type* if the induced pure Hodge structure on each $\mathrm{Gr}_l^W H^i(X,\mathbb{Q})$ is of pure Hodge–Tate type.

1.3.1. *Purity.* — The MHS on $H^i(X,\mathbb{Q})$ is *pure* if the weight filtration is trivial with weight equal to the cohomological degree; that is, $0 = W_{i-1}H^i(X,\mathbb{Q}) \subset W_iH^i(X,\mathbb{Q}) = H^i(X,\mathbb{Q})$ so that $H^i(X,\mathbb{Q}) = \mathrm{Gr}_i^W H^i(X,\mathbb{Q})$ has pure Hodge structure of weight i.

For example, via the compactification $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$, the MHS on $H^1(\mathbb{G}_m,\mathbb{Q})$ is

$$W_1H^1(\mathbb{G}_m,\mathbb{Q}) = \mathrm{Im}(H^1(\mathbb{P}^1,\mathbb{Q}) \to H^1(\mathbb{G}_m,\mathbb{Q})) = 0 \subset W_2H^1(\mathbb{G}_m,\mathbb{Q}) = H^1(\mathbb{G}_m,\mathbb{Q})$$

which has weight 2 in cohomological degree 1, and so the MHS is not pure, but it is of mixed Hodge–Tate type. This can be confusing, as we have an isomorphism $H^1(\mathbb{G}_m,\mathbb{Q}) \cong H^2(\mathbb{P}^1,\mathbb{Q})$ of abstract mixed Hodge structures: for $\mathbb{G}_m$ the weight does not match the cohomological degree, whereas for $\mathbb{P}^1$ it does.

From the final statement of Theorem 1.6, for a smooth variety X and a smooth compactification $\overline{X}$, we can define the *pure part* of the cohomology of X as follows

$$H^*_{\mathrm{pure}}(X,\mathbb{Q}) := \mathrm{Im}(H^*(\overline{X},\mathbb{Q}) \to H^*(X,\mathbb{Q})),$$

whose induced MHS is pure. The pure part of the cohomology is preserved under pullbacks and proper pushfowards. As explained in (HMMS, §2.2), the restriction map $H^*(\overline{X},\mathbb{Q}) \to H^*_{\mathrm{pure}}(X,\mathbb{Q})$ factors as

$$H^*(\overline{X},\mathbb{Q}) \xrightarrow{\mathrm{pr}_1^*} H^*(\overline{X}\times X,\mathbb{Q}) \xrightarrow{\cup[\Delta]} H^*(\overline{X}\times X,\mathbb{Q}) \xrightarrow{\mathrm{pr}_{2*}} H^*(X,\mathbb{Q})$$

where $\Delta \subset \overline{X}\times X$ denotes the diagonal. Hence the pure cohomology $H^*_{\mathrm{pure}}(X,\mathbb{Q})$ is spanned by the classes in any Künneth decomposition of Δ. Later, for certain smooth moduli spaces with natural tautological classes, we will use this to deduce that the tautological and pure cohomology rings coincide.

1.3.2. *Purity on the Dolbeault side.* — There is a $\mathbb{G}_m$-action on the Dolbeault moduli space of Higgs bundles given by scaling the Higgs field. The Hitchin map is $\mathbb{G}_m$-equivariant for the $\mathbb{G}_m$-action on $\mathcal{A}$ with weight i on $H^0(C, \omega_C^{\otimes i})$. We will use this action to show that the cohomology of $\mathcal{M}_{n,d} = \mathcal{M}_{n,d}(C)$ is pure when n and d are coprime. Hitchin (1987b) used this $\mathbb{G}_m$-action to compute the Betti numbers of $\mathcal{M}_{2,1}$.

Proposition 1.7. — *If $(n,d) = 1$, then $H^*(\mathcal{M}_{n,d}, \mathbb{Q})$ has a pure Hodge structure.*

Proof. — Since $\mathbb{G}_m$ acts on $\mathcal{A}$ with positive weights, every point $a \in \mathcal{A}$ flows to $0 \in \mathcal{A}$ under the $\mathbb{G}_m$-action as $t \to 0$. Thus, as $h\colon \mathcal{M}_{n,d} \to \mathcal{A}$ is $\mathbb{G}_m$-equivariant and proper, the limit of the $\mathbb{G}_m$-action on all points in $\mathcal{M}_{n,d}$ exists as $t \to 0$ and lies in $h^{-1}(0)$.

Since $\mathcal{M}_{n,d}$ is smooth, the $\mathbb{G}_m$-fixed locus is smooth (see Fogarty, 1973) and the fixed locus is proper, as it is closed in the proper scheme $h^{-1}(0)$. By work of Białynicki-Birula (1973), the downward $\mathbb{G}_m$-flow is a deformation retract $\mathcal{M}_{n,d} \to (\mathcal{M}_{n,d})^{\mathbb{G}_m}$ and the MHS on $H^*(\mathcal{M}_{n,d}, \mathbb{Q})$ is pure, as it is a direct sum of (pure Tate twists of) cohomology groups of smooth proper varieties (namely the connected components of the fixed locus). $\square$

In contrast, the MHS on the Betti moduli space is genuinely mixed. For example, $\mathcal{M}^{\mathrm{B}}_{1,0} \cong (\mathbb{G}_m)^{2g}$ has non-pure Hodge structure, as we explained for $\mathbb{G}_m$ above.

1.4. The decomposition theorem and perverse filtrations

In this section, we will introduce the necessary results needed from the theory of perverse sheaves (Beĭlinson, Bernstein, and Deligne, 1982); an excellent reference for this material is the survey of de Cataldo and Migliorini (2009), as well as the Bourbaki seminar of Williamson (2017).

For a complex variety X, we let $D^b_c(X, \mathbb{Q})$ be the constructible derived category of sheaves of $\mathbb{Q}$-vector spaces. This category has a six operation formalism given by the four operations $f^{-1}, Rf_*, Rf_!, f^!$ for a morphism f of complex varieties, as well as a tensor product and internal homomorphism $R\mathcal{H}om$. Moreover, there is a Verdier duality $\mathbb{D}\mathcal{K} := R\mathcal{H}om(\mathcal{K}, \omega_X)$, where $\omega_X = \pi^!\mathbb{Q}$ is the dualising complex for the structure map $\pi\colon X \to \mathrm{Spec}(\mathbb{C})$. Verdier duality gives $\mathcal{H}^i(\mathbb{D}\mathcal{K}) \simeq (\mathcal{H}^{-i}_c\mathcal{K})^*$ which recovers Poincaré duality when X is smooth, as then $\omega_X \simeq \mathbb{Q}_X[2\dim X]$.

For a morphism $f\colon X \to Y$ of complex varieties, the cohomology sheaves of the complex $Rf_*\mathbb{Q}_X \in D^b_c(Y, \mathbb{Q})$ compute the higher direct images $\mathcal{H}^i(Rf_*\mathbb{Q}_X) = R^if_*\mathbb{Q}_X$.

Theorem 1.8 (Deligne's Decomposition Theorem). — *Let $f\colon X \to Y$ be a smooth projective morphism of smooth complex varieties. Then there is a decomposition*

$$Rf_*\mathbb{Q}_X \simeq \bigoplus_{i=0}^{2d} R^if_*\mathbb{Q}_X[-i]$$

and the sheaves $R^if_*\mathbb{Q}_X[-i]$ are semisimple local systems on Y.

In particular, $H^*(X,\mathbb{Q}_X) \simeq \bigoplus_{i=0}^{2d} H^{*-i}(X, R^i f_*\mathbb{Q}_X)$. In general, one can compute the cohomology of X from the Leray spectral sequence associated to f and Deligne showed that this degenerates on the E_2 page if f is smooth and projective. To generalise Deligne's decomposition theorem to the singular case, one must replace cohomology with intersection cohomology and work with perverse sheaves.

1.4.1. *Perverse sheaves.* — The category $\mathcal{P}(X,\mathbb{Q})$ of perverse sheaves on X is the full subcategory of $D^b_c(X,\mathbb{Q})$ consisting of complexes $\mathcal{K}$ such that

$$\dim\operatorname{supp}\mathcal{H}^i(\mathcal{K}) \leq -i \quad \text{and} \quad \dim\operatorname{supp}\mathcal{H}^i(\mathbb{D}\mathcal{K}) \leq -i$$

for all $i \in \mathbb{Z}$. These conditions are known as the support and cosupport conditions and determine subcategories ${}^{\mathfrak{p}}D^{\leq 0}_X$ and ${}^{\mathfrak{p}}D^{\geq 0}_X$ respectively which define the perverse t-structure on $D^b_c(X,\mathbb{Q})$ whose heart is $\mathcal{P}(X,\mathbb{Q})$. In particular, this is an abelian category and the perverse cohomology sheaves ${}^{\mathfrak{p}}\mathcal{H}^i := {}^{\mathfrak{p}}\tau_{\geq 0} \circ {}^{\mathfrak{p}}\tau_{\leq 0}[i]$ can be computed in terms of the perverse truncation functors

$${}^{\mathfrak{p}}\tau_{\leq i-1}\mathcal{K} \longrightarrow {}^{\mathfrak{p}}\tau_{\leq i}\mathcal{K} \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^i\mathcal{K}[-i] \xrightarrow{+1} .$$

The category $\mathcal{P}(X,\mathbb{Q})$ of perverse sheaves is Noetherian and Artinian with simple objects given by intersection cohomology complexes associated to local systems on locally closed irreducible subvarieties. For a local system L on a locally closed smooth subvariety $S \hookrightarrow X$, the *intersection cohomology complex* is defined as

$$\mathcal{IC}_X(\overline{S}, L) := i_* j_{!*} L[\dim S]$$

where $i \colon \overline{S} \hookrightarrow X$ is the closed embedding of the closure of S in X and $j \colon S \hookrightarrow \overline{S}$ is the open embedding and $j_{!*}\mathcal{K} := \operatorname{Im}({}^{\mathfrak{p}}\mathcal{H}^0 j_!\mathcal{K} \to {}^{\mathfrak{p}}\mathcal{H}^0 j_*\mathcal{K})$ is the intermediate extension of Goresky–MacPherson. By definition $j^* i^* \mathcal{IC}_X(\overline{S}, L) = L[\dim S]$. The scheme $\overline{S}$ is called the *support* of this intersection cohomology complex.

The intersection cohomology complex of X is defined as $\mathcal{IC}_X := \mathcal{IC}_X(\overline{X_{\mathrm{reg}}}, \mathbb{Q}_{X_{\mathrm{reg}}})$. The intersection cohomology groups $IH^*(X,\mathbb{Q})$ are the (shifted) cohomology sheaves of $\mathcal{IC}_X$. If X is smooth of dimension d_X, we have $\mathcal{IC}_X(X,\mathbb{Q}) = \mathbb{Q}_X[d_X]$.

1.4.2. *Decomposition theorem.* — Deligne's decomposition generalises as follows.

Theorem 1.9 (Beilinson–Bernstein–Deligne–Gabber Decomposition Theorem (Beĭlinson, Bernstein, and Deligne, 1982))

For a proper morphism $f \colon X \to Y$ of complex varieties, there is a decomposition

$$Rf_*\mathcal{IC}_X \simeq \bigoplus_{i\in\mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathcal{IC}_X)[-i]. \tag{3}$$

The perverse sheaves on the right are semisimple, so are sums of intersection complexes $\mathcal{IC}_X(\overline{S_\alpha}, L_\alpha)$ for simple local systems L_α on locally closed smooth subvarieties S_α.

In particular, $IH^*(X,\mathbb{Q}) = \bigoplus_\alpha H^{*-d_\alpha}(X, \mathcal{IC}(\overline{S_\alpha}, L_\alpha))$ where the sum ranges over finitely many local systems L_α over locally closed subset $S_\alpha \subset X$. The decomposition theorem over the complex numbers was originally deduced from a corresponding decomposition over finite fields in the derived category of ℓ-adic sheaves (Beĭlinson, Bernstein, and Deligne, 1982); see also de Cataldo and Migliorini (2009, §3.3) for a geometric proof in the complex setting.

The supports of $Rf_*\mathcal{IC}_X$ are by definition the supports $\overline{S_\alpha}$ of the simple summands in the decomposition theorem for f. We say f has *full supports*, if these supports $\overline{S_\alpha}$ are all equal to Y. In this case, ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathcal{IC}_X)$ is determined by its restriction to a dense open subset: if $j\colon U \hookrightarrow Y$ is a dense open, then ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathcal{IC}_X) = j_{!*}j^*({}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathcal{IC}_{X_U}))$. If f is a small morphism (see de Cataldo and Migliorini, 2009, Remark 4.2.4; for example, the Grothendieck–Springer resolution of a semisimple Lie algebra is small), then $Rf_*\mathbb{Q}[d_X]$ is the intersection complex of a semisimple local system with full support Y.

If X is smooth and f has relative dimension d, then the Goresky–MacPherson inequality states that any support Z of $Rf_*\mathbb{Q}_X$ has codimension at most d and if equality is achieved, then $R^{2d}f_*\mathbb{Q}_X$ has a direct summand over Z. A key idea of Ngô is that this inequality can be improved when f is a *weak abelian fibration* in the sense of Ngô (2010, Section 7), which in particular means there is a smooth commutative group scheme $\mathcal{P}/Y$ acting on f such that f is generically a $\mathcal{P}$-torsor. If f is a weak abelian fibration, the idea is to bound the codimension of a support by the minimal value of a δ-*function* on that support and if equality is achieved and f has irreducible fibres, then this support is a full support. Ngô uses this δ-inequality to prove a support theorem for δ-regular abelian fibrations and we will explain in §3.3.1 how he applies this to show the Hitchin fibration restricted to the elliptic locus has full supports (for both classical G-Higgs bundles and D-twisted G-Higgs bundles).

1.4.3. *Perverse filtrations.* — The decomposition (3) is not canonical (see de Cataldo and Migliorini, 2009, §1.4.2), but there is a canonical perverse Leray filtration for $f\colon X \to Y$ whose graded pieces describe the cohomology of the perverse sheaves in the decomposition theorem.

For any $\mathcal{K} \in D^b_c(X, \mathbb{Q})$, we can define (up to a choice of shift) a perverse filtration $P^f_\bullet H(X, \mathcal{K})$ by taking the images of $H^*(Y, {}^{\mathfrak{p}}\tau_{\leq \bullet} Rf_*\mathcal{K}) \to H^*(X, \mathcal{K})$. We will make a shift so that the perverse filtration is graded only in non-negative degrees, so we can naturally compare it with the weight filtration. For a proper morphism $f\colon X \to Y$ between smooth quasi-projective varieties, the decomposition theorem gives

$$Rf_*\mathbb{Q}_X[d_X - r_f] = \bigoplus_{i=0}^{2r_f} {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathbb{Q}_X[d_X - r_f])[-i]$$

where $r_f = \dim X \times_Y X - \dim X$ is the defect of semismallness; see de Cataldo and Migliorini (2009, Definition 4.2.2) for the definition of a semismall morphism. The perverse filtration of f is defined by

$$P^f_i H^j(X, \mathbb{Q}) = \operatorname{Im}\left(H^{j-d_X+r_f}(Y, {}^{\mathfrak{p}}\tau_{\leq i} Rf_*\mathbb{Q}_X[d_X - r_f]) \to H^j(X, \mathbb{Q})\right)$$

which is supported in degrees $[0, 2r_f]$ and whose graded pieces compute the cohomology of the summands in the decomposition theorem:

$$\mathrm{Gr}_i^{P,f} H^*(X,\mathbb{Q}) = P_i^f H^*(X,\mathbb{Q})/P_{i-1}^f H^*(X,\mathbb{Q}) = H^*(Y, {}^{\mathfrak{p}}\mathcal{H}^{-i}(Rf_*\mathbb{Q}_X[d_X - r_f])).$$

We say a class $\gamma \in H^*(X,\mathbb{Q})$ has *perversity* i if $\gamma \in P_i^f H^*(X,\mathbb{Q})$ and $\gamma \notin P_{i-1}^f H^*(X,\mathbb{Q})$.

In the case where the target is an affine space (as, for example, is the case for the Hitchin map), de Cataldo and Migliorini (2010) proved that the perverse filtration is given by a flag filtration:

$$P^i H^j(X,\mathbb{Q}) = \ker\left(H^i(X,\mathbb{Q}) \to H^i(f^{-1}(\Lambda^{j-i-1}),\mathbb{Q})\right)$$

where $\Lambda^{j-i-1} \subset Y$ is a general affine subspace of dimension $j-i-1$. However, this concrete description does not seem helpful to tackle the $P=W$ conjecture in general.

Example 1.10. — We have $\mathcal{M}_{1,d}(C) \simeq \mathrm{Pic}^d(C) \times H^0(C,\omega_C)$ and the Hitchin map is identified with the projection onto the second factor. Consequently the perverse filtration coincides with the Leray filtration and is trivial:

$$0 = P_{l-1}H^l(\mathcal{M}_{1,d},\mathbb{Q}) \subset P_l H^l(\mathcal{M}_{1,d},\mathbb{Q}) = H^l(\mathcal{M}_{1,d},\mathbb{Q}).$$

1.4.4. *Relative Hard Lefschetz.* — The final ingredient that we need concerning perverse sheaves is the relative Hard Lefschetz Theorem. For a smooth projective manifold and an ample class $\eta \in H^2(X,\mathbb{Q})$, the classical Hard Lefschetz theorem states that $\eta^i\colon H^{d_X-i}(X) \to H^{d_X+i}(X)$ is an isomorphism. This extends to the relative setting and can also be generalised to allow singular fibres provided one works with perverse cohomology sheaves as follows.

Theorem 1.11 (Relative Hard Lefschetz). — *Let $f\colon X \to Y$ be a projective morphism of smooth complex varieties and $\omega = c_1(\mathcal{L}) \in H^2(X,\mathbb{Q})$ be a relatively ample class. Then*

$${}^{\mathfrak{p}}\mathcal{H}^{-i}(Rf_*\mathbb{Q}_X[d_X]) \xrightarrow{\omega^i} {}^{\mathfrak{p}}\mathcal{H}^{i}(Rf_*\mathbb{Q}_X[d_X])$$

is an isomorphism for all $i \geq 0$.

In terms of the perverse filtration associated to f, which we shifted to be concentrated in degrees $[0, 2r_f]$, the Relative Hard Lefschetz Theorem gives isomorphisms

$$\omega^i\colon \mathrm{Gr}_{r_f-i}^{P,f} H^*(X,\mathbb{Q}) \cong \mathrm{Gr}_{r_f+i}^{P,f} H^*(X,\mathbb{Q}).$$

Remark 1.12. — The cup product with the relatively ample class ω defines a Lefschetz operator, which we also denote by ω. On the associated graded vector space $\mathrm{Gr}^P H^*(X,\mathbb{Q}) := \bigoplus_i \mathrm{Gr}_i^P H^*(X,\mathbb{Q}) \cong H^*(X,\mathbb{Q})$, the Lefschetz operator is nilpotent of degree 2 and so, by the Jacobson–Morozov Theorem, extends to an $\mathfrak{sl}_2$-triple $\langle \mathfrak{e},\mathfrak{f},\mathfrak{h}\rangle$ acting on $\mathrm{Gr}^P H^*(X,\mathbb{Q})$ such that $\mathfrak{e}$ equals the Lefschetz operator ω and the $\mathfrak{h}$-graded pieces are the P-graded pieces $\mathrm{Gr}_i^P H^*(X,\mathbb{Q})$ up to a shift in indices (see HMMS, Proposition 8.2 which gives a more abstract formulation in terms of Lefschetz structures).

1.4.5. *Functoriality of perverse filtrations.* — In this section, we briefly state (without proof) the results of (HMMS, §8.2) concerning the compatibility of perverse filtrations with cup products, pullbacks, Gysin maps and correspondences.

PROPOSITION 1.13. — *Assume we have a commutative diagram of quasi-projective smooth complex varieties*

$$\begin{tikzcd} X_1 \arrow[dr, "f_1"'] & X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "f"] & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

where f and f_i are projective. Then the following statements hold.

(i) *The cup product with $\alpha \in H^i_{\mathrm{pure}}(X)$ shifts the perverse filtration by i:*

$$\alpha \cup P^f_k H^*_{\mathrm{pure}}(X, \mathbb{Q}) \subset P^f_{k+i} H^*_{\mathrm{pure}}(X, \mathbb{Q}).$$

In particular, α has perversity bounded by i.

(ii) *The pullback map for π_i satisfies*

$$\pi_i^* P^{f_i}_k H^*_{\mathrm{pure}}(X_i, \mathbb{Q}) \subset P^f_{k+d_{X_i}-d_X} H^*_{\mathrm{pure}}(X, \mathbb{Q}).$$

(iii) *The Gysin map for π_i satisfies*

$$\pi_{i*} P^f_k H^*_{\mathrm{pure}}(X, \mathbb{Q}) \subset P^{f_i}_{k+d_{X_i}-d_X} H^{*+2d_{X_i}-2d_X}_{\mathrm{pure}}(X_i, \mathbb{Q}).$$

(iv) *The correspondence $[X]_\sharp$ given by $\alpha \mapsto \pi_{2*}(\pi_1^*(\alpha) \cup [X]^{\mathrm{vir}})$ satisfies*

$$[X]_\sharp P^{f_1}_k H^*(X_1, \mathbb{Q}) \subset P^{f_2}_{k+d_{X_1}+d_{X_2}-2d_X+2l} H^{*+2d_{X_2}-2d_X+2l}(X_2, \mathbb{Q}),$$

where $[X]^{\mathrm{vir}} \in H^{2l}_{\mathrm{pure}}(X)$ is the virtual fundamental class (see HMMS, §2.4). If X_i are algebraic symplectic and $X \subset X_1 \times X_2$ is Lagrangian (so $2d_X - 2l = d_{X_1} + d_{X_2}$), then $[X]_\sharp$ respects the perversity.

1.5. Statement and history of the $P = W$ conjecture

We can now state the $P = W$ conjecture of de Cataldo, Hausel, and Migliorini (2012). Recall that the weight filtration $W_\bullet H^*(\mathcal{M}^{\mathrm{B}}_{n,d}, \mathbb{Q})$ is concentrated in weights $[0, 2N]$ where $N = \dim \mathcal{M}^{\mathrm{B}}_{n,d}$ and satisfies $W_{2i} = W_{2i+1}$, whereas the perverse Leray filtration $P^h_\bullet H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d}, \mathbb{Q})$ is concentrated in degrees $[0, 2r_h]$ where $r_h = \dim \mathcal{M}^{\mathrm{Dol}}_{n,d} - \dim \mathcal{A}_n = \frac{1}{2}N$.

THEOREM 1.14 ($P = W$ Conjecture for GL_n). — *For n and d coprime, the perverse filtration for the Hitchin map $h\colon \mathcal{M}^{\mathrm{Dol}}_{n,d} \to \mathcal{A}_n$ on the Dolbeault moduli space coincides with the weight filtration on the cohomology of the Betti moduli space $\mathcal{M}^{\mathrm{B}}_{n,d}$ under the isomorphism on cohomology induced by non-abelian Hodge theory:*

$$P^h_i H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d}, \mathbb{Q}) = W_{2i} H^*(\mathcal{M}^{\mathrm{B}}_{n,d}, \mathbb{Q}) = W_{2i+1} H^*(\mathcal{M}^{\mathrm{B}}_{n,d}, \mathbb{Q}).$$

This was proved in rank 2 by de Cataldo, Hausel, and Migliorini (2012) and in arbitrary rank independently by Maulik and Shen (MS) and Hausel, Mellit, Minets, Schiffmann (HMMS). More recently an alternative proof was given by Maulik, Shen, and Yin (2023).

1.5.1. *History of the $P = W$ conjecture.* — The origins of the $P = W$ conjecture can be traced back to work of Hausel and Rodriguez-Villegas (2008) on computing the mixed Hodge structures of twisted GL_n-character varieties by point counting over finite fields. As well as writing down a conjectural formula for the mixed Hodge polynomials of character varieties, they observed that the E-polynomials of character varieties are palindromic and thus satisfy a curious Poincaré duality-type symmetry. This symmetry is traced back to Alvis–Curtis duality for the character theory of the general linear group over a finite field. Based on this, Hausel and Rodriguez-Villegas formulated a curious Hard Lefschetz conjecture for character varieties and proved this in rank $n = 2$ (Hausel and Rodriguez-Villegas, 2008); this is now a theorem of Mellit (2019) in arbitrary rank and coprime degree.

Theorem 1.15 (Curious Hard Lefschetz (Hausel and Rodriguez-Villegas, 2008; Mellit, 2019))

For n and d coprime, there is a class $\alpha \in H^2(\mathcal{M}^B_{n,d}, \mathbb{Q})$ such that α^i induces an isomorphism

$$\mathrm{Gr}^W_{N-2i} H^*_c(\mathcal{M}^B_{n,d}, \mathbb{C}) \xrightarrow{\sim} \mathrm{Gr}^W_{N+2i} H^{*+2i}_c(\mathcal{M}^B_{n,d}, \mathbb{C})$$

where $N = \dim \mathcal{M}^B_{n,d}$.

The name *Curious* Hard Lefschetz points to two surprising features: i) the Betti moduli space is affine and ii) the Lefschetz operator α is a type $(2, 2)$ class rather than a $(1, 1)$ class as in the classical Hard Lefschetz Theorem, so all indices are doubled and the cup product with α increases the cohomological degree by 2 and the weight by 4. We give an outline of the methods in Mellit's proof in §1.5.2.

The curious Hard Lefschetz conjecture lead de Cataldo, Hausel, and Migliorini (2012) to propose the $P = W$ conjecture, as it would explain the curious Hard Lefschetz symmetry for the Betti moduli space as coming from the relative Hard Lefschetz Theorem for the perverse Leray filtration for the Hitchin map.

One fundamental obstruction to proving this conjecture in general is that although the weight filtration is multiplicative (i.e. compatible with the cup product), the perverse Leray filtration is not in general. In rank 2, de Cataldo, Hausel, and Migliorini (2012) prove the $P = W$ conjecture for GL_2, PGL_2 and SL_2 by restricting to a dense open in the Hitchin base over which they prove the perverse Leray filtration coincides with the Leray filtration, which is multiplicative. This enables them to describe the perversity of generators of the cohomology (see §2.2.2 below).

Let us mention a few other cases of the $P = W$ conjecture that were established before the general case for GL_n was proved by Maulik and Shen (MS) and Hausel, Mellit, Minets, and Schiffmann (HMMS).

There is a class of parabolic Higgs moduli spaces labelled by the affine Dynkin diagrams which are Hilbert schemes of points over elliptic fibrations described by Groechenig (2014). For these parabolic Higgs moduli spaces, Shen and Zhang (2021) prove a $P = W$ conjecture by showing tautological generation of the cohomology and proving that the perverse filtration is multiplicative.

The topology of Lagrangian fibrations on compact hyperkähler manifolds was studied by Shen and Yin (2022) and Harder, Li, Shen, and Yin (2021), where they prove a numerical $P = W$ statement. Subsequently, de Cataldo, Maulik, and Shen (2022a) study the $P = W$ conjecture by considering an embedding $j\colon C \hookrightarrow A$ of C in an abelian surface in order to realise the Hitchin fibration as a degeneration of a family of Lagrangian fibrations on moduli spaces of sheaves on abelian surfaces, which are compact hyperkähler manifolds. They study the specialisation map on cohomology associated to this degeneration, which in general loses a lot of information, but they can describe the behaviour of tautological classes lying in the image of the restriction map j^*. By the Abel–Jacobi map, a genus 2 curve can be embedded in its Jacobian, so that the restriction on cohomology is surjective; consequently, they deduce that the $P = W$ conjecture holds in genus $g = 2$. Since j^* always contains the even part of the cohomology of C, they deduce that $P = W$ holds for the subring of even tautological classes.

1.5.2. *Mellit's proof of curious Hard Lefschetz.* — Since Maulik and Shen use the Curious Hard Lefschetz Theorem in their proof of the $P = W$ conjecture, we will give an outline of the ideas involved in this section.

In fact, Mellit proved a Curious Hard Lefschetz Theorem for GL_n-character varieties with several punctures provided the monodromies around these punctures are diagonalisable and 'generic' in the sense of Mellit (2019, Definition 4.6.1). His proof can be divided into the following steps:

(i) Describe the form $\omega \in H^2(\mathcal{M}^{\mathrm{B}}_{n,d}, \mathbb{C})$ explicitly to show it has type $(2, 2)$.
(ii) If one local monodromy is regular semisimple, find a homotopy replacement given by a unipotent character variety, which is cellular, and deduce curious Hard Lefschetz from the corresponding statements on the cells.
(iii) Deduce the general case (which includes the case of $\mathcal{M}^{\mathrm{B}}_{n,d}$ for n and d coprime) from the previous case by adding an extra puncture and varying the monodromy at that puncture to produce a family of character varieties such that the cohomology of the fibres can be related via a natural Weyl group action.

For (1), Mellit uses the evaluation map for the character stack

$$\mathrm{ev}^*\colon H^*(B\mathrm{GL}_n) \to H^*(\mathcal{M}^{\mathrm{B}}_{n,d} \times C)$$

and considers $\omega = \int_{[C]} \mathrm{ev}^*(\omega_{\mathrm{GL}_n}) \in H^2(\mathcal{M}^{\mathrm{B}}_{n,d})$ for a class $\omega_{\mathrm{GL}_n} \in H^*(B\mathrm{GL}_n)$ constructed from certain differential forms on the bar construction of GL_n (Mellit, 2019, Proposition 4.5.1). For (2), let us assume for simplicity of exposition that we consider a character variety $\mathcal{M}^{\mathrm{B}}$ for a curve with one puncture and fixed regular semisimple monodromy, so the eigenvectors can be used to construct a basis. The homotopy replacement is a unipotent character variety which is a fibre bundle $\widetilde{\mathcal{M}}^{\mathrm{B}} \to \mathcal{M}^{\mathrm{B}}$ with fibre the group U_n of upper triangular unipotent matrices, where $\widetilde{\mathcal{M}}^{\mathrm{B}}$ is constructed as a quotient of the space of tuples $(A_1, \ldots, A_g, B_1, \ldots, B_g) \in \mathrm{GL}_n^{2g}$ such that the difference between the commutator $\prod_{i=1}^n [A_i, B_i]$ and the monodromy is strictly upper triangular. The cellular decomposition on $\widetilde{\mathcal{M}}^{\mathrm{B}}$ is induced by the Bruhat decomposition and it suffices to show the corresponding statement on each cell by Mellit (2019, Proposition 2.6.4). For (3), by allowing the monodromy to vary at an additional puncture, Mellit shows the cohomology of $\mathcal{M}^{\mathrm{B}}_{n,d}$ (which corresponds to the fibre where the monodromy at the extra puncture is trivial) is the sign component of the cohomology of a character variety where the extra puncture has regular semisimple monodromy. This involves relating these character varieties to the Grothendieck–Springer resolution for GL_n.

Since $\omega \in F^2H^2(\mathcal{M}^{\mathrm{B}}, \mathbb{C})$, the cup product with ω gives a nilpotent operator whose associated filtration coincides with the Hodge filtration and splits the weight filtration. Moreover, for n and d coprime, the mixed Hodge structure on the twisted GL_n-character variety $\mathcal{M}^{\mathrm{B}}_{n,d}$ splits (Mellit, 2019, Corollary 1.5.4).

In fact, Mellit shows Curious Hard Lefschetz holds for any 'log-canonical' non-degenerate 2-form ω on a torus and moreover the Lefschetz operator ω has an adjoint, which together with ω forms part of an $\mathfrak{sl}_2$-triple. This is a shadow of the idea employed in (HMMS) to prove the $P = W$ conjecture.

1.5.3. *Other versions of $P = W$.* — Since non-abelian Hodge theory holds after replacing GL_n by any complex reductive group G, one can also predict a version of $P = W$ in these cases. We discuss the groups PGL_n and SL_n and their relation with GL_n in further detail in §2.4. One reason the case of GL_n (and PGL_n) is accessible is that $H^*(\mathcal{M}_{n,d}, \mathbb{Q})$ is tautologically generated when n and d are coprime (see §2.2.2).

Non-abelian Hodge theory also extends to parabolic Higgs bundles, which are Higgs bundles with flag structures in some fibres that are compatible with the Higgs field, and so one can also formulate a version of $P = W$ in the parabolic case.

Returning to GL_n but in the non-coprime case, the Betti and Dolbeault moduli spaces are singular, but still homeomorphic via non-abelian Hodge theory. In this case, Relative Hard Lefschetz fails in general and the Curious Hard Lefschetz property fails for the Betti moduli spaces, as their E-polynomial are no longer palindromic. Instead one can either formulate $P = W$ on the isomorphism between intersection cohomology groups given by non-abelian Hodge theory (as in de Cataldo and Maulik, 2020, Question 4.1.7) or consider $P = W$ on a resolution of singularities by lifting the non-abelian Hodge isomorphism (as in Felisetti and Mauri, 2022). When the character variety admits a symplectic resoluton, Felisetti and Mauri (2022) prove the intersection cohomology (or equivalently resolution) version of $P = W$.

There is no natural lift of the non-abelian Hodge isomorphism to the level of stacks (except in genus 0), as the non-abelian Hodge isomorphism relies on the *polystability* of the Higgs bundle in an essential way. However Davison (2023) and Davison, Hennecart, and Schlegel Mejia (2023) show that the stack of n-dimensional representations of $\pi_1(C)$ and the stack of semistable degree 0 rank n Higgs bundles have isomorphic Borel–Moore homologies (with appropriate coefficients), by using the theory of cohomological Hall algebras to express both sides in terms of a BPS algebra, which is a free algebra encoding the intersection cohomology of the good moduli spaces. Davison (2023) also formulates a stacky version of the $P = W$ conjecture (provided the perverse filtration is appropriately defined), which via the above Hall algebra result is equivalent to the intersection $P = W$ conjecture.

One can also replace the curve C by a higher-dimensional smooth projective variety, where non-abelian Hodge theory applies and thus one can also ask whether $P = W$ holds. In the case of an abelian variety, where the topology of the Hitchin map is much simpler and both filtrations P and W are trivial, this has been verified in Bolognese, Küronya, and Ulirsch (2023). Moreover, $P = W$ phenomena have been studied on generalisations of the Dolbeault and Betti moduli spaces over a curve, such as cluster varieties (Zhang, 2021) and hyperkähler manifolds (Harder, Li, Shen, and Yin, 2021).

A *geometric* $P = W$ conjecture of Katzarkov, Noll, Pandit, and Simpson (2015) aims to understand the cohomological $P = W$ phenomena by giving a geometric description of neighbourhoods of infinity in natural compactifications of the Dolbeault and Betti moduli spaces up to homotopy. Under certain technical assumptions, the geometric $P = W$ conjecture implies the cohomological $P = W$ conjecture at the highest weight (Mauri, Mazzon, and Stevenson, 2022, Theorem A).

2. TAUTOLOGICAL CLASSES

The aim of this section is to reduce the $P = W$ conjecture to statements entirely on the Dolbeault side about the interaction of the perverse filtration with cup products of tautological classes, constructed from the universal bundle.

2.1. Universal families

For a general complex reductive group G, the Dolbeault and Betti moduli stacks naturally come with universal families, which are given by a universal G-bundle that is equipped with a universal Higgs field in the Dolbeault case and a universal flat connection in the Betti case. The non-abelian Hodge correspondence only concerns the corresponding moduli spaces, which only admit universal families if they are fine moduli spaces, and moreover these universal families may be non-unique.

On the Betti stack $\mathfrak{M}_G^{\mathrm{B}}$ of G-local systems there is a universal flat G-bundle

$$(\mathcal{P}_G, \Delta_G) \to \mathfrak{M}_G^B \times C$$

where $\mathcal{P}_G = [\widetilde{C} \times \mathrm{Rep}(\pi_1(C), G) \times G/\pi_1(C) \times G]$ and $\widetilde{C}$ is a universal cover of C. Forgetting the flat structure Δ_G, the topological G-bundle $\mathcal{P}_G^{\mathrm{an}}$ determines a classifying map

$$\nu\colon (\mathfrak{M}_G^B)^{\mathrm{an}} \times C^{\mathrm{an}} \to BG^{\mathrm{an}}.$$

On the Dolbeault side, for $G = \mathrm{GL}_n$ and a coprime degree d, the Higgs moduli space $\mathcal{M}_{n,d}^{\mathrm{Dol}}$ is a fine moduli space. In particular, there is a universal Higgs bundle

$$(\mathcal{E}, \Theta\colon \mathcal{E} \to \mathcal{E} \otimes \pi_C^* \omega_C) \to \mathcal{M}_{n,d}^{\mathrm{Dol}} \times C.$$

that is unique up to tensoring with pullbacks of line bundles on $\mathcal{M}_{n,d}^{\mathrm{Dol}}$. The non-uniqueness of the universal family on $\mathcal{M}_{n,d}^{\mathrm{Dol}}$ can be dealt with in several different ways: i) by passing from GL_n-Higgs bundles to PGL_n-Higgs bundles which have a unique universal family, ii) by projectivising the universal bundle as in Hausel and Thaddeus (2004, §5), iii) by considering universal families twisted by a cohomology class with appropriately normalised first Chern class as in de Cataldo, Maulik, and Shen (2022a, §0.3), or iv) by using a normalised (virtual) universal family similarly to Atiyah and Bott (1983, p. 582). All these perspectives are closely related: the PGL_n-bundle $\mathbb{P}(\mathcal{E})$ on $\mathcal{M}_{n,d}^{\mathrm{Dol}} \times C$ is isomorphic to the pullback of the universal PGL_n-Higgs bundle $\mathcal{E}_{\mathrm{PGL}_n}$ on $\mathfrak{M}_{\mathrm{PGL}_n,d}^{\mathrm{Dol}} \times C$ via the quotient map $q\colon \mathcal{M}_{n,d}^{\mathrm{Dol}} \to \mathfrak{M}_{\mathrm{PGL}_n,d}^{\mathrm{Dol}}$.

The non-abelian Hodge correspondence for PGL_n gives a diffeomorphism of orbifolds

$$\mathfrak{M}_{\mathrm{PGL}_n,d}^{\mathrm{B}} \simeq \mathfrak{M}_{\mathrm{PGL}_n,d}^{\mathrm{Dol}}$$

and an isomorphism of the underlying universal PGL_n-bundles $\mathcal{P}_{\mathrm{PGL}_n} \simeq \mathcal{E}_{\mathrm{PGL}_n}$. For GL_n, the diffeomorphism $\mathcal{M}_{n,d}^{\mathrm{Dol}} \simeq \mathcal{M}_{n,d}^{\mathrm{B}}$ gives an isomorphism of universal PGL_n-bundles $\mathbb{P}(\mathcal{E}) \simeq (q_B \times \mathrm{Id}_C)^* \mathcal{P}_{\mathrm{PGL}_n}$ where $q_B\colon \mathcal{M}_{n,d}^B \to \mathfrak{M}_{\mathrm{PGL}_n,d}^{\mathrm{B}}$ is the quotient map (see Hausel and Thaddeus, 2004, §5 for further details).

2.2. Cohomology of Higgs moduli spaces and tautological generation

For the general linear group GL_n and d coprime to n, the cohomology of the Higgs moduli space $\mathcal{M}_{n,d}$ is generated by certain tautological classes constructed from the universal bundle. We will see that the same is also true for the PGL_n-Higgs moduli space, whose cohomology differs from that of $\mathcal{M}_{n,d}$ by a tensor factor of the cohomology of the Jacobian of C. However, for SL_n, the tautological ring is a proper subring of the cohomology of the SL_n-Higgs moduli space.

2.2.1. *Relating the cohomology of Higgs moduli spaces for different groups.* — Let us begin by describing the relationship between the cohomology of the moduli spaces $\mathcal{M}_{G,d}$ of G-Higgs bundles for $G = \mathrm{GL}_n$, SL_n and PGL_n of degree d coprime to n. First, as the PGL_n-moduli space is the quotient of the SL_n-moduli space by the action of $\Gamma = \mathrm{Jac}(C)[n]$, we have

$$H^*(\mathcal{M}_{\mathrm{PGL}_n,d}, \mathbb{Q}) \cong H^*(\mathfrak{M}_{\mathrm{PGL}_n,d}, \mathbb{Q}) \cong H^*(\mathcal{M}_{\mathrm{SL}_n,d}, \mathbb{Q})^{\Gamma},$$

as $\mathfrak{M}_{\mathrm{PGL}_n,d} = [\mathcal{M}_{\mathrm{SL}_n,d}/\Gamma]$ is a Deligne–Mumford stack.

We note that the Γ-action on the cohomology of $\mathcal{M}_{\mathrm{SL}_n,d}$ is non-trivial, as was already observed by Hitchin (1987b) in rank $n = 2$. The corresponding endoscopic decomposition associated to this Γ-action plays an important role in topological mirror symmetry for

SL_n and PGL_n-Higgs bundles (Groechenig, Wyss, and Ziegler, 2020; Maulik and Shen, 2021) and was carefully described by Maulik and Shen (2021). The variant part of the cohomology is related to the cohomology of endoscopic moduli spaces attached to non-trivial characters of Γ which can be described as Higgs moduli spaces on a Galois cover of C determined by the character of Γ.

On the geometric level, we have an isomorphism of varieties

$$\mathcal{M}_{\mathrm{SL}_n,d} \times^{\Gamma} T^* \operatorname{Jac}(C) \to \mathcal{M}_{\mathrm{GL}_n,d}$$

and thus the cohomology of $\mathcal{M}_{\mathrm{GL}_n,d}$ is the Γ-invariant cohomology of $\mathcal{M}_{\mathrm{SL}_n,d} \times T^* \operatorname{Jac}(C)$. Since the Γ-action on the cohomology of $\operatorname{Jac}(C)$ is trivial, we obtain isomorphisms of $\mathbb{Q}$-algebras

$$H^*(\mathcal{M}_{\mathrm{GL}_n,d}) \cong H^*(\mathcal{M}_{\mathrm{SL}_n,d})^{\Gamma} \otimes H^*(\operatorname{Jac}(C)) \cong H^*(\mathcal{M}_{\mathrm{PGL}_n,d}) \otimes H^*(\operatorname{Jac}(C)). \tag{4}$$

In particular, to provide generators for the cohomology of $\mathcal{M}_{\mathrm{GL}_n,d}$ it suffices to find generators for the cohomology of $\mathcal{M}_{\mathrm{PGL}_n,d}$.

2.2.2. *Tautological generation.* — For d coprime to n, a choice of universal Higgs bundle $(\mathcal{E}, \Theta) \to \mathcal{M}_{n,d} \times C$ determines *tautological classes*, which are defined as the Künneth components of the Chern character of the underlying universal bundle $\mathcal{E}$. These tautological classes generate a subring of $H^*(\mathcal{M}_{G,d}, \mathbb{Q})$ called the *tautological ring*.

First, we note that often tautological classes are defined as the Künneth components of the Chern *classes* of the universal bundle, but we will use the Chern character as in (MS) due to its multiplicativity properties; since we are working with $\mathbb{Q}$-coefficients, this does not change the tautological ring as we can determine the components $\mathrm{ch}_k(\mathcal{E})$ of the Chern character from the Chern classes $c_k(\mathcal{E})$ and vice versa. In (HMMS) a different choice of tautological generators is used (see §4.2.2).

Second, as the universal family on $\mathcal{M}_{n,d} \times C$ is non-unique, the tautological classes will only be uniquely defined if they are appropriately normalised. This normalisation is not important for tautological generation of the cohomology, but it is crucial to compute the perversity and weights of the tautological classes. There are different ways to correctly normalise the tautological classes: work with the universal PGL_n-bundle or the PGL_n-moduli space as in (MS), or consider twisted universal families which are normalised as in de Cataldo, Maulik, and Shen (2022a, §0.3) or work with a normalised rational virtual universal bundle of degree 0.

For concreteness, we define the tautological classes using the universal PGL_n-bundle $\mathcal{E}_{\mathrm{PGL}_n}$ on $\mathcal{M}_{\mathrm{PGL}_n,d} \times C$ when n and d are coprime. For all $\gamma \in H^i(C, \mathbb{Q})$ we define

$$c_k(\gamma) := p_{\mathcal{M}*}(\mathrm{ch}_k(\mathcal{E}_{\mathrm{PGL}_n}) \cup p_C^*(\gamma)) \in H^{i+2k-2}(\mathcal{M}_{\mathrm{PGL}_n,d}, \mathbb{Q})$$

where p_C and $p_{\mathcal{M}}$ denote the natural projections.

In rank $n = 2$, Hausel and Thaddeus (2004) showed that the tautological classes generate $\mathcal{M}_{\mathrm{PGL}_2,1}$, and combining these with the classes coming from $\operatorname{Jac}(C)$, they obtained generators for $\mathcal{M}_{\mathrm{GL}_2,1}$. Markman (2002) generalised this to higher rank by viewing Higgs bundles as sheaves on a surface T^*C via the spectral correspondence and using an idea of Ellingsrud and Strømme (1993) and Beauville (1995) to express the diagonal class of the moduli space in terms of the Chern classes of the universal family.

Theorem 2.1 (Markman's tautological generation). — *For n and d coprime, the following statements hold.*

(i) *The classes $c_k(\gamma)$ generate $H^*(\mathcal{M}_{\mathrm{PGL}_n,d},\mathbb{Q})$ as a $\mathbb{Q}$-algebra.*
(ii) *$H^*(\mathcal{M}_{\mathrm{GL}_n,d},\mathbb{Q})$ as in (4) is generated as a $\mathbb{Q}$-algebra by $c_k(\gamma)$ and the pullbacks $\xi_1,\dots,\xi_{2g}$ under $\det\colon \mathcal{M}_{\mathrm{GL}_n,d}\to\mathcal{M}_{\mathrm{GL}_1,d}\cong \mathrm{Jac}^d(C)\times H^0(C,\omega_C)$ of the generators of $H^*(\mathrm{Jac}(C),\mathbb{Q})$.*

The tautological classes $c_k(\gamma)$ can also be directly defined using Chern character components of a normalised virtual universal family as follows. Let $\mathcal{E}\to\mathcal{M}_{\mathrm{GL}_n,d}\times C$ denote a universal family, then as we are working with $\mathbb{Q}$-coefficients we can take a virtual nth root of $\det(\mathcal{E})^{-1}$ in the (rational) Grothendieck group of vector bundles. The rational Chern characters $\overline{\mathrm{ch}}_k := \mathrm{ch}_k(\mathcal{E}\otimes\det(\mathcal{E})^{\otimes -1/n})$ of the degree 0 virtual bundle $\mathcal{E}\otimes\det(\mathcal{E})^{\otimes -1/n}$ are independent of $\mathcal{E}$ and coincide with the Chern characters of the pullback of $\mathcal{E}_{\mathrm{PGL}_n}\to\mathcal{M}_{\mathrm{PGL}_n,d}\times C$.

The cohomology of the SL_n-Higgs moduli space is not tautologically generated, as all tautological classes are Γ-invariant and so they cannot describe the variant cohomology. This means the $P=W$ conjecture is substantially more complicated for SL_n and is only known when n is prime (see §2.4).

For certain other smooth fine moduli spaces $\mathcal{M}$ of sheaves on surfaces, one can define a tautological ring using the Künneth components of the universal sheaf. The tautological ring is contained in the pure part of the cohomology of $\mathcal{M}$ and they coincide in certain nice situations: for example, since $H^*_{\mathrm{pure}}(\mathcal{M},\mathbb{Q})$ is generated by the Künneth components of the diagonal in $\overline{\mathcal{M}}\times\mathcal{M}$ for a smooth compactification (see §1.3), it suffices to express the diagonal in terms of tautological classes as in Markman (2002) in order to conclude the tautological and pure rings coincide; this argument is employed in (HMMS), see §4.3.

2.3. Weights of tautological classes on the Betti moduli space

Recall that the universal G-bundle $\mathcal{P}_G\to\mathfrak{M}_G^{\mathrm{B}}\times C$ determines a classifying map

$$\nu\colon (\mathfrak{M}_G^{\mathrm{B}})^{\mathrm{an}}\times C^{\mathrm{an}}\to BG^{\mathrm{an}}$$

and a corresponding pullback on cohomology $\nu^*\colon H^*(BG,\mathbb{Q})\to H^*(\mathfrak{M}_G^B,\mathbb{Q})\otimes H^*(C,\mathbb{Q})$. One then obtains tautological Betti classes by taking the Künneth components of the images of the generators of $H^*(BG,\mathbb{Q})$.

More concretely, for GL_n, we have

$$H^*(B\mathrm{GL}_n,\mathbb{Q})\cong H^*(B\mathbb{G}_m^n,\mathbb{Q})^{S_n}\cong\mathbb{Q}[\alpha_1,\dots,\alpha_n]^{S_n}\cong\mathbb{Q}[c_1,\dots,c_n]\cong\mathbb{Q}[\mathrm{ch}_1,\dots,\mathrm{ch}_n],$$

where c_i are the elementary symmetric polynomials in the Chern roots α_i. The Chern roots can be viewed as generators of the different copies of $H^*(B\mathbb{G}_m, \mathbb{Q})$. For a GL_n-bundle $P \to X$ corresponding to a classifying map $\nu_P \colon X \to B\mathrm{GL}_n$, the corresponding cohomological pullback $\nu_P^* \colon H^*(B\mathrm{GL}_n, \mathbb{Q}) \to H^*(X)$ sends c_k to $c_k(P)$.

The weights of the tautological classes on the Betti stack of G-local systems on C (or in fact a more general topological space X of finite type) were calculated by Shende (2017). Since the pullback on cohomology along an algebraic morphism respects Hodge structures, Shende replaces the non-algebraic classifying map by an algebraic evaluation map of simplicial schemes by replacing C with a simplicial set Δ_C whose geometric realisation is homotopy equivalent to C. By viewing Δ_C as a constant simplicial scheme, its mixed Hodge structure is of weight zero. Since the classifying stack BG has a pure Hodge structure of Hodge–Tate type

$$H^*(BG, \mathbb{Q}) = \bigoplus_{k \geq 0} {}^k \operatorname{Hod}^{2k}(BG),$$

Shende concludes that the cup product with $\gamma \in H^k(C, \mathbb{Q})$ increases the cohomological degree by k but does not change the Hodge degree.

For $G = \mathrm{PGL}_n$ and coprime degree d, the Betti moduli stack $\mathfrak{M}^B_{\mathrm{PGL}_n,d}$ is a Deligne–Mumford stack whose underlying coarse moduli space $\mathcal{M}^{\mathrm{B}}_{\mathrm{PGL}_n,d}$ has the same rational cohomology. The tautological Betti classes

$$c_k^{\mathrm{B}}(\gamma) = \int_\gamma \mathrm{ch}_k(\mathcal{P}_{\mathrm{PGL}_n}) := (p_{\mathfrak{M}})_*(p_C^*(\gamma) \cup \mathrm{ch}_k(\mathcal{P}_{\mathrm{GL}_n})) \in H^*(\mathfrak{M}^{\mathrm{B}}_{\mathrm{PGL}_n,d}, \mathbb{Q})$$

correspond to the tautological (Dolbeault) classes $c_k(\gamma)$ as the underlying universal (complex) PGL_n-bundles are isomorphic (see §2.1). We can now precisely state Shende's result for PGL_n-moduli spaces.

Theorem 2.2 (Shende (2017)). — *For $\gamma \in H^*(C, \mathbb{Q})$, we have $c_k(\gamma) \in {}^k\operatorname{Hod}^*(\mathfrak{M}^B_{\mathrm{PGL}_n,d})$. Hence, the tautological class $c_k(\gamma)$ has homogeneous weight $2k$ and type (k,k) which is independent of the cohomological degree of γ.*

For $G = \mathrm{GL}_n$, where the universal family on $\mathcal{M}_{n,d}$ is non-unique, only appropriately normalised tautological classes have the correct weight as explained in de Cataldo, Maulik, and Shen (2022a, Lemma 4.6). In fact by Shende (2017), the weight 2 Hodge classes in the second degree cohomology of $\mathcal{M}_{n,d} \times C$ are precisely the middle Künneth components

$${}^1 \operatorname{Hod}^2(C \times \mathcal{M}^{\mathrm{B}}_{n,d}) = H^1(C, \mathbb{Q}) \otimes H^1(\mathcal{M}^{\mathrm{B}}_{n,d}, \mathbb{Q})$$

and so the normalised Chern characters of the universal bundle must be defined so the first Chern character lies in this Künneth component and thus has weight 2.

For $G = \mathrm{PGL}_n$, the cohomology is generated by tautological classes $c_k(\gamma)$ of even weight $2k$ of type (k,k). Hence, we conclude $W_{2k+1}H^*(\mathcal{M}_{G,d}, \mathbb{Q}) = W_{2k}H^*(\mathcal{M}_{G,d}, \mathbb{Q})$. Since the weight filtration is multiplicative, we have

$$\prod_{i=1}^{s} c_{k_i}(\gamma_i) \in W_{\sum_{i=1}^s 2k_i} H^*(\mathcal{M}^{\mathrm{B}}_{\mathrm{PGL}_n,d}, \mathbb{Q}).$$

2.4. Equivalent formulations of $P = W$

In the case of $G = \mathrm{GL}_n$ and PGL_n, we can exploit the fact that the cohomology of the Dolbeault moduli space is tautologically generated and the weights of the generators are known to reduce the $P = W$ conjecture to statements about the perversity of the tautological classes and their products.

Proposition 2.3. — *For n and d coprime, the following statements are equivalent:*

(i) *The $P = W$ conjecture holds for GL_n and degree d.*
(ii) *The $P = W$ conjecture holds for PGL_n and degree d.*
(iii) *The tautological classes $c_k(\gamma) \in H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d}, \mathbb{Q})$ have perversity k and the perverse filtration is multiplicative: $P_i \cup P_j \subset P_{i+j}$.*
(iv) *For any $k_1, \ldots, k_s \in \mathbb{Z}_{>0}$ and $\gamma_1, \ldots, \gamma_s \in H^*(C, \mathbb{Q})$, we have*
$$\prod_{i=1}^{s} c_{k_i}(\gamma_i) \in P_{\sum_{i=1}^{s} k_i} H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d}, \mathbb{Q}).$$
(v) *For all k, we have $W_{2k} H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d}, \mathbb{Q}) \subset P_k H^*(\mathcal{M}^{\mathrm{B}}_{n,d}, \mathbb{Q})$.*

Proof. — The cohomology of GL_n and PGL_n-Higgs moduli spaces are related by Eq. (4) and, by Theorem 2.1, the (tautological) generators of the GL_n-Higgs moduli space are the (tautological) generators of the PGL_n-Higgs moduli space together with the pullbacks $\xi_1, \ldots, \xi_{2g}$ of the generators of $H^*(\mathrm{Jac}(C), \mathbb{Q})$ via the (smooth) determinant map in the following diagram

$$\begin{tikzcd} \mathcal{M}^{\mathrm{Dol}}_{n,d} \arrow[r, "\det"] \arrow[d, "h_n"'] & \mathcal{M}^{\mathrm{Dol}}_{1,d} \arrow[d, "h_1"'] \\ \mathcal{A}_n = \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}) \arrow[r] & \mathcal{A}_1 = H^0(C, \omega_C). \end{tikzcd}$$

Since the rank 1 Hitchin fibration h_1 is smooth and in fact a trivial fibration, the classes ξ_i have weight 2 and perversity 1 (see de Cataldo, Hausel, and Migliorini, 2012, Proposition 1.4.1). By realising $\mathcal{M}^{\mathrm{Dol}}_{\mathrm{PGL}_n,d}$ as the quotient of the moduli space of trace-free GL_n-Higgs bundles by the Jacobian of C as in de Cataldo, Hausel, and Migliorini (2012, §2.4), one obtains as in Eq. (2.4.14) of *loc. cit.*
$$P_{\leq k} H^i(\mathcal{M}^{\mathrm{Dol}}_{n,d}, \mathbb{Q}) = \bigoplus_{j \geq 0} \left(P_{\leq k-j} H^{i-j}(\mathcal{M}^{\mathrm{Dol}}_{\mathrm{PGL}_n,d}, \mathbb{Q}) \otimes \bigwedge^{j} H^1(C, \mathbb{Q}) \right),$$
from which the equivalence of (i) and (ii) follows.

Since the weights of the tautological classes are known and the weight filtration is multiplicative, we see (ii) $\implies$ (iii) $\implies$ (iv). Since the cohomology is generated by tautological classes whose weights are known we conclude (iv) $\implies$ (v) for $\mathcal{M}^{\mathrm{Dol}}_{\mathrm{PGL}_n,d}$, but then this gives the corresponding statement for $\mathcal{M}^{\mathrm{Dol}}_{\mathrm{GL}_n,d}$ as the cohomology only differs by a factor $H^*(\mathrm{Jac}(C), \mathbb{Q})$ whose weights and perversity were described above. Finally, to deduce (v) $\implies$ (i), one can compare dimensions and inductively see that

Relative Hard Lefschetz and Curious Hard Lefschetz (Theorems 1.11 and 1.15) force certain inequalities to become equalities. For example, we deduce $P_0 = W_0$ as

$$\dim W_{2N}/W_{2N-1} \overset{\text{CHL}}{=} \dim W_0 \leq \dim P_0 \overset{\text{RHL}}{=} \dim P_N/P_{N-1} \leq W_{2N}/W_{2N-1},$$

where $N = \dim \mathcal{M}^{\mathrm{B}} = 2r_h$, then one proceeds inductively to show $P_i = W_{2i}$. □

The perversity of the tautological classes is computed by de Cataldo, Maulik, and Shen (2022a), who show

$$c_k(\gamma) \in P_k(\mathcal{M}^{\mathrm{Dol}}_{n,d}, \mathbb{Q}),$$

although this is not used in either (MS) or (HMMS). This means it suffices to prove the multiplicativity of the perverse filtration for the Hitchin map. We note that in general, the perverse filtration is not multiplicative (for example, see de Cataldo, 2017b, Exercise 5.6.8) and so the $P = W$ proofs heavily rely on the specific geometry of the Hitchin fibration.

Let us finally remark on the additional difficulties concerning the $P = W$ conjecture for SL_n (with coprime degree). The $P = W$ conjecture for PGL_n (or equivalently GL_n) gives the $P = W$ conjecture for the invariant part of the cohomology of the SL_n-Higgs moduli space under the action of $\Gamma = \mathrm{Jac}(C)[n]$. The variant part of the cohomology is described by endoscopic moduli spaces attached to non-trivial characters of Γ which can be realised as subvarieties of $\mathrm{GL}_{n/r}$-Higgs moduli spaces on a degree r Galois cover of C determined by the character of Γ as in Hausel and Pauly (2012) and Maulik and Shen (2021). If $n = p$ is prime, the endoscopic moduli spaces parametrise rank 1 vector bundles on a degree n Galois cover of C; consequently they are the total space of cotangent bundles of Prym varieties whose associated perverse filtrations are trivial. By using topological mirror symmetry (Hausel and Thaddeus, 2003; Groechenig, Wyss, and Ziegler, 2020; Maulik and Shen, 2021) to explicitly compute the E-polynomials of the variant part on the Betti and Dolbeault side for $n = p$ prime, de Cataldo, Maulik, and Shen (2022b, Theorem 1.5) show that the $P = W$ conjecture for SL_p is equivalent to the $P = W$ conjecture for GL_p; in particular, the $P = W$ conjecture for SL_p with p prime is now a theorem.

2.4.1. *Perverse equals Chern.* — We have seen that via tautological generation, the $P = W$ conjecture can be entirely phrased on the Dolbeault moduli space as the interaction of specific tautological classes with the perverse filtration. More precisely, it suffices to show the perverse filtration coincides with the *Chern filtration* (as explained in Maulik, Shen, and Yin, 2023 and denoted $P = C$), where the Chern filtration is defined as

$$C_k := \mathrm{Span}\left(\prod_i \psi_{k_i}(\gamma_i) : \sum_i k_i \leq k\right)$$

for a fixed choice of tautological generators $\psi_k(\gamma) := p_{\mathcal{M}*}(f_k(\mathcal{E}) \cup p_C^*(\gamma)) \in H^{2k+\deg(\gamma)-2}(\mathcal{M}^{\mathrm{Dol}}_{n,d})$ defined using degree k homogeneous polynomials f_k.

One advantage of thinking in this framework is that this makes sense in a broader context. For example, there is no Betti-version of D-twisted Higgs moduli spaces, so we cannot ask about $P = W$, but we can ask whether $P = C$ holds.

Remark 2.4. — In all three proofs of the $P = W$ conjecture for GL_n, non-abelian Hodge theory plays a minimal role due to tautological generation. More precisely, the tautological cohomology rings on the Dolbeault and Betti moduli spaces can both be viewed as quotients of the same ring (the Fock space, see §4.4.2) which has a natural filtration by Chern degree. Shende shows for any G that the image of the Chern filtration agrees (up to rescaling indices) with the weight filtration on the tautological cohomology of the Betti moduli space. All existing proofs for GL_n (or PGL_n) involve showing P contains (or in (HMMS) is equal to) the image of the Chern filtration C on the tautological cohomology of the Dolbeault moduli space. Only at the end, non-abelian Hodge theory is combined with Markman's tautological generation (and additionally Curious Hard Lefschetz in (MS) and Maulik, Shen, and Yin, 2023) to conclude.

3. THE PROOF OF MAULIK AND SHEN

3.1. Strong perversity and overview of the proof

Consider the perverse filtration of a proper morphism $f\colon X \to Y$ between smooth irreducible quasi-projective varieties over $\mathrm{k} = \mathbb{C}$. A class $\gamma \in H^l(X, \mathbb{Q})$ can be viewed as a morphism $\gamma\colon \mathbb{Q}_X \to \mathbb{Q}_X[l]$. For $c \geq 0$, we say as in (MS) that $\gamma \in H^l(X, \mathbb{Q})$ has *strong perversity* c (with respect to f) if for all i the composition

$$^{\mathfrak{p}}\tau_{\leq i} Rf_*\mathbb{Q}_X \hookrightarrow Rf_*\mathbb{Q}_X \xrightarrow{Rf_*\gamma} Rf_*\mathbb{Q}_X[l]$$

has image contained in $^{\mathfrak{p}}\tau_{\leq i+(c-l)}(Rf_*\mathbb{Q}_X[l])$. As the perverse truncation $^{\mathfrak{p}}\tau_{\leq i}$ is functorial, $\gamma \in H^l(X, \mathbb{Q})$ automatically has strong perversity l and so this notion is only interesting for $c < l$. Moreover, if γ has strong perversity c, then it has perversity at most c.

The following proposition collects many useful consequences of this notion that are used in the proof of Maulik and Shen.

Proposition 3.1. — *For a proper morphism $f\colon X \to Y$ between smooth irreducible quasi-projective varieties, the following statements hold.*

i) *If $\gamma \in H^l(X, \mathbb{Q})$ has strong perversity c, then $\gamma \cup P_i^f H^*(X, \mathbb{Q}) \subset P_{i+c}^f H^{*+l}(X, \mathbb{Q})$.*
ii) (Strong perversity is multiplicative). *If $\gamma_i \in H^*(X, \mathbb{Q})$ has strong perversity c_i for $i = 1, \ldots, s$, then $\prod_{i=1}^s \gamma_i$ has strong perversity $\sum_{i=1}^s c_i$.*
iii) (Splitting principle for strong perversity). *If $E \to X$ is a vector bundle with a filtration whose subquotients are line bundles L_i whose first Chern classes have strong perversity 1, then $c_k(E)$ and $\mathrm{ch}_k(E)$ have strong perversity k for all k.*

iv) *A class* $\gamma \in H^l(X,\mathbb{Q})$ *has strong perversity* $l-1$ *if and only if*

$$^{\mathfrak{p}}\mathcal{H}^i(Rf_*\gamma)\colon\ ^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathbb{Q}_X) \to {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathbb{Q}_X[l])$$

vanishes for all i*. In particular, if* f *has full supports, one can check that* $^{\mathfrak{p}}\mathcal{H}^i(Rf_*\gamma)$ *vanishes over any non-empty open subset* $U \subset Y$*.*

Proof. — For i), by strong perversity of γ, we have a commutative square

$$\begin{tikzcd} H^*(Y, {}^{\mathfrak{p}}\tau_{\leq i} Rf_*\mathbb{Q}_X) \arrow[r, "\cup\gamma"] \arrow[d] & H^*(Y, {}^{\mathfrak{p}}\tau_{\leq i+c-l} Rf_*\mathbb{Q}_X[l]) \arrow[d] \\ H^*(Y, Rf_*\mathbb{Q}_X) \arrow[r, "\cup\gamma"] & H^*(Y, Rf_*\mathbb{Q}_X[l]) \end{tikzcd}$$

where $H^*(Y, {}^{\mathfrak{p}}\tau_{\leq i+c-l}Rf_*\mathbb{Q}_X[l]) = H^{*+l}(Y {}^{\mathfrak{p}}\tau_{\leq i+c}Rf_*\mathbb{Q}_X)$. Since the images of the vertical maps are $P_iH^*(Y,Rf_*\mathbb{Q}_X)$ and $P_{i+c}H^{*+l}(Y,Rf_*\mathbb{Q}_X)$ respectively, if a class lies in the image of the left vertical map, then after applying γ it lies in the image of the right vertical map.

For ii), we use the commutative diagram

$$\begin{tikzcd} {}^{\mathfrak{p}}\tau_{\leq i}Rf_*\mathbb{Q}_X \arrow[r, "Rf_*\gamma_1"] \arrow[d, hook] & {}^{\mathfrak{p}}\tau_{\leq i+c_1-l_1}Rf_*\mathbb{Q}_X[l_1] \arrow[r, "Rf_*\gamma_2"] \arrow[d, hook] & {}^{\mathfrak{p}}\tau_{\leq i+c_1+c_2-l_1-l_2}Rf_*\mathbb{Q}_X[l_1+l_2] \arrow[d, hook] \\ Rf_*\mathbb{Q}_X \arrow[r, "Rf_*\gamma_1"] & Rf_*\mathbb{Q}_X[l_1] \arrow[r, "Rf_*\gamma_2"] & Rf_*\mathbb{Q}_X[l_1+l_2]. \end{tikzcd}$$

For iii), we note that a linear combination of classes with strong perversity c has strong perversity c and a k-fold product of classes with strong perversity 1 has strong perversity k by ii). Since $c_k(E)$ and $\mathrm{ch}_k(E)$ are homogeneous degree k polynomials in the first Chern classes $c_1(L_i)$ (e.g. $c_k(E) = e_k(c_1(L_1),\dots,c_1(L_n))$ for the elementary symmetric polynomials e_k), we conclude $c_k(E)$ and $\mathrm{ch}_k(E)$ have strong perversity k.

Finally iv) holds as we have a distinguished triangle

$$^{\mathfrak{p}}\tau_{\leq i-1}(Rf_*\mathbb{Q}_X[l]) \to {}^{\mathfrak{p}}\tau_{\leq i}(Rf_*\mathbb{Q}_X[l]) \to {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathbb{Q}_X[l])[-i] \xrightarrow{+1}$$

and thus $^{\mathfrak{p}}\mathcal{H}^i(Rf_*\gamma) = 0$ if and only if $^{\mathfrak{p}}\tau_{\leq i} \circ Rf_*\gamma$ factors via $^{\mathfrak{p}}\tau_{\leq i-1}(Rf_*\mathbb{Q}_X[l])$. □

As a relatively straightforward corollary of the first two statements, one sees that it suffices to show the tautological classes $c_k(\gamma) \in H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d})$ have strong perversity k to prove the $P=W$ conjecture for PGL_n (or equivalently GL_n by Proposition 2.3). In fact, Maulik and Shen show it suffices to prove that $\mathrm{ch}_k(\mathcal{E}_{\mathrm{PGL}_n}) \in H^*(\mathcal{M}_{\mathrm{PGL}_n,d},\mathbb{Q})$ has strong perversity k with respect to the morphism $h_{\mathrm{PGL}_n} \times \mathrm{Id}_C\colon \mathcal{M}_{\mathrm{PGL}_n,d} \times C \to \mathcal{A}_{\mathrm{PGL}_n} \times C$, where $\mathcal{E}_{\mathrm{PGL}_n} \to \mathcal{M}_{\mathrm{PGL}_n,d} \times C$ is the universal bundle for n and d coprime.

Lemma 3.2. — *Let* d *be coprime to* n*. If* $\mathrm{ch}_k(\mathcal{E}_{\mathrm{PGL}_n})$ *has strong perversity* k *with respect to* $h_{\mathrm{PGL}_n} \times \mathrm{Id}_C\colon \mathcal{M}_{\mathrm{PGL}_n,d} \times C \to \mathcal{A}_{\mathrm{PGL}_n} \times C$*, then for* $\gamma \in H^*(C,\mathbb{Q})$*, the cup product with* $c_k(\gamma)$ *increases the perversity with respect to* h_{PGL_n} *by at most* k*. In particular, the* $P=W$ *conjecture holds for* PGL_n *(and* GL_n*) with* d *coprime to* n*.*

Proof. — Fix a homogeneous basis $\Pi = \{\gamma_0, \dots, \gamma_{2g+2}\}$ of $H^*(C, \mathbb{Q})$ with Poincaré dual basis $\gamma_0^\vee, \dots, \gamma_{2g+2}^\vee$ and write

$$\mathrm{ch}_k(\mathcal{E}) = \sum_{\gamma \in \Pi} \gamma^\vee \otimes c_k(\gamma).$$

Since $\mathrm{ch}_k(\mathcal{E})$ has strong perversity k, we have

$$(5)\quad - \cup \mathrm{ch}_k(\mathcal{E})\colon P_i^{h_{\mathrm{PGL}_n} \times \mathrm{Id}_C} H^*(\mathcal{M}_{\mathrm{PGL}_n,d} \times C, \mathbb{Q}) \to P_{i+k}^{h_{\mathrm{PGL}_n} \times \mathrm{Id}_C} H^*(\mathcal{M}_{\mathrm{PGL}_n,d} \times C, \mathbb{Q})$$

by Proposition 3.1. Using the Künneth isomorphism

$$P_i^{h_{\mathrm{PGL}_n} \times \mathrm{Id}_C} H^*(\mathcal{M}_{\mathrm{PGL}_n,d} \times C, \mathbb{Q}) \cong H^*(C, \mathbb{Q}) \otimes P_i^h H^*(\mathcal{M}_{\mathrm{PGL}_n,d}, \mathbb{Q})$$

we can apply the morphism (5) to $\gamma \otimes P_i^{h_{\mathrm{PGL}_n}} H^*(\mathcal{M}_{\mathrm{PGL}_n,d}, \mathbb{Q})$ to conclude

$$- \cup c_k(\gamma)\colon P_i^{h_{\mathrm{PGL}_n}} H^*(\mathcal{M}_{\mathrm{PGL}_n,d}, \mathbb{Q}) \to P_{i+k}^{h_{\mathrm{PGL}_n}} H^*(\mathcal{M}_{\mathrm{PGL}_n,d}, \mathbb{Q})$$

as claimed. The final claim follows by Proposition 2.3. □

3.1.1. *Idea behind the proof of Maulik and Shen.* — We have just seen it suffices to show the kth component of the Chern character of the universal bundle $\mathcal{E}_{\mathrm{PGL}_n}$ has strong perversity k for $h \times \mathrm{Id}_C$. Moreover, the last two statements in Proposition 3.1 provide some insight into the idea behind Maulik and Shen's approach.

Inspired by Yun's global Springer theory (Yun, 2011, 2012), Maulik and Shen instead work with a moduli space of parabolic Higgs bundles, where the word parabolic indicates the additional data of a full flag in a fibre of the underlying vector bundle over some point in C. This extra data gives rise to a flag on the universal bundle, so the above splitting principle could be applied and it suffices to know the first Chern classes of the tautological line bundles have strong perversity one. Yun (2012) proves the first Chern classes of these universal line bundles (at least in the PGL_n case) have strong perversity one over an open set in the Hitchin base. It remains to show this strong perversity extends over the full Hitchin base, which would be automatic if the parabolic Hitchin map had full supports. However, this Hitchin map does not have full supports.

The fix for this problem is understood by previous work of Maulik and Shen (e.g. Maulik and Shen, 2021), which builds on work of Ngô (2006, 2010) and Chaudouard and Laumon (2016): rather than working with classical Higgs bundles, one must use D-twisted Higgs bundles $(E, \theta\colon E \to E \otimes \omega_C(D))$ for an effective divisor D. One should show the Hitchin map has full supports if D has sufficiently high degree and then use Maulik and Shen's already established vanishing cycles trick to pass from D-twisted Higgs bundles to classical Higgs bundles. The key new ingredient is a parabolic support theorem.

3.1.2. *Key steps in the proof of Maulik and Shen.* — We divide the proof as follows.

(A) Pass to a moduli space of parabolic D-twisted Higgs bundles appearing in Yun's global Springer theory, where the splitting principle can be applied as the universal bundle is filtered by tautological line bundles.

(B) The first Chern classes of the tautological line bundles on parabolic Higgs moduli spaces have strong perversity 1 over an open in the Hitchin base by work of Yun.
(C) Prove the parabolic D-twisted Hitchin map has full supports when D has sufficiently high degree to deduce strong perversity over the whole Hitchin base.
(D) Use vanishing cycles to go from D-twisted Higgs bundles to classical Higgs bundles.

3.2. Steps (A) – (B): Parabolic moduli spaces and global Springer theory

Let us start by recalling Ngo's description (Ngô, 2006, Section 2) of the stack of G-Higgs bundles and the Hitchin map in terms of the quotient of the adjoint action of G on its Lie algebra $\mathfrak{g}$, before then turning to Yun's global Springer theory which involves incorporating the Grothendieck–Springer resolution $\widetilde{\mathfrak{g}} \to \mathfrak{g}$.

For a reductive group G, the stack of G-Higgs bundles over a point can be identified with the stack $[\mathfrak{g}/G]$. The good moduli space of this stack is the affine GIT quotient $[\mathfrak{g}/G] \to \mathfrak{g}/\!/G \simeq \mathfrak{t}/\!/W$ which is isomorphic to the spectrum of the Weyl invariant functions on the Lie algebra $\mathfrak{t}$ of a maximal torus $T < G$ by the Chevalley restriction theorem. This is a polynomial ring in $n = \mathrm{rk}(T)$ variables of homogeneous degrees d_i which are one more than the exponents of G. If $\mathbb{G}_m$ acts on $\mathfrak{g}$ by scalar multiplication and on $\mathfrak{g}/\!/G$ with weights $(d_1, \dots, d_n)$, then the GIT quotient $\chi\colon \mathfrak{g} \to \mathfrak{c} := \mathfrak{g}/\!/G$ is $\mathbb{G}_m$-equivariant. Furthermore, the restriction to the regular elements $\mathfrak{g}^{\mathrm{reg}} \to \mathfrak{g}/\!/G$ is an orbit space and Kostant constructed sections $\epsilon\colon \mathfrak{g}/\!/G \to \mathfrak{g}^{\mathrm{reg}}$ associated to $\mathfrak{sl}_2$-triples (e, f, h) where e is regular nilpotent.

3.2.1. *Ngô's description of the Hitchin fibration.* — The above picture can be globalised from a point to a curve C by noting that a morphism

$$C \to [\mathfrak{g}/G]$$

corresponds to a G-bundle E on C with a G-equivariant map $E \to \mathfrak{g}$ or equivalently a section of $\mathrm{ad}(E) = E \times^G \mathfrak{g}$. For an effective divisor D on C, a D-twisted G-Higgs bundle on C is a pair (E, θ) given by a G-bundle E and $\theta \in H^0(C, \mathrm{ad}(E) \otimes \omega_C(D))$; this pair corresponds to a section

$$C \to [\mathfrak{g}/G]_D := [\mathfrak{g}/G] \times^{\mathbb{G}_m} \rho_D,$$

where ρ_D is the $\mathbb{G}_m$-torsor on C corresponding to the line bundle $\omega_C(D)$. Thus the stack $\mathfrak{Higgs}_G^D$ of all D-twisted G-Higgs bundles over C classifies sections $C \to [\mathfrak{g}/G]_D$ and the universal family corresponds to an evaluation map

$$\mathrm{ev}_{\mathfrak{H}}\colon \mathfrak{Higgs}_G^D \times C \to [\mathfrak{g}/G]_D. \tag{6}$$

The Hitchin base $\mathcal{A}_G^D = \oplus_{i=1}^n H^0(C, \omega_C(D)^{\otimes d_i})$ parametrises sections $C \to (\mathfrak{g}/\!/G)_D := \mathfrak{g}/\!/G \times^{\mathbb{G}_m} \rho_D$. Thus the Hitchin map $\mathfrak{h}_{\mathfrak{H}}$ on the whole stack is induced by the good moduli space morphism $[\mathfrak{g}/G] \to \mathfrak{g}/\!/G$ in the sense that the following square is cartesian

$$\begin{tikzcd} \mathfrak{Higgs}_G^D \times C \arrow[r, "\mathrm{ev}_{\mathfrak{H}}"] \arrow[d, "\mathfrak{h}_{\mathfrak{H}} \times \mathrm{Id}_C"] & {[\mathfrak{g}/G]_D} \arrow[d] \\ \mathcal{A}_G^D \times C \arrow[r, "\mathrm{ev}_{\mathcal{A}}"] & (\mathfrak{g}/\!/G)_D. \end{tikzcd}$$

If $\deg(\omega_C(D)) \geq 2g$, then $\omega_C(D)^{\otimes i}$ is globally generated for all $i \geq 1$ and thus the evaluation map $\mathrm{ev}_{\mathcal{A}}$ is smooth, as it is a surjective bundle map over C.

We assume that $\omega_C(D)$ has a square root, in order to obtain a Hitchin–Kostant section $\epsilon^D\colon \mathcal{A}_G^D \to \mathfrak{Higgs}_G^D$ from a Kostant section $\mathfrak{g}/\!/G \to \mathfrak{g}$ as in Ngô (2006, Proposition 2.5).

3.2.2. Step (A): *Yun's Global Springer Theory.* — By combining Ngô's description of the Hitchin fibration with the Grothendieck–Springer resolution, Yun introduces a global version of Springer theory. Recall that the Grothendieck–Springer resolution for a reductive group G with fixed Borel subgroup $B < G$ is given by

$$\pi_{\mathfrak{g}}\colon \tilde{\mathfrak{g}} := \{(x, \mathfrak{b}') \in \mathfrak{g} \times \mathcal{B} : x \in \mathfrak{b}'\} \to \mathfrak{g}$$

where $\mathcal{B} = G/B$ is the flag variety, which we identify with the space of Borel subalgebras $\mathfrak{b}' \subset \mathfrak{g}$. The map $\pi_{\mathfrak{g}}\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$ is a small map and a torsor under the Weyl group W over a dense open, thus

$$(R\pi_{\mathfrak{g}*}\mathbb{Q}_{\tilde{\mathfrak{g}}})^W \cong \mathbb{Q}_{\mathfrak{g}}.$$

Yun's global Springer theory describes the moduli stack of parabolic D-twisted G-Higgs bundles as the fibre product of the stack quotient of the Grothendieck–Springer resolution with the evaluation morphism appearing in Equation (6). The stack $\mathfrak{Higgs}_G^{D,\mathrm{par}}$ classifies tuples (E, θ, x, E_x^B) where $(E, \theta) \in \mathfrak{Higgs}_G^D$ and $x \in C$ and E_x^B denotes a reduction of the fibre E_x to a fixed Borel subgroup $B < G$. By Yun (2011, Lemma 2.1.2), the natural forgetful map $\pi\colon \mathfrak{Higgs}_G^{D,\mathrm{par}} \to \mathfrak{Higgs}_G^D \times C$ fits into a Cartesian square

$$\begin{tikzcd} \mathfrak{Higgs}_G^{D,\mathrm{par}} \arrow[r, "\mathrm{ev}_{\mathfrak{H}}^{\mathrm{par}}"] \arrow[d, "\pi_{\mathfrak{H}}"] & {[\tilde{\mathfrak{g}}/G]_D} \arrow[d, "\pi_{\mathfrak{g}}"] \\ \mathfrak{Higgs}_G^D \times C \arrow[r, "\mathrm{ev}_{\mathfrak{H}}"] & {[\mathfrak{g}/G]_D,} \end{tikzcd}$$

where $[\tilde{\mathfrak{g}}/G]_D := [\tilde{\mathfrak{g}}/G] \times^{\mathbb{G}_m} \rho_D$.

Since the stack of all Higgs bundles is not smooth, we will restrict to the open stable locus. By the spectral correspondence, Higgs bundles whose spectral curve is integral are automatically stable, so over the elliptic locus the stack of all Higgs bundles coincides with the stack of stable Higgs bundles.

Let $\mathfrak{M}_G^D \hookrightarrow \mathfrak{Higgs}_G^D$ denote the substack of stable D-twisted G-Higgs bundles on C and let $\mathfrak{M}_G^{D,\mathrm{par}} \hookrightarrow \mathfrak{Higgs}_G^{D,\mathrm{par}}$ be corresponding open substack which fits into a commutative diagram

$$
\begin{tikzcd}
 & \mathfrak{M}_G^{D,\mathrm{par}} \arrow[r] \arrow[dl, "\mathfrak{h}^{\mathrm{par}}"'] \arrow[d, "\pi"] \arrow[rr, bend left, "\mathrm{ev}^{\mathrm{par}}"] & \mathfrak{Higgs}_G^{D,\mathrm{par}} \arrow[r, "\mathrm{ev}_{\mathfrak{H}}^{\mathrm{par}}"'] \arrow[d, "\pi_{\mathfrak{H}}"'] & {[\tilde{\mathfrak{g}}/G]_D} \arrow[d, "\pi_{\mathfrak{g}}"] \\
\mathcal{A}_G^D \times C & \mathfrak{M}_G^D \times C \arrow[l, "\mathfrak{h}\times \mathrm{Id}_C"] \arrow[r] \arrow[rr, bend right, "\mathrm{ev}"'] & \mathfrak{Higgs}_G^D \times C \arrow[r, "\mathrm{ev}_{\mathfrak{H}}"] & {[\mathfrak{g}/G]_D}
\end{tikzcd} \tag{7}
$$

where both squares are Cartesian. We will refer to $\mathfrak{M}_G^{D,\mathrm{par}}$ as the stack of 'stable' parabolic D-twisted G-Higgs bundles, where stability is determined by the stability of the underlying Higgs bundle. In fact, for parabolic Higgs bundles, there are different notions of stability depending on a choice of weights (see Mehta and Seshadri, 1980).

By construction, the pullback $\mathcal{F} := \pi^*\mathcal{E}$ of the universal Higgs bundle $(\mathcal{E}, \Phi)$ on $\mathfrak{M}_G^D \times C$ admits a Borel reduction $\mathcal{F}^B$. Hence, there is a tautological T-torsor $\mathcal{L}^T = \mathcal{F}^B \times^B T$ over $\mathfrak{M}_G^D$ and from any character $\chi\colon T \to \mathbb{G}_m$, the $\mathbb{G}_m$-torsor $\mathcal{L}^T \times^\chi \mathbb{G}_m$ defines a universal line bundle $\mathcal{L}_\chi$ on $\mathfrak{Higgs}_G^{D,\mathrm{par}}$.

Example 3.3. — The GIT quotient for the adjoint action of $G = \mathrm{GL}_n$

$$\chi\colon \mathfrak{gl}_n \twoheadrightarrow \mathfrak{gl}_n/\!/\mathrm{GL}_n \cong \mathrm{Spec}(k[x_1, \dots, x_n]^{S_n}) \cong \mathrm{Spec}(k[\sigma_1, \dots, \sigma_n]) \cong \mathbb{A}^n$$

sends a matrix to the coefficients of its characteristic polynomial. The generators of this polynomial ring have homogeneous degrees $(1, \dots, n)$. By identifying $\mathcal{B} := \mathrm{GL}_n/B$ with the space of full flags in k^n, we can view the stack $[\widetilde{\mathfrak{gl}}_n/\mathrm{GL}_n]$ as the stack of endomorphisms $\theta\colon k^n \to k^n$ with a full flag $0 = V_0 \subset V_1 \subset \cdots V_n = k^n$ that is preserved by θ; this pair defines is a parabolic Higgs bundle (with a full flag) over a point. The Grothendieck–Springer resolution is a S_n-torsor over the set of matrices with n distinct eigenvalues: a flag preserved by this matrix is given by an ordering of the distinct 1-dimensional eigenspaces or equivalently an ordering of the eigenvalues. The stack $\mathfrak{Higgs}_{\mathrm{GL}_n}^{\mathrm{par}}$ parametrises GL_n-Higgs bundles together with a point $x \in C$ and a full flag in the fibre over x. Consequently the pullback of the universal bundle $\mathcal{F} := \pi^*\mathcal{E}$ admits a full flag $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ whose successive quotients are line bundles $\mathcal{L}_i = \mathcal{L}_{\chi_i}$ associated to the characters $\chi_i\colon T \to \mathbb{G}_m$ corresponding to the standard basis.

Let us explain how to descend strong perversity statements for the stack of parabolic Higgs bundles to the stack of Higgs bundles via the map $\pi\colon \mathfrak{M}_G^{D,\mathrm{par}} \to \mathfrak{M}_G^D \times C$ in Diagram (7) obtained as a base change of the Grothendieck–Springer resolution.

For $G = \mathrm{PGL}_n, \mathrm{SL}_n$ or GL_n, the evaluation morphism $\mathrm{ev}\colon \mathfrak{M}_G^D \times C \to [\mathfrak{g}/G]_D$ on the stable locus is smooth by the deformation theory argument in (Maulik and Shen, 2021, Proposition 4.1). Consequently the morphism π is the pullback of the Grothendieck–Springer resolution along a smooth map and so π is also a small morphism and a

W-torsor over a dense open set and thus

$$(R\pi_*\mathbb{Q}_{\mathfrak{M}_G^{D,\mathrm{par}}})^W \simeq \mathbb{Q}_{\mathfrak{M}_G^D\times C}.$$

From this equality and the fact that $\mathfrak{h}^{\mathrm{par}} = (\mathfrak{h}\times \mathrm{Id}_C)\circ\pi$, we obtain the following result.

Lemma 3.4 (MS, Lemma 3.3). — *For $G = \mathrm{PGL}_n, \mathrm{SL}_n$ or GL_n and $\gamma\in H^*(\mathfrak{M}_G^D\times C,\mathbb{Q})$, the pullback $\pi^*\gamma$ has strong perversity k for $\mathfrak{h}^{\mathrm{par}}$ if and only if γ has strong perversity k for $\mathfrak{h}\times\mathrm{Id}_C$.*

3.2.3. Step (B): *Strong perversity of universal line bundles over an open set.* — The following result of Yun (in the case of $G = \mathrm{PGL}_n$) is vital for Maulik and Shen's proof.

Proposition 3.5 (Yun, 2012, Lemma 3.2.3). — *If G is semisimple and $\deg(\omega_C(D)) > 2g$, there is a Zariski dense open subset of $C\times\mathcal{A}_G^D$ over which*

$$^{\mathfrak{p}}\mathcal{H}^i_{\mathfrak{h}^{\mathrm{par}}}(c_1(\mathcal{L}_\chi))\colon\ ^{\mathfrak{p}}\mathcal{H}^i_{\mathfrak{h}^{\mathrm{par}}}(R\mathfrak{h}^{\mathrm{par}}_*\mathbb{Q}_{\mathfrak{M}_G^{D,\mathrm{par}}}) \to{}^{\mathfrak{p}}\mathcal{H}^i_{\mathfrak{h}^{\mathrm{par}}}(R\mathfrak{h}^{\mathrm{par}}_*\mathbb{Q}_{\mathfrak{M}_G^{D,\mathrm{par}}}[2])$$

vanishes. In particular, $c_1(\mathcal{L}_\chi)$ has strong perversity one for $\mathfrak{h}^{\mathrm{par}}$ over this open set.

Throughout the rest of this subsection, as G and D are fixed, we will drop them from the notation. We will explain the proof of this proposition over a smaller open subset than Yun does (as for our purposes it suffices to know it over any dense open subset). In particular, we will only need to care about ordinary cohomology rather than perverse cohomology, as the fibres of the Hitchin map are smooth on this open locus.

The proof of this proposition involves identifying the Higgs stack with a Picard stack $\mathcal{P}/\mathcal{A}$ over an open set in the Hitchin base (via a spectral correspondence) and using the fact that G is semisimple to see that the cup product with $c_1(\mathcal{L}_\chi)$ is determined by its action on the relative cohomology of the neutral component $\mathcal{P}^0$. Once this is known, the proof really boils down to the fact that multiplication by N on an abelian group scheme acts by multiplication by N^i on the ith cohomology. After introducing the Picard stack of Ngô and the enhanced parabolic Hitchin fibration of Yun, we will outline Yun's proof.

Let us introduce the Picard stack $\mathcal{P}/\mathcal{A}$ following Ngô (2010, §4.3.1). In Ngô (2010, §2.1), he constructs a group scheme J of *regular centralisers* over $\mathfrak{c} := \mathfrak{g}/\!/G$ with a G-equivariant homomorphism $\chi^*J\to I$ to the universal centraliser group scheme $I/\mathfrak{g}$, such that $\chi^*J|_{\mathfrak{g}^{\mathrm{reg}}}\simeq I|_{\mathfrak{g}^{\mathrm{reg}}}$. In particular $[\mathfrak{g}^{\mathrm{reg}}/G]\to\mathfrak{c}$ is a J-gerbe. Since $J\to\mathfrak{c}$ is $\mathbb{G}_m$-equivariant, we can twist by the $\mathbb{G}_m$-torsor ρ_D to produce $J_D\to\mathfrak{c}_D$. The fibre $\mathcal{P}_a$ of $\mathcal{P}$ over $a\in\mathcal{A}$ is the stack of torsors over C under the group $J_a := \mathrm{ev}_a^*J_D$. Then $\mathcal{P}$ acts fibrewise on $\mathfrak{Higgs}\to\mathcal{A}$ and moreover $\mathfrak{Higgs}^{\mathrm{reg}}$ (the locus mapping to $[\mathfrak{g}^{\mathrm{reg}}/G]$ under $\mathrm{ev}_{\mathfrak{H}}$) is a $\mathcal{P}$-torsor over the Hitchin base (Ngô, 2010, Proposition 4.3.3). For example, if $G=\mathrm{GL}_n$, then $\mathcal{P}$ is the Picard stack for the spectral curve $\mathcal{C}_n^D/\mathcal{A}_n^D$ and $\mathcal{P}$ has components indexed by all $d\in\mathbb{Z}$. For $G=\mathrm{SL}_n$, the Picard stack $\mathcal{P}$ is the relative Prym stack for the spectral curve $\mathcal{C}^D_{\mathrm{SL}_n}/\mathcal{A}^D_{\mathrm{SL}_n}$ and $\mathcal{P}$ has n components.

Yun works with the enhanced parabolic Hitchin fibration $\tilde{\mathfrak{h}}$ which fits in the diagram

$$\begin{tikzcd}
\mathcal{P} \times_{\mathcal{A}} \widetilde{\mathcal{A}} =: \widetilde{\mathcal{P}} \arrow[r, "\widetilde{\tau}"] \arrow[dr, "\tilde{g}"'] & \mathfrak{Higgs}^{\mathrm{par}} \arrow[r, "\pi"] \arrow[d, shift left, "\widetilde{\mathfrak{h}}"] \arrow[dr, "\mathfrak{h}^{\mathrm{par}}"] & \mathfrak{Higgs} \times C \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathfrak{h} \times \mathrm{Id}_C"] & \mathfrak{Higgs} \arrow[d, shift left, "\mathfrak{h}"] & \mathcal{P} \arrow[l, "\tau"'] \arrow[dl, "g"] \\
 & \widetilde{\mathcal{A}} \arrow[r, "q"] \arrow[u, bend left, "\widetilde{\epsilon}"] \arrow[rr, bend right, "\widetilde{p}"'] & \mathcal{A} \times C \arrow[r, "p=\mathrm{pr}_1"] & \mathcal{A} \arrow[u, bend left, "\epsilon"] &
\end{tikzcd}$$

where $q\colon \widetilde{\mathcal{A}} \to \mathcal{A} \times C$ is the *enhanced Hitchin base* (also known as the *universal cameral cover*) which is a branched W-cover defined as the base change of $\mathfrak{t}_D \to (\mathfrak{t}/\!/W)_D \simeq (\mathfrak{g}/\!/G)_D$ along $\mathrm{ev}_{\mathcal{A}}$. As $\deg(\omega_C(D)) \geq 2g$, the evaluation map $\mathrm{ev}_{\mathcal{A}}$ is smooth and so is $\widetilde{\mathcal{A}}$ by Yun (2011, Lemma 2.2.3). Let $q_a\colon \widetilde{C}_a = q^{-1}(\{a\} \times C) \to C$ which is a branched W-cover known as the cameral cover. For $G = \mathrm{GL}_n$, the cameral cover factors via the spectral curve as $q_a\colon \widetilde{C}_a \to \mathcal{C}_a \to C$.

By Yun (2011, Lemma 2.3.3), $\widetilde{\mathcal{P}} := \mathcal{P} \times_{\mathcal{A}} \widetilde{\mathcal{A}}$ acts fibrewise on $\mathfrak{Higgs}^{\mathrm{par}} \to \widetilde{\mathcal{A}}$. The Hitchin–Kostant section $\epsilon\colon \mathcal{A} \to \mathfrak{Higgs}^{\mathrm{reg}}$ combined with the Picard stack action determine a morphism $\tau\colon \mathcal{P} \to \mathfrak{Higgs}$, which is an isomorphism over a dense open in the Hitchin base. By base change, we get a section $\tilde{\epsilon}\colon \widetilde{\mathcal{A}} \to \mathfrak{Higgs}^{\mathrm{reg}} \times_{\mathcal{A}} \widetilde{\mathcal{A}} \simeq \mathfrak{Higgs}^{\mathrm{par,reg}}$ and similarly a morphism $\widetilde{\tau}\colon \widetilde{\mathcal{P}} \to \mathfrak{Higgs}^{\mathrm{par}}$, which is also an isomorphism over a dense open.

Let us specify the dense open set that we will work with more carefully. As in Ngô (2010, §4.7), assuming $\deg(\omega_C(D)) > 2g$, there is a dense open locus $\mathcal{A}^{\diamond} \subset \mathcal{A}$ over which the cameral curve $\widetilde{C}_a$ is transversal to the discriminant divisor in $\mathfrak{t}_D$ (and thus is smooth and connected). For an $\mathcal{A}$-scheme Y, we denote its base change to $\mathcal{A}^{\diamond} \subset \mathcal{A}$ by $Y^{\diamond}$. By Ngô (2010, Proposition 4.7.7), $\mathfrak{Higgs}^{\diamond}$ is a torsor under $\mathcal{P}^{\diamond}$ and by Yun (2011, Lemma 2.6.1)

$$\mathfrak{Higgs}^{\mathrm{par},\diamond} \simeq \mathfrak{Higgs}^{\diamond} \times_{\mathcal{A}^{\diamond}} \widetilde{\mathcal{A}}^{\diamond}.$$

Before returning to the strong perversity statement, we need one further ingredient from Yun's global Springer theory: by Yun (2012, Lemma 4.13), there is a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{P}} \times_{\widetilde{\mathcal{A}}} \mathfrak{Higgs}^{\mathrm{par}} \arrow[r, "\mathrm{act}"] \arrow[d, "\mathcal{Q}^T \times \mathcal{L}^T"] & \mathfrak{Higgs}^{\mathrm{par}} \arrow[d, "\mathcal{L}^T"] \\
BT \times BT \arrow[r, "\mathrm{mult}"] & BT
\end{tikzcd} \tag{8}$$

where the bottom arrow is induced by multiplication on T and the vertical arrows are induced by universal T-torsors $\mathcal{L}^T \to \mathfrak{Higgs}^{\mathrm{par}}$ and $\mathcal{Q}^T \to \widetilde{\mathcal{P}}$. More precisely, there is a natural morphism $j\colon \mathcal{P} \to \mathcal{P}_T(\widetilde{\mathcal{A}}/\mathcal{A})^W$ to the stack of W-equivariant T-torsors on $\widetilde{\mathcal{A}}/\mathcal{A}$ built from homomorphisms $j_a\colon q_a^* J_a \to T \times \widetilde{C}_a$ constructed from the trivial T-torsor on $\mathfrak{t}$, and $\mathcal{Q}^T$ is defined to be the pullback of the universal T-torsor on $\mathcal{P}_T(\widetilde{\mathcal{A}}/\mathcal{A})^W \times_{\mathcal{A}} \widetilde{\mathcal{A}}$ via the base change to $\widetilde{\mathcal{A}}$ of j; see Ngô (2010, §2.4) and Yun (2012, Construction 4.1.1) for details. For $\chi \in X^*(T)$, we let $\mathcal{Q}_\chi := \mathcal{Q}^T \times^{T,\chi} \mathbb{G}_m$ be the associated line bundle, which satisfies for $N \in \mathbb{Z}$

$$(\mathrm{Id}_{\widetilde{\mathcal{A}}} \times [N])^* \mathcal{Q}_\chi \cong \mathcal{Q}_{N\chi} \cong \mathcal{Q}_\chi^{\otimes N}. \tag{9}$$

Having introduced the enhanced Hitchin fibration and associated Picard stacks, we now turn to explaining the proof of Proposition 3.5. First, we note that it suffices to show ${}^{\mathfrak{p}}\mathcal{H}^i_{\tilde{\mathfrak{h}}}(c_1(\mathcal{L}_\chi))$ vanishes over a dense open, as

$${}^{\mathfrak{p}}\mathcal{H}^i_{\mathfrak{h}^{\mathrm{par}}}(c_1(\mathcal{L}_\chi)) = q_* \, {}^{\mathfrak{p}}\mathcal{H}^i_{\tilde{\mathfrak{h}}}(c_1(\mathcal{L}_\chi)).$$

Hence we can restrict our attention to the enhanced parabolic Hitchin map $\tilde{\mathfrak{h}}$ over $\widetilde{\mathcal{A}}^\diamond$. Since $\tilde{\tau}$ is an isomorphism over $\widetilde{\mathcal{A}}^\diamond$, we can pull back $\mathcal{L}_\chi$ via $\tilde{\tau} = \mathrm{act} \circ (\mathrm{Id} \times \tilde{\epsilon})$, which using the commutativity of Diagram (8) gives

$$\tilde{\tau}^* \mathcal{L}_\chi \simeq \mathcal{Q}_\chi \otimes \tilde{g}^* \tilde{\epsilon}^* \mathcal{L}_\chi. \tag{10}$$

Since $p\colon \widetilde{\mathcal{A}} \to \mathcal{A}$ has relative dimension 1 and is smooth over $\mathcal{A}^\diamond$, by adjunction (Yun, 2012, Remark 4.4.2), the restricted morphism $c_1(\mathcal{L}_\chi)^\diamond\colon R\tilde{\mathfrak{h}}_*\mathbb{Q} \to R\tilde{\mathfrak{h}}_*\mathbb{Q}(1)[2]$ over $\widetilde{\mathcal{A}}^\diamond$ gives a morphism

$$c_1(\mathcal{L}_\chi)^\diamond_\sharp\colon \tilde{p}_!\tilde{p}^!\mathbb{Q} \to Rh_*\mathbb{Q}(1)[2] \xrightarrow{\cup} \mathrm{End}(Rh_*\mathbb{Q})(1)[2].$$

After pulling back along the isomorphism $\tau\colon \mathcal{P}|_{\mathcal{A}^\diamond} \cong \mathfrak{Higgs}|_{\mathcal{A}^\diamond}$, Equation (10) gives a decomposition in $D^b_c(\mathcal{A}^\diamond, \mathbb{Q})$

$$c_1(\tilde{\tau}^*\mathcal{L}_\chi)^\diamond_\sharp = c_1(\mathcal{Q}_\chi)^\diamond_\sharp \oplus c_1(\tilde{g}^*\tilde{\epsilon}^*\mathcal{L}_\chi)^\diamond_\sharp\colon \tilde{p}_!\tilde{p}^!\mathbb{Q} \longrightarrow Rg_*\mathbb{Q}(1)[2]$$

and Yun's proof of Proposition 3.5 then reduces to showing the following two claims:

(i) $c_1(\tilde{g}^*\tilde{\epsilon}^*\mathcal{L}_\chi)^\diamond_\sharp$ factors via $R^0g_*\mathbb{Q}(1)[2]$,
(ii) $c_1(\mathcal{Q}_\chi)^\diamond_\sharp$ factors via $R^1g_*\mathbb{Q}(1)[2]$.

The first claim follows as $\tilde{\epsilon}^*\mathcal{L}_\chi$ is fibrewise trivial over $\widetilde{\mathcal{A}}$. The second claim is where the semisimplicity of G is needed. Let $g_0\colon \mathcal{P}^0 \to \mathcal{A}^\diamond$ denote the neutral component of $g\colon \mathcal{P}^\diamond \to \mathcal{A}^\diamond$, then $Rg_{0*}\mathbb{Q}(1)[2]$ can be identified with a direct summand of $Rg_*\mathbb{Q}(1)[2]$ where the group scheme $\pi_0(\mathcal{P})$ acts trivially; Yun refers to this as the stable summand. By Yun (2012, Lemma 4.6.1), $\mathcal{Q}_\chi$ is the pullback of the Poincaré line bundle on $\widetilde{\mathcal{A}} \times_{\mathcal{A}} \mathcal{P}(\widetilde{\mathcal{A}}/\mathcal{A})$ via the base change of a morphism

$$j_\chi\colon \mathcal{P} \xrightarrow{j_{\mathcal{P}}} \mathcal{P}_T(\widetilde{\mathcal{A}}/\mathcal{A}) \xrightarrow{(-)\times^{T,\chi}\mathbb{G}_m} \mathcal{P}(\widetilde{\mathcal{A}}/\mathcal{A})$$

along $\tilde{p}\colon \widetilde{\mathcal{A}} \to \mathcal{A}$. Since G is semisimple, for each $a \in \mathcal{A}^\diamond$ we know that $\pi_0(\mathcal{P}_a)$ is finite, but $\pi_0(\mathrm{Pic}(\widetilde{C}_a)) \cong \mathbb{Z}$, and thus the morphism $c_1(\mathcal{Q}_\chi)^\diamond_\sharp$ must factor via the neutral component $\mathcal{P}^0$ (i.e. via the direct summand $Rg_{0*}\mathbb{Q}(1)[2]$). Furthermore, for any $N \in \mathbb{Z}$, Equation (9) implies we have a commutative diagram

$$\begin{tikzcd}
\tilde{p}_!\tilde{p}^!\mathbb{Q} \arrow[r, "c_1(\mathcal{Q}_\chi)^\diamond_\sharp"] \arrow[rr, bend right, "Nc_1(\mathcal{Q}_\chi)^\diamond_\sharp"'] & Rg_{0*}\mathbb{Q}(1)[2] \arrow[r, "{[N]^*}"] & Rg^0_*\mathbb{Q}(1)[2]
\end{tikzcd}$$

and as $[N]^*$ acts by multiplication by N^i on $R^i g^0_* \mathbb{Q}(1)[2]$ because g_0 is an abelian scheme over $\mathcal{A}^\diamond$, we deduce the second claim. Combining these two claims with adjunction (Yun, 2012, Remark 4.4.2), Yun deduces Proposition 3.5; in fact, he shows this holds over an even larger subset by applying a support theorem on this larger open (see Yun, 2012, Theorem B).

Remark 3.6. — Yun uses the tautological line bundles $\mathcal{L}_\chi$ for each $\chi \in X^*(T)$ (or strictly speaking the cup product with their first Chern classes) to define an action of $X^*(T)$ on $R\mathfrak{h}^{\mathrm{par}}_* \mathbb{Q}$. He further shows (Yun, 2011, Theorem A) that there is an action of the extended affine Weyl group $\widetilde{W} = X_*(T) \rtimes W$ on $R\mathfrak{h}^{\mathrm{par}}_* \mathbb{Q}$ constructed using cohomological parabolic Hecke correspondences. These Hecke correspondences give an algebro-geometric construction of the (gauge theoretic) *'t Hooft operators* appearing in Kapustin and Witten (2007) and the geometric Langlands program, whereas the operators given by Chern class cup-products indexed by characters $\chi \in X^*(T)$ are shadows of the *Wilson operators* (given by tensoring with tautological bundles). In fact, these operators are predicted to be exchanged under a form of Langlands duality expressed as a homological mirror symmetry between Higgs bundles for G and its Langlands dual group. By combining these actions, Yun produces an action of the graded double affine Hecke algebra $\mathbb{H}$ on $R\mathfrak{h}^{\mathrm{par}}_* \mathbb{Q}$ (see Yun, 2011, Theorem B). In §4, we will see that analogous operators are used in the $P = W$ proof of Hausel, Mellit, Minets and Schiffmann (HMMS). Recall from the discussion towards the end of the introduction that Fourier duality is also a central technique in Maulik, Shen, and Yin (2023).

3.3. Step (C): Support theorems

Maulik and Shen's parabolic support theorem is a key new ingredient in their $P = W$ proof. We begin by explaining the non-parabolic analogue for D-twisted GL_n-Higgs bundles, where first Ngô (2010) showed the Hitchin map restricted to the elliptic locus has full supports (for any $D \geq 0$) and then Chaudouard and Laumon (2016) proved for $D > 0$ the whole Hitchin map has full supports by showing the generic point of each support lies in the elliptic locus. In the parabolic case, the first part of this procedure is completed by work of Yun (2012), who showed over the elliptic locus any strict support is a component of the endoscopic loci. In particular, for $G = \mathrm{PGL}_n$ or $G = \mathrm{GL}_n$, the restriction of the parabolic Hitchin map to the elliptic locus has full supports for D of sufficiently large degree. Then Maulik and Shen extend this to the full Hitchin base using a relative dimension bound.

3.3.1. *Review of support theorems for Higgs bundles (after Ngô, Chaudouard–Laumon).* — Let us outline the key ideas in the support theorems for twisted non-parabolic Hitchin fibrations of Ngô (2006, 2010) (see also Hales, 2012) and Chaudouard and Laumon (2016).

For an effective divisor D on C, let $\mathcal{M}^D := \mathcal{M}^D_{n,d}$ be the moduli space of semistable D-twisted Higgs bundles $(E, \theta \colon E \to E \otimes \omega_C(D))$ with Hitchin map $h^D \colon \mathcal{M}^D \to \mathcal{A}^D$

and corresponding universal spectral curve $\mathcal{C}^D/\mathcal{A}^D$. We define open sets

$$\mathcal{A}^{D,\mathrm{sm}} \subset \mathcal{A}^{D,\mathrm{ell}} \subset \mathcal{A}^D$$

where $a \in \mathcal{A}^{D,\mathrm{sm}}$ (resp. $\mathcal{A}^{D,\mathrm{ell}}$) if its corresponding spectral curve is smooth and connected (resp. integral). Over the elliptic locus, the relative Jacobian $\mathcal{P} := \mathcal{P}_0(\mathcal{C}^{D,\mathrm{ell}}/\mathcal{A}^{D,\mathrm{ell}})$ acts on the Higgs moduli space by tensorisation using the spectral correspondence $\mathcal{M}^{D,\mathrm{ell}}_{n,d} \simeq \overline{\mathcal{P}}_e(\mathcal{C}^{D,\mathrm{ell}}/\mathcal{A}^{D,\mathrm{ell}})$ to identify the elliptic locus with a relative compactified Jacobian.

Theorem 3.7 (Support theorems for D-twisted Hitchin maps)
Let D be an effective divisor on C and assume n and d are coprime.

(i) (Ngô) *For $D \geq 0$, the elliptic Hitchin map $h^{D,\mathrm{ell}} \colon \mathcal{M}^{D,\mathrm{ell}} \to \mathcal{A}^{D,\mathrm{ell}}$ has full supports.*
(ii) (Chaudouard–Laumon) *For $D > 0$, the Hitchin map $h^D \colon \mathcal{M}^D \to \mathcal{A}^D$ has full supports.*

The proof of the first part due to Ngô applies a general result (Ngô, 2010, Théorème 7.2.1, Proposition 7.2.2) improving the Goresky–MacPherson inequality on the codimensions of supports in the case of weak abelian fibrations (i.e. there is an action of a smooth commutative group scheme with nice properties). For the Hitchin map, the action of the relative Jacobian on the elliptic locus is used as follows.

For $a \in \mathcal{A}^{D,\mathrm{ell}}$, as the spectral curve $\mathcal{C}^D_a$ is integral with locally planar singularities, $\mathcal{P}_a = \mathrm{Pic}^0(\mathcal{C}^D_a)$ is a semi-abelian variety appearing as an extension

$$1 \to R_a \to \mathcal{P}_a = \mathrm{Pic}^0(\mathcal{C}^D_a) \to B_a := \mathrm{Pic}^0(\mathcal{C}^{D,\nu}_a) \to 1 \tag{11}$$

where $\mathcal{C}^{D,\nu}_a \to \mathcal{C}^D_a$ is the normalisation and thus B_a is an abelian variety, and the kernel R_a is an affine group scheme. Associated to such a family of integral curves, there is a δ-function

$$\delta \colon \mathcal{A}^{D,\mathrm{ell}} \to \mathbb{N}, \quad a \mapsto \delta(a) = \dim R_a.$$

Ngô proves a δ-inequality for any support Z of $h^{D,\mathrm{ell}}$:

$$\mathrm{codim}(Z) \leq \delta_Z := \min\{\delta(a) : a \in Z\}. \tag{12}$$

Suppose for each $a \in Z$, one can find a neighbourhood $a \in U \subset \mathcal{A}^{D,\mathrm{ell}}$ and a section $B_U \to \mathcal{P}_U$ of the map in Eq. (11), then B_U acts on $\mathcal{M}^{D,\mathrm{ell}}_U$ via this section and we obtain a factorisation

$$h^{D,\mathrm{ell}}_U \colon \mathcal{M}^{D,\mathrm{ell}}_U \to [\mathcal{M}^{D,\mathrm{ell}}_U/B_U] \to U$$

where the first morphism is smooth and proper, so pushing forward along this map does not change the supports. Then the second map is smooth and proper of relative dimension δ_Z and thus we can use the Goresky–Macpherson inequality for the second morphism to deduce the above δ-inequality. As the generic fibre of $\mathcal{P}$ is an irreducible abelian variety, it is not possible to find such local sections $B_U \to \mathcal{P}_U$, but Ngô instead imitates this proof on the level of homology to conclude the desired inequality.

By the Severi inequality for the relative compactified Jacobian associated to the family $\mathcal{C}^{D,\mathrm{ell}}/\mathcal{A}^{D,\mathrm{ell}}$ of integral spectral curves, one obtains a so-called *δ-regularity inequality*

$$\operatorname{codim}(Z) \geq \delta_Z \tag{13}$$

and thus we must have an equality for any support Z of $h^{D,\mathrm{ell}}$. As in the case of the Goresky–MacPherson inequality, this implies that any support Z is detected by $R^{2r_D}h_*^{D,\mathrm{ell}}\mathbb{Q}$ where r_D denotes the relative dimension of h^D. Since the compactified Jacobian of an integral spectral curve $\mathcal{C}_a$ is also integral (as the singularities are planar) and $R^{2r_D}h_*^{D,\mathrm{ell}}\mathbb{Q}$ sees precisely the irreducible components of the fibres of $h^{D,\mathrm{ell}}$, it follows that $R^{2r_D}h_*^{D,\mathrm{ell}}\mathbb{Q}$ is a rank 1 local system and so its only support is $Z = \mathcal{A}^{D,\mathrm{ell}}$, which completes the proof of the first statement.

For classical GL_n-Higgs bundles (i.e. $D = 0$), the Hitchin map does not have full supports: its supports on the locus of reduced spectral curves are described by work of de Cataldo, Heinloth, and Migliorini (2021, Theorem 6.11).

For the second part of the support theorem, Chaudouard and Laumon (2016) show the generic points of the supports Z of $h^D\colon \mathcal{M}^D \to \mathcal{A}^D$ are contained in the elliptic locus. If a general point $a \in Z$ has corresponding spectral curve $\mathcal{C}_a$ with l irreducible components which are non-reduced thickenings of order m_i of a degree n_i-cover $X_i \to C$ (so $\sum_{i=1}^{l} m_i n_i = n$), the idea is to apply the Severi inequality to each integral curve X_i to deduce that this forces $l = 1$ and $n_1 = n$ and $m_1 = 1$ (i.e. $a \in \mathcal{A}^{D,\mathrm{ell}}$). For this, we use the dimension formulas

$$\phi_n := \dim \mathcal{M}^D_{n,d} - \dim \mathcal{A}^D_n = r^D_n - \dim \mathcal{A}^D_n = \begin{cases} 0 & \text{if } D = 0, \\ 1 - n\deg(D) & \text{if } D > 0. \end{cases} \tag{14}$$

Ngô's δ-inequality (12) combined with the Severi inequalities (13) for each X_i give

$$\delta_Z \geq \operatorname{codim}(Z) \geq \left(\sum_{i=1}^{l} \phi_{n_i} - \phi_n\right) + \delta_Z$$

and so we obtain $1 - l \geq (n - \sum_{i=1}^{s} n_i)\deg(D)$ from the above dimension formulas (14) if $D > 0$. Since the right hand side is non-negative, we must have $l = 1$ and thus $n_1 = n$ and $m_1 = 1$.

For D-twisted SL_n-Higgs moduli spaces (with fixed determinant of coprime degree), the supports over the elliptic locus were described by Ngô (2006, 2010): there are finitely many non-full supports related to the (non-trivial) endoscopy of SL_n. Recall that each $\gamma \in \Gamma = \mathrm{Jac}(C)[n]$ determines a cyclic étale cover of C, then $a \in \mathcal{A}^{\mathrm{ell}}_{\mathrm{SL}_n}$ lies in the endoscopic locus $\mathcal{A}^{\mathrm{ell},\gamma}_{\mathrm{SL}_n}$ if the normalisation of the spectral curve factors via the étale cover corresponding to γ (see Hausel and Pauly, 2012). In fact, for $D > 0$, there are no new supports when passing to the full Hitchin base by a result of de Cataldo (2017a).

3.3.2. *Maulik and Shen's parabolic support theorem.* — Recall the parabolic Hitchin map

$$\mathfrak{h}^{\mathrm{par}}_G \colon \mathfrak{M}^{D,\mathrm{par}}_G \to \mathcal{A}^D_G \times C$$

on the 'stable' stack of parabolic D-twisted G-Higgs bundles from Diagram (7). For $G = \mathrm{PGL}_n$, this stack is a smooth Deligne–Mumford stack (see MS, Proposition 3.1) and the parabolic Hitchin map is proper; thus we can apply the decomposition theorem.

THEOREM 3.8 (Parabolic Support Theorem, MS, Theorem 3.2)
For an effective divisor D on C with $\deg(D) > 2g$, the parabolic Hitchin map $\mathfrak{h}^{\mathrm{par}}_{\mathrm{PGL}_n} : \mathfrak{M}^{D,\mathrm{par}}_{\mathrm{PGL}_n,d} \to \mathcal{A}^D_{\mathrm{PGL}_n} \times C$ has full supports.

For a semisimple group G, Yun (2012) followed Ngô's technique (Ngô, 2010) to describe the supports of the parabolic Hitchin map over the elliptic locus $\mathcal{A}^{D,\mathrm{ell}}_G \times C$: any non-full support is a component of the endoscopic loci. In particular, for $G = \mathrm{PGL}_n$ whose endoscopic theory is trivial, the restriction of the parabolic Hitchin map to the elliptic locus has full supports. Consequently, similar to the proof of Chaudouard and Laumon (2016), it suffices to show there is no support whose generic point lies outside of the elliptic locus $\mathcal{A}^{D,\mathrm{ell}}_G \times C$ by proving an analogous dimension formula.

Since parabolic bundles are easiest to understand for the groups GL_n and SL_n, where the parabolic structure is given by a flag in a fibre of a vector bundle and stability is given by verifying an inequality of slopes, Maulik and Shen prove the above theorem by working with SL_n-parabolic moduli spaces instead and then taking the Γ-invariant piece to get back to PGL_n. In this case, one can actually work with the moduli space rather than the moduli stack. Fix a degree d line bundle L, then there is a diagram

$$\begin{tikzcd}
\mathcal{M}^{D,\mathrm{par}}_{\mathrm{SL}_n,L} \arrow[r, "/\Gamma"] \arrow[d, "\pi_{\mathrm{SL}_n}"] \arrow[dd, bend right=60, "h^{\mathrm{par}}_{\mathrm{SL}_n}"'] & \mathfrak{M}^{D,\mathrm{par}}_{\mathrm{PGL}_n,d} \arrow[d, "\pi_{\mathrm{PGL}_n}"] \arrow[dd, bend left=60, "\mathfrak{h}^{\mathrm{par}}_{\mathrm{PGL}_n}"] \\
\mathcal{M}^{D}_{\mathrm{SL}_n,L} \times C \arrow[r, "/\Gamma"] \arrow[d] & \mathfrak{M}^{D}_{\mathrm{PGL}_n,d} \times C \arrow[d] \\
\mathcal{A}^D_{\mathrm{SL}_n} \times C \arrow[r, "\sim"] & \mathcal{A}^D_{\mathrm{PGL}_n} \times C
\end{tikzcd}$$

By work of de Cataldo (2017a), the SL_n-Hitchin map $h_{\mathrm{SL}_n} : \mathcal{M}^D_{\mathrm{SL}_n,L} \to \mathcal{A}^D_{\mathrm{SL}_n}$ admits a weak abelian fibration structure for the relative Prym variety $\mathrm{Prym}^0(\mathcal{C}^D_{\mathrm{SL}_n}/\mathcal{A}^D_{\mathrm{SL}_n})$ acting by tensor product. Maulik and Shen show $h^{\mathrm{par}}_{\mathrm{SL}_n} : \mathcal{M}^{D,\mathrm{par}}_{\mathrm{SL}_n,L} \to \mathcal{A}^D_{\mathrm{SL}_n} \times C$ admits a weak abelian fibration for the pullback P of this relative Prym variety to $\mathcal{A}^D_{\mathrm{SL}_n} \times C$. Then using δ-regularity for the relative Prym variety (de Cataldo, 2017a, Corollary 5.4.4) and an adaption of the argument above in the non-parabolic case, it suffices to show each closed fibre of the parabolic Hitchin map $h^{\mathrm{par}}_{\mathrm{SL}_n}$ has dimension

$$d = \dim \mathcal{M}^{D,\mathrm{par}}_{\mathrm{SL}_n,L} - \dim(\mathcal{A}^D_{\mathrm{SL}_n} \times C) = \frac{n(n-1)}{2}\deg(D) + (n^2-1)(g-1).$$

Since $h^{\mathrm{par}}_{\mathrm{SL}_n}$ is surjective and $\mathbb{G}_m$-equivariant for the scaling action, by semicontinuity it suffices to bound the dimension of the fibre over $(0,x)$ for each $x \in C$ by d. Fix $x \in C$, then as $\deg(D) > 2g$, we can write $D = x_0 + \cdots + x_r$ as a reduced effective divisor containing x. Maulik and Shen prove their dimension formula using a dimension bound for the nilpotent cone of *strongly* parabolic D-twisted SL_n-Higgs bundles, as the strongly parabolic D-twisted SL_n-Hitchin fibration is Lagrangian (Faltings, 1993), so the dimension of its nilpotent cone is the dimension of its Hitchin base. Since a similar method relating parabolic and strongly parabolic Higgs moduli spaces also appears in (HMMS), we will explain this in §4.3.

3.3.3. *Deductions on strong perversity.* — By combining the above support theorem with Yun's result about the Chern classes of the universal line bundles on the parabolic moduli space having strong perversity 1 over a dense open set (Proposition 3.5), we obtain the following result.

Corollary 3.9. — *For a divisor D of degree $\deg(D) > 2g$, the kth Chern character component $\mathrm{ch}_k(\mathcal{E}^D)$ of the universal family $\mathcal{E}^D \to \mathfrak{M}^D_{\mathrm{PGL}_n,d} \times C$ has strong perversity k with respect to $\mathfrak{h}_{\mathrm{PGL}_n} \times \mathrm{Id}_C$.*

Proof. — By construction, the pullback $\mathcal{F}^D = \pi^*_{\mathrm{PGL}_n}\mathcal{E}^D$ of $\mathcal{E}^D$ to $\mathfrak{M}^{D,\mathrm{par}}_{\mathrm{PGL}_n,d}$ admits a reduction to a Borel $B < \mathrm{PGL}_n$; thus the Chern roots of $\mathcal{F}^D$ are the first Chern classes of the associated tautological line bundles $\mathcal{L}_\chi$ for some $\chi \in X^*(T)$. By Proposition 3.5, we know that $c_1(\mathcal{L}_\chi)$ has strong perversity one with respect to $\mathfrak{h}^{\mathrm{par}}$ over a dense open in the Hitchin base. By Theorem 3.8, as $\deg(D) > 2g$, the parabolic Hitchin fibration $\mathfrak{h}^{\mathrm{par}}_{\mathrm{PGL}_n}$ has full supports and thus we can conclude $c_1(\mathcal{L}_\chi)$ has strong perversity one over the whole Hitchin base by Proposition 3.1. Since $c_1(\mathcal{L}_\chi)$ are the Chern roots of $\mathcal{F}^D$, we deduce that $\mathrm{ch}_k(\mathcal{F}^D)$ has strong perversity k with respect to $\mathfrak{h}^{\mathrm{par}}_{\mathrm{PGL}_n}$. Finally, we conclude $\mathrm{ch}_k(\mathcal{E}^D)$ has strong perversity k with respect to $\mathfrak{h}_{\mathrm{PGL}_n} \times \mathrm{Id}_C$ by Lemma 3.4. $\square$

3.4. Step (D): Vanishing cycles

Vanishing cycles were originally used to study the topology of hypersurface singularities and degenerations of families of projective manifolds. In modern enumerative geometry, vanishing cycles are frequently employed to study the topology of some 'tricky' moduli space which occurs as the critical locus of a regular function on some 'easier' moduli space, whose enumerative geometry can be more readily described. Often the tricky moduli space is singular and the easier moduli space is smooth; however, for us both will be smooth Higgs moduli spaces (or stacks), but the tricky space is for classical Higgs bundles ($D = 0$), where the Hitchin map has many supports, and the easier space is the one for D-twisted Higgs moduli space, where the Hitchin map has full supports.

3.4.1. *Vanishing cycles and strong perversity.* — For a morphism $\mu \colon X \to \mathbb{A}^1$ on a smooth irreducible variety of dimension d_X, there is a vanishing cycles functor

$$\varphi_\mu \colon D^b_c(X, \mathbb{Q}) \to D^b_c(X_0, \mathbb{Q})$$

where $X_0 = \mu^{-1}(0)$; for details, see for example de Cataldo and Migliorini (2009, §5.5). We assume that φ_μ is normalised by an appropriate shift so it respects the perverse t-structures and induces a functor between categories of perverse sheaves on X and X_0.

The image of the intersection complex $\varphi_\mu(\mathcal{IC}_X) = \varphi_\mu(\mathbb{Q}_X[d_X])$ is a perverse sheaf supported on the critical locus of μ; we let X' denote the support of this perverse sheaf.

Proposition 3.10 (MS, Proposition 1.4). — *Let $f\colon X \to Y$ be a proper morphism between smooth irreducible varieties and suppose we have a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \arrow[dr, "\mu"] & \\ Y' \arrow[r] & Y \arrow[r, "\nu"] & \mathbb{A}^1 \end{tikzcd}$$

where X' is the support of $\varphi_\mu(\mathcal{IC}_X) = \varphi_\mu(\mathbb{Q}_X[d_X])$. Assume that X' is smooth and $\varphi_\mu(\mathcal{IC}_X) \simeq \mathcal{IC}_{X'} = \mathbb{Q}_{X'}[d_{X'}]$. If $\gamma \in H^l(X,\mathbb{Q})$ has strong perversity c with respect to f, then its restriction $i^*\gamma \in H^l(X',\mathbb{Q})$ has strong perversity c with respect to f'.

The proof uses proper base change for φ, the fact that the vanishing cycles functor respects the perverse truncations and the assumption that $\varphi_\mu(\mathbb{Q}_X[d_X]) \simeq \mathbb{Q}_{X'}[d_{X'}]$.

3.4.2. *Vanishing cycles for Higgs moduli spaces.* — In Maulik and Shen's work on endoscopic decompositions of the cohomology of moduli spaces of SL_n-Higgs bundles (Maulik and Shen, 2021), they relate D-twisted and classical Higgs bundles using vanishing cycles for functions constructed from evaluating at a point in D and using the Killing form on $\mathfrak{sl}_n$.

We fix d coprime to n and $L \in \mathrm{Pic}^d(C)$ throughout this section. The D-twisted SL_n-Higgs moduli space $\mathcal{M}^D_{\mathrm{SL}_n,L}$ is the closed subvariety of $\mathcal{M}^D_{n,d}$ consisting of Higgs bundles with determinant isomorphic to L and trace-free Higgs field. There is a Hitchin map $h^D_{\mathrm{SL}_n}\colon \mathcal{M}^D_{\mathrm{SL}_n,L} \to \mathcal{A}^D_{\mathrm{SL}_n} := \oplus_{i=2}^n H^0(C, \omega_C(D)^{\otimes i})$.

Maulik and Shen show that if p is a point in the support of the effective divisor D, then moduli spaces (and stacks) of D-twisted and $(D-p)$-twisted SL_n-Higgs bundles fit into a commutative diagram

$$\begin{tikzcd} \mathfrak{M}^{D-p}_{\mathrm{SL}_n,L} \arrow[r, hook, "\iota_{\mathfrak{M}}"] \arrow[d, "\delta^{D-p}_L"'] & \mathfrak{M}^D_{\mathrm{SL}_n,L} \arrow[r, "\mathrm{ev}_p"] \arrow[d, "\delta^D_L"] \arrow[rr, bend left, "\mu_{\mathfrak{M}}"] & \mathfrak{M}_{\mathrm{SL}_n}(p) \simeq [\mathfrak{sl}_n/\mathrm{SL}_n] \arrow[d] \arrow[r, "\widehat{\mu}"] & \mathbb{A}^1 \\ \mathcal{M}^{D-p}_{\mathrm{SL}_n,L} \arrow[r, hook, "\iota_{\mathcal{M}}"] \arrow[d, "h^{D-p}_L"'] & \mathcal{M}^D_{\mathrm{SL}_n,L} \arrow[r] \arrow[d, "h^D_L"] \arrow[rr, bend right, "\mu_{\mathcal{M}}"'] & \mathcal{M}_{\mathrm{SL}_n}(p) \simeq \mathfrak{sl}_n /\!/ \mathrm{SL}_n \arrow[d, "\wr"'] \arrow[r, "\mu"] & \mathbb{A}^1 \\ \mathcal{A}^{D-p}_{\mathrm{SL}_n} \arrow[r, hook, "\iota_{\mathcal{A}}"] & \mathcal{A}^D_{\mathrm{SL}_n} \arrow[r] \arrow[rru, bend right, "\mu_{\mathcal{A}}"'] & \mathcal{A}_{\mathrm{SL}_n}(p) \simeq \mathfrak{t} /\!/ S_n & \end{tikzcd} \tag{15}$$

with the following properties.

(i) The function μ is induced by the Killing form

$$\mathfrak{sl}_n \to \mathbb{A}^1, \quad g \mapsto \mathrm{tr}(g^2).$$

(ii) The closed embedding $\iota_{\mathfrak{M}}$ is the critical locus of $\mu_{\mathfrak{M}}$ by Maulik and Shen (2021, Theorem 4.5(a)).

(iii) The *evaluation at p* morphism ev_p on the stack is smooth by a deformation theory argument (Maulik and Shen, 2021, Proposition 4.1). Hence, using the elementary computation of the vanishing cycles functor for the Killing quadratic form $\mathfrak{sl}_n \to \mathbb{A}^1$ together with smooth and proper base change, one obtains

$$\mu_{\mathcal{M}}(\mathcal{IC}_{\mathcal{M}^D_{\mathrm{SL}_n,L}}) \simeq \mathcal{IC}_{\mathcal{M}^{D-p}_{\mathrm{SL}_n,L}}.$$

(iv) The closed embedding $\iota_{\mathcal{M}}$ is equivariant for the natural action of $\Gamma := \mathrm{Jac}(C)[n]$, whose quotients give the PGL_n-moduli stacks. Moreover, the PGL_n- and SL_n-Hitchin maps have isomorphic bases; thus there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{M}^{D-p}_{\mathrm{PGL}_n,d} \arrow[r, "\iota_{\mathfrak{M}_{\mathrm{PGL}_n}}"] \arrow[d, "\mathfrak{h}^{D-p}_{\mathrm{PGL}_n}"'] & \mathfrak{M}^{D}_{\mathrm{PGL}_n,d} \arrow[d, "\mathfrak{h}^{D}_{\mathrm{PGL}_n}"'] \arrow[dr, "\mu_{\mathfrak{M}_{\mathrm{PGL}_n}}"] & \\
\mathcal{A}^{D-p}_{\mathrm{PGL}_n} \arrow[r] & \mathcal{A}^{D-p}_{\mathrm{PGL}_n} \arrow[r, "\mu_{\mathcal{A}}"] & \mathbb{A}^1
\end{tikzcd}$$

and consequently (see MS, Proposition 2.5)

$$\mu_{\mathfrak{M}_{\mathrm{PGL}_n}}(\mathcal{IC}_{\mathfrak{M}^D_{\mathrm{PGL}_n,L}}) \simeq \mathcal{IC}_{\mathfrak{M}^{D-p}_{\mathrm{PGL}_n,L}}.$$

By Proposition 3.10, they can relate strong perversity of the Chern character components of the universal families $\mathcal{E}^D_{\mathrm{PGL}_n} \to \mathfrak{M}^D_{\mathrm{PGL}_n,d} \times C$ for D and $D-p$ as follows.

Proposition 3.11 (MS, §2.3). — *If p is in the support of D and $\mathrm{ch}_k(\mathcal{E}^D_{\mathrm{PGL}_n})$ has strong perversity k with respect to $\mathfrak{h}^D_{\mathrm{PGL}_n} \times \mathrm{Id}_C$, then $\mathrm{ch}_k(\mathcal{E}^{D-p}_{\mathrm{PGL}_n})$ has strong perversity k with respect to $\mathfrak{h}^{D-p}_{\mathrm{PGL}_n} \times \mathrm{Id}_C$.*

By inductively combining this proposition with Corollary 3.9 and then applying Lemma 3.2 and Proposition 2.3, one immediately obtains the following result.

Corollary 3.12. — *For all $D \geq 0$ (and in particular $D = 0$), the class $\mathrm{ch}_k(\mathcal{E}^D_{\mathrm{PGL}_n})$ has strong perversity k with respect to $h^D_{\mathrm{PGL}_n} \times \mathrm{Id}_C$. In particular, the $P = W$ conjecture holds for PGL_n and coprime degree d (and thus also GL_n and coprime degree).*

4. THE PROOF OF HAUSEL, MELLIT, MINETS & SCHIFFMANN

4.1. Overview of the proof

Although the proof in (HMMS) grew from the study of cohomological Hall algebras (CoHAs) of surfaces in Mellit, Minets, Schiffmann, and Vasserot (2023), one starting point for understanding this proof is to consider $\mathfrak{sl}_2$-triples coming from ample classes via the cohomological Relative Hard Lefschetz Theorem. Recall from Remark 1.12 that a relatively ample tautological class $\omega \in H^2(\mathcal{M}_{n,d})$ determines an $\mathfrak{sl}_2$-triple $\langle \overline{\mathbf{e}}, \overline{\mathbf{f}}, \overline{\mathbf{h}} \rangle$ acting on $\mathrm{Gr}^P_h H^*(\mathcal{M}_{n,d})$, where $\overline{\mathbf{e}}$ is induced by the cup product with ω and the perverse graded pieces are the $\overline{\mathbf{h}}$-graded pieces. The idea is to lift this to an $\mathfrak{sl}_2$-triple $\langle \mathbf{e}, \mathbf{f}, \mathbf{h} \rangle$ acting on $H^*(\mathcal{M}_{n,d})$ such that $\mathbf{h}$ behaves with respect to appropriate tautological generators as predicted by $P = W$; that is, we would like to find an $\mathfrak{sl}_2$-triple $\langle \mathbf{e}, \mathbf{f}, \mathbf{h}\rangle$ and tautological generators $\psi_k(\gamma)$ such that

(i) $W_{2k}H^*(\mathcal{M}^{\mathrm{B}}_{n,d})$ is the span of products $\prod_{i=1}^s \psi_{k_i}(\gamma_i)$ such that $\sum_{i=1}^s k_i \leq k$, and
(ii) we have $[\mathbf{h}, \psi_k(\gamma)] = k\psi_k(\gamma)$ as operators on the cohomology $H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d})$.

Here we view the tautological classes as tautological operators via their cup products. Statement (i) gives the equality $W = C$, where C is the Chern filtration for these tautological generators (see §2.4.1), which follows from work of Shende (2017) and Markman (2002) (see Theorems 2.1 and 2.2). From Statement (ii), one inductively deduces that $P_kH^*(\mathcal{M}^{\mathrm{Dol}}_{n,d})$ is spanned by products $\prod_{i=1}^s \psi_{k_i}(\gamma_i)$ with $\sum_{i=1}^s k_i \leq k$, which implies $P = C$. Together these statements imply the equality $P = W$. In particular, unlike the proof in (MS) (see Proposition 2.3), this does not use Curious Hard Lefschetz but rather provides a new proof.

The key question is: how can we construct such a lift of the $\mathfrak{sl}_2$-triple coming from Relative Hard Lefschetz? For this, we need to choose the ample class defining $\mathbf{e}$ and find a suitable $\mathbf{f}$. A key idea of (HMMS) is to construct $\mathbf{f}$ using Hecke correspondences, which are natural cohomological correspondences on moduli of Higgs bundles. More generally, for certain moduli stacks of objects in an abelian category, Hecke correspondences are used in the construction of CoHAs. The goal is then to describe the interaction of the tautological classes with the perverse filtration, by instead understanding their interaction with Hecke correspondences.

In fact, the tautological and Hecke operations can be used to produce an action of a much larger Lie algebra. Hausel, Mellit, Minets and Schiffmann show the Lie algebra $\mathcal{H}_2$ of polynomial Hamiltonian vector fields on the plane acts on $H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d})[x,y] \simeq H^*(\mathfrak{M}^{\mathrm{Dol}}_{n,d})[x]$. The main work involves computing the relations between the generators; this goes much beyond the simple commutator relations needed for the $\mathfrak{sl}_2$-triple. In (HMMS), the authors explain that the expectation for such an action of $\mathcal{H}_2$ arose from similarities with the homology of links. For simplicity, we will focus our discussion on the $\mathfrak{sl}_2$-triples inside $\mathcal{H}_2$ that are needed for the proof of the $P = W$ conjecture, as these act on $H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d})$ without needing to add the extra variables x and y.

Let us point out a few subtleties concerning Hecke correspondences:

(i) Hecke modifications do not preserve (semi)stability in general.
(ii) Hecke modifications shift the invariants (e.g. Chern character) and so do not naturally act on a single moduli space, but rather on all moduli spaces together.
(iii) Hecke correspondences are naturally defined at the stack level.

The first point is overcome in (HMMS) by instead working with elliptic loci, which are preserved by Hecke correspondences. In order to relate the (tautological) cohomology of the whole moduli space with an elliptic locus, moduli spaces of parabolic Higgs bundles are used. The second issue is handled by noting that Higgs moduli spaces are periodic: tensoring by a degree one line bundle produces an isomorphism $\mathcal{M}_{n,d} \to \mathcal{M}_{n,d+n}$. Hence a specific Hecke operator is used to identify the cohomology of moduli spaces for different Chern characters and thus prove a cohomological χ-independence statement for the tautological cohomology of elliptic (parabolic) Higgs moduli spaces. The final point is dealt with by first defining Hecke operators at the stack level and then defining reduced operators which act on the cohomology of the moduli spaces.

Since the tautological and Hecke operators do not quite have the correct symmetry properties required to obtain an $\mathfrak{sl}_2$-triple, the periodicity of the cohomology is combined with a degeneration procedure to obtain suitable operators that satisfy the correct relations for an $\mathfrak{sl}_2$-triple.

We should point out that many results in (HMMS) and Mellit, Minets, Schiffmann, and Vasserot (2023) concern a much more general set-up and study operators on the cohomology of moduli spaces of sheaves on surfaces. Higgs bundles are a special case, as by the spectral correspondence we can realise Higgs moduli spaces as moduli space of pure 1-dimensional sheaves on the (open) surface T^*C as in Remark 1.3.

4.1.1. *Key steps in the proof.* — We will divide the proof as follows.

- **(A)** Define Hecke and tautological operators on the cohomology of elliptic loci. Then relate the cohomology of the whole Higgs moduli space with the cohomology of elliptic parabolic moduli spaces, so it suffices to prove $P = C$ on the latter.
- **(B)** Compute the commutator relations between the Hecke operators and tautological operators on the cohomology of moduli spaces of sheaves on surfaces.
- **(C)** Prove cohomological χ-independence for the pure cohomology of elliptic parabolic Higgs moduli stacks, to view the Hecke operators as acting on the cohomology of a fixed stack. By degeneration, produce operators with better symmetry properties.
- **(D)** Pass from the cohomology of the stack to its good moduli space by defining reduced operators that act on the elliptic (parabolic) Higgs moduli spaces.
- **(E)** Construct an $\mathfrak{sl}_2$-triple from certain (reduced) tautological and Hecke operators, then appropriately modify this triple to get a suitable $\mathfrak{sl}_2$-triple to prove $P = C$.

4.1.2. *Notation.* — Throughout this section, we let S denote a smooth complex surface, which may be compact or non-compact. We let t_1 and t_2 denote the Chern roots of TS and write $(c_1, c_2) = (t_1 + t_2, t_1 t_2)$ and $(s_1, s_2) = (c_1, c_1^2 - c_2)$.

The stack of coherent sheaves on S decomposes as

$$\mathfrak{Coh}_S = \bigsqcup_{\alpha} \mathfrak{Coh}_{S,\alpha}$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in H^0(S) \oplus H^2(S) \oplus H^4(S)$ denotes the Chern character. For each α, fix a smooth open substack $\mathfrak{M}_{S,\alpha} \subset \mathfrak{Coh}^{ss}_{S,\alpha}$ of the stack of semistable sheaves (for some notion of semistability) with universal sheaf $\mathcal{F}_\alpha \to \mathfrak{M}_{S,\alpha} \times S$. For $\delta = (0, 0, [s])$, we have that $\mathfrak{Coh}_{S,\delta} \cong S \times B\mathbb{G}_m$ is the stack of length 1 sheaves.

In (HMMS), the authors use cohomology with $\mathbb{C}$-coefficients, although it seems sufficient to work with $\mathbb{Q}$-coefficients. We suppress the coefficients and let $H := H^*_{\mathrm{pure}}(S)$.

4.2. Natural operators on cohomology

We introduce two natural operators on the cohomology of the moduli stacks $\mathcal{M}_{S,\alpha}$:

(i) Tautological operators given by cup products with tautological cohomology classes constructed from the universal family.
(ii) Hecke operators constructed as a cohomological Hecke correspondence given by a length 1 modification of a sheaf at a single point.

The first operators act on the cohomology of a moduli space of sheaves with fixed Chern character, whereas the second operators change the Chern character.

We first define the operators when S is compact (see §4.2.3 for the open case).

4.2.1. *Hecke operators.* — We will only need 0-dimensional Hecke modifications, which are modifications of a sheaf by a length 1 torsion sheaf supported at a single point; these sheaves are parametrised by the stack $\mathfrak{M}_{S,\delta} = \mathfrak{Coh}_{S,\delta} \simeq S \times B\mathbb{G}_m$. There is a stack of extensions which we view as a correspondence

$$\begin{tikzcd} & \mathfrak{Ext}_\alpha \arrow[dl, "\pi_1\times\pi_2"'] \arrow[dr, "\pi_3"] & \\ \mathfrak{Coh}_{S,\delta} \times \mathfrak{Coh}_{S,\alpha-\delta} & & \mathfrak{Coh}_{S,\alpha} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{E} \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{T} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (\mathcal{T}, \mathcal{E}) & & \mathcal{F}. \end{tikzcd}$$

We will assume:

(i) Hecke correspondences respect the substacks $\mathfrak{M}_{S,\alpha} \subset \mathfrak{Coh}_{S,\alpha}$: if $\mathcal{F} \in \mathfrak{M}_{S,\alpha}$ appears as an extension $\mathcal{E} \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{T}$ for a length 1 sheaf $\mathcal{T}$, then $\mathcal{E} \in \mathfrak{M}_{S,\alpha-\delta}$.
(ii) The open substacks $\mathfrak{M}_{S,\alpha}$ are sufficiently nice that their universal sheaf $\mathcal{F}_\alpha$ can be resolved by a two term complex of vector bundles $\mathcal{E}_1 \to \mathcal{E}_0$.

The first assumption is essential for us to have Hecke correspondence diagrams between the substacks $\mathfrak{M}_{S,\alpha}$ of the form

$$\begin{tikzcd} & \mathcal{Z}_\alpha \arrow[dl, "\pi_1\times\pi_2"'] \arrow[dr, "\pi_3"] & \\ \mathfrak{M}_{S,\delta} \times \mathfrak{M}_{S,\alpha-\delta} & & \mathfrak{M}_{S,\alpha}. \end{tikzcd} \tag{16}$$

The second assumption is imposed in order to describe length one Hecke correspondences as the projectivisation of a complex: more precisely: $\mathcal{Z}_\alpha$ can be realised as the zero set of a canonical section on $\pi^*\mathcal{E}_1^\vee(1) \to \mathbb{P}(\mathcal{E}_0)$ for $\pi\colon \mathbb{P}(\mathcal{E}_0) \to S \times \mathfrak{M}_{S,\alpha}$ and thus one can define a virtual fundamental class $[\mathcal{Z}_\alpha]^{\mathrm{vir}}$ (see Mellit, Minets, Schiffmann, and Vasserot, 2023, §2.4 and Neguţ, 2019, §2.4). We then define a Hecke operator

$$\begin{array}{rccl} T\colon & H^*(\mathfrak{M}_{S,\delta}) \otimes H^*(\mathfrak{M}_{S,\alpha-\delta}) & \to & H^*(\mathfrak{M}_{S,\alpha}) \\ & \xi u^n \otimes \eta & \mapsto & T(\xi u^n)(\eta) := \pi_{3*}(\pi_1^*(\xi u^n) \cup \pi_2^*(\eta) \cup [\mathcal{Z}_\alpha]^{\mathrm{vir}}), \end{array}$$

where we identify $H^*(\mathfrak{M}_{S,\delta}) \cong H^*(S) \otimes H^*(B\mathbb{G}_m) \cong H^*(S)[u]$ and consider elements ξu^n for $\xi \in H^*(S)$. Note that the morphism π_3 is proper, as it is a base change of a proper map. We will write $T_n(\xi) := T(\xi u^n)\colon H^*(\mathfrak{M}_{S,\alpha-\delta}) \to H^*(\mathfrak{M}_{S,\alpha})$.

4.2.2. *Tautological operators.* — The construction of tautological classes in §2.2.2 can be generalised to associate to any symmetric function f and any class $\gamma \in H^*(S)$, a tautological class $f^\alpha(\gamma) \in H^*(\mathfrak{M}_{S,\alpha})$ using the universal sheaf $\mathcal{F}_\alpha$ as follows.

Let Λ be the ring of symmetric functions in infinitely many variables $z_1, z_2, \ldots$, which we can write as a polynomial ring in different generators:

$$\Lambda = \mathbb{Q}[p_1, p_2, \ldots] = \mathbb{Q}[e_1, e_2, \ldots] = \mathbb{Q}[h_1, h_2, \ldots]$$

where $p_k = z_1^k + z_2^k + \cdots$ are the power sums, $e_k = z_1 z_2 \ldots z_k + \cdots$ are the elementary symmetric functions, and $h_k = z_1^k + z_1^{k-1} z_2 + \cdots$ are the complete homogeneous symmetric functions. We define $p_k(\mathcal{F}_\alpha) \in H^{2k}(\mathfrak{M}_{S,\alpha} \times S)$ by the following generating series:

$$\operatorname{ch}(\mathcal{F}_\alpha) = \operatorname{rk} \mathcal{F}_\alpha + \sum_{k=1}^{\infty} \frac{p_k(\mathcal{F}_a)}{k!}$$

and set $p_0(\mathcal{F}_\alpha) := \operatorname{rk} \mathcal{F}_\alpha$. The assignment $p_k \mapsto p_k(\mathcal{F}_\alpha)$ determines a ring homomorphism

$$\Lambda = \mathrm{k}[p_1, p_2, \ldots] \to H^*(\mathfrak{M}_{S,\alpha} \times S), \quad f \mapsto f(\mathcal{F}_\alpha).$$

For any $\gamma \in H^k(S)$ and $f \in \Lambda$, we define an associated tautological class

$$f^\alpha(\gamma) := \pi_{\mathfrak{M}*}(f(\mathcal{F}_\alpha) \cup \pi_S^*(\gamma)) \in H^{i+2k-4}(\mathfrak{M}_{S,\alpha}).$$

Together these classes generate a *tautological subring* $H^*_{\mathrm{taut}}(\mathfrak{M}_{S,\alpha}) \subset H^*(\mathfrak{M}_{S,\alpha})$. We use the same notation for the corresponding *tautological operators*

$$f^\alpha(\gamma) \colon H^j(\mathfrak{M}_{S,\alpha}) \xrightarrow{\cup f^\alpha(\gamma)} H^{j+i+2k-4}(\mathfrak{M}_{S,\alpha}).$$

Let us introduce the *tautological ψ-classes* which are used as tautological generators in (HMMS) due to their simplified commutator relations with $T_n(\xi)$, as described in Equations (20) and (22) below. The ψ-classes are defined by inserting the Todd class, which was also used by Li, Qin, and Wang (2002). Recall that t_1 and t_2 denote the Chern roots of TS and thus $\mathrm{Td}_S = t_1 t_2/(1-e^{-t_1})(1-e^{-t_2})$. For each $\gamma \in H^*(S)$ and $n \geq 0$, we define $\psi_n^\alpha(\gamma)$ by the generating series

$$\sum_{n\geq 0} \frac{x^n \psi_n}{n!} = \left(\sum_{n\geq 0} \frac{x^n p_n}{n!}\right) \frac{t_1 t_2 x}{(1-e^{-t_1 x})(1-e^{-t_2 x})}.$$

More precisely, if $\mathrm{Td}_{S,k} \in H^{2k}(S)$ denote the Todd class components, then

$$\psi_n^\alpha(\gamma) := \sum_{1\leq m\leq n+1} \frac{n!}{m!} p_m^\alpha(\mathrm{Td}_{S,n+1-m}\,\gamma). \tag{17}$$

4.2.3. *The case of an open surface.* — If $S_0 \subset S$ is an open surface, then as explained in (HMMS, §3.5), the above applies to stacks $\mathcal{M}_{S,\alpha}$ which consist of sheaves supported in S_0. One difference is that one should replace $H^*(S)$ with $H^*_{\mathrm{pure}}(S_0)$, since the tautological class $f_\alpha(\gamma)$ only depends on the image of γ in $H^*_{\mathrm{pure}}(S_0)$.

For D-twisted Higgs bundles, we consider the open subset $S_0 := \operatorname{Tot}(\omega_C(D))$ of $S := \mathbb{P}(\mathcal{O}_C \oplus \omega_C(D)^*)$. Then $H^*_{\mathrm{pure}}(S_0) = H^*(C)$. Let $\pi\colon \operatorname{Tot}(\omega_C(D)) \to C$ denote the projection; then the Todd class of S_0 is given by

$$\mathrm{Td}_{S_0} = \pi^*(\mathrm{Td}_C) \cdot \pi^*(\mathrm{Td}_C(\omega_C(D))) = \pi^*(1 + c_1(\mathcal{T}_C))\pi^*(1 + c_1(\omega_C(D)))$$

whose Chern roots are $t_1 = \pi^* c_1(\mathcal{T}_C)$ and $t_2 = \pi^* c_1(\omega_C(D))$. In particular, $t_1 + t_2 = \deg(D)\pi^*[x]$, where $[x]$ is the class of a point on C. In the classical Higgs case ($D = 0$), we have $t_1 + t_2 = 0$ and $\mathrm{Td}_{S_0} = 1$.

Henceforth S denotes a possibly open smooth surface and $H := H^*_{\mathrm{pure}}(S) \subset H^*(S)$.

4.3. Step (A): Operators on elliptic loci and parabolic Higgs moduli spaces

The spectral correspondence describes D-twisted Higgs bundles on C as pure 1-dimensional sheaves on the open surface $S = \operatorname{Tot}(\omega_C(D))$. Unfortunately, Hecke modifications do not preserve semistability, but they do preserve the spectral curve (which is the support of the pure 1-dimensional sheaf on S). Indeed, for a Hecke modification $\mathcal{E} \hookrightarrow \mathcal{F} \to \mathcal{T}$ where $\mathcal{E}$ and $\mathcal{F}$ are pure 1-dimensional sheaves on S and $\mathcal{T}$ is a length 1 sheaf, this sequence cannot split as $\mathcal{F}$ is pure. It follows that $\mathcal{E}$ and $\mathcal{F}$ have the same support and thus the same spectral curve.

Over the elliptic locus, where the spectral curve is integral, all Higgs bundles are stable, thus we have Hecke correspondences

$$\begin{tikzcd} & \mathcal{Z}_{n,d} \arrow[dl] \arrow[dr] & \\ (S \times B\mathbb{G}_m) \times \mathfrak{M}^{D,\mathrm{ell}}_{n,d-1} & & \mathfrak{M}^{D,\mathrm{ell}}_{n,d}. \end{tikzcd}$$

The universal bundle can be locally resolved by a 2-term complex of vector bundles (for example, see Faltings, 1993, §1). Hence as in §4.2, we can define Hecke operators $T_n(\xi)$ and tautological operators $\psi_k(\gamma)$ on $H^*_{\mathrm{taut}}(\mathfrak{M}^{D,\mathrm{ell}}_{n,d})$.

Moreover, Markman's argument (Markman, 2002) for tautological generation should generalise to show that the Künneth components of the diagonal can be expressed in terms of the tautological classes and thus for n and d coprime, we have

$$H^*_{\mathrm{taut}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d}) = H^*_{\mathrm{pure}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d})$$

as in the end of §1.3.1.

In the next subsections, we will see that general results about Hecke actions on moduli spaces of sheaves on surfaces (following Mellit, Minets, Schiffmann, and Vasserot, 2023) will produce an action of the Lie algebra $\mathcal{H}_2$ on $H^*_{\mathrm{taut}}(\mathfrak{M}^{D,\mathrm{ell}}_{n,d})[x] \cong H^*_{\mathrm{taut}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d})[x,y]$ and eventually find an $\mathfrak{sl}_2$-triple to prove $P = W$.

The rest of this subsection is devoted to describing how to go from the cohomology of the elliptic locus to the cohomology of the moduli space of all semistable Higgs bundles. In (HMMS), this is achieved using certain parabolic Higgs moduli spaces (and stacks).

Remark 4.1. — Using deep results about CoHAs, the authors of (HMMS) hope to eventually be able to remove the need to pass via parabolic Higgs bundles, which should streamline the argument. However, one advantage to the current argument is that it provides an approach to proving the $P = W$ conjecture for parabolic Higgs bundles. Furthermore, it involves a nice geometric argument relating elliptic loci with (parabolic) moduli spaces, which might be of future interest (e.g. Maulik, Shen, and Yin, 2023 use this in the third $P = W$ proof).

The parabolic Higgs bundle moduli spaces appearing in (HMMS) are slightly different from the universal one appearing in (MS) and §3.2, which came from Yun's global Springer theory. Recall that Yun considered parabolic Higgs bundles consisting of a Higgs bundle together with a point on C and a flag over that point which was preserved by the Higgs field. In particular, the point where we have the flag structure can move freely in C. It is more standard for parabolic bundles to have flags at a fixed number of parabolic points x_i which are encoded by a reduced effective divisor $D = x_1 + \cdots + x_r$. These are the parabolic Higgs bundles considered in (HMMS) and we will refer to them as *D-parabolic Higgs bundles* (to distinguish from the parabolic D-twisted Higgs bundles used in §3.2). To prove the (non-parabolic) $P = W$ conjecture, it suffices to take $D = x$.

The stack $\mathfrak{Higgs}_{n,d}^{D\text{-par}}$ of all D-parabolic Higgs bundles parametrises tuples $(E, \theta \colon E \to E \otimes \omega_C(D), E^{\bullet}_{x_i})$, where $(E, \theta) \in \mathfrak{M}^D_{n,d}$ is a D-twisted Higgs bundle and $E^{\bullet}_{x_i}$ is an (increasing) full flag in the fibre of each $x_i \in D$ that is preserved by the Higgs field (that is, $\mathrm{res}_{x_i}(\theta)(E^j_{x_i}) \subset E^j_{x_i}$ for all j). If $\mathrm{res}_{x_i}(\theta)(E^j_{x_i}) \subset E^{j-1}_{x_i}$ for all j and all x_i, the Higgs field is called *strongly parabolic* and we let $\mathfrak{Higgs}_{n,d}^{D\text{-spar}}$ denote the substack of strongly D-parabolic Higgs bundles.

There are different notions of stability for D-parabolic Higgs bundles (arising from different choices in their GIT construction, see Mehta and Seshadri, 1980), but we will consider an open elliptic locus that is contained in all stable loci. Let $\mathfrak{M}_{n,d}^{D\text{-par,ell}}$ be the substack of D-parabolic Higgs bundles whose spectral curve is integral and such that $\mathrm{res}_{x_i}(\theta)$ has distinct eigenvalues for each $x_i \in D$. In particular, the parabolic structure is just given by an ordering of the eigenvalues, and so there is a forgetful map

$$\mathfrak{M}_{n,d}^{D\text{-par,ell}} \to \mathfrak{M}_{n,d}^{D,\mathrm{ell}}$$

which is a $S_n^{|D|}$-torsor over its image. For any notion of stability arising from a choice of parabolic stability parameters (as in Mehta and Seshadri, 1980), all D-parabolic Higgs bundles in $\mathfrak{M}_{n,d}^{D\text{-par,ell}}$ are stable. Hence, we fix one generic stability parameter (i.e. semistability and stability coincide) and let $\mathfrak{M}_{n,d}^{D\text{-par}}$ denote the corresponding stack of (semi)stable D-parabolic Higgs bundles. Then $\mathfrak{M}_{n,d}^{D\text{-par,ell}} \subset \mathfrak{M}_{n,d}^{D\text{-par}}$ and we also have an inclusion $\mathcal{M}_{n,d}^{D\text{-par,ell}} \subset \mathcal{M}_{n,d}^{D\text{-par}}$ of the corresponding moduli spaces.

We will relate the cohomology of the Higgs moduli space with elliptic loci (for D-twisted Higgs bundles) using D-parabolic Higgs bundles and different fibres of the map

$$\chi_D \colon \mathcal{M}_{n,d}^{D\text{-par}} \to \mathcal{A}_{D,n} := \mathbb{A}^{n|D|-1}$$

which records the ordered eigenvalues $\mu_{i,j}$ of $\mathrm{res}_{x_i}(\theta)$ for all $x_i \in D$. Note that the sum of all these eigenvalues is zero by the residue theorem: $\sum_i \mathrm{res}_{x_i} \mathrm{tr}(\theta) = 0$. Furthermore, $\mathcal{A}_{D,n}$ is a direct summand in the Hitchin base for $\mathcal{M}_{n,d}^{D\text{-par}}$ and χ_D is the composition of the Hitchin fibration with the projection onto $\mathcal{A}_{D,n}$.

Lemma 4.2 (HMMS, Proposition 8.16). — *For $\mu \in \mathcal{A}_{D,n}$ generic, $\chi_D^{-1}(\mu) \subset \mathcal{M}_{n,d}^{D\text{-par,ell}}$.*

Proof. — Let $\mu = (\mu_{i,j}) \in \mathcal{A}_{D,n}$ be chosen so that the $\mu_{i,j}$ are all distinct and their sum is zero, but no proper subset sums to zero. Then any parabolic Higgs bundle in the fibre $\chi_D^{-1}(\mu)$ is simple: if it had a subbundle, then the sum of the residues of the trace of the Higgs field on this subbundle would be zero, which would force some subset of the $\mu_{i,j}$'s to sum to zero. Hence, any parabolic Higgs bundle in $\chi_D^{-1}(\mu)$ has integral spectral curve and the eigenvalues $\mu_{i,j}$ of $\mathrm{res}_{x_i}(\theta)$ are distinct, which proves the claim. □

In particular, we have the following commutative diagram

$$\begin{tikzcd}
\chi_D^{-1}(\mu) \arrow[r, hook, "\iota_\mu"] \arrow[d, hook] & \mathcal{M}_{n,d}^{D\text{-par}} \arrow[d, "\chi_D"] & \chi_D^{-1}(0) = \mathcal{M}_{n,d}^{D\text{-spar}} \arrow[l, hook', "\iota_0"'] \\
\mathcal{M}_{n,d}^{D\text{-par,ell}} \arrow[ur, hook, "j"] \arrow[d] & \mathbb{A}^{n|D|-1} = \mathcal{A}_{D,n} & \widetilde{\mathcal{M}}_{n,d} \arrow[u, hook, "i"] \arrow[d, "\pi"', "(\mathrm{GL}_n/B)^{|D|}-\text{bundle}"] \\
\mathcal{M}_{n,d}^{D,\text{ell}} & & \mathcal{M}_{n,d},
\end{tikzcd}$$

where $\widetilde{\mathcal{M}}_{n,d}$ is the subspace of $(E, \theta, E_{x_i}^\bullet) \in \chi_D^{-1}(0)$ such that $\mathrm{res}_{x_i}(\theta) = 0$ for all $x_i \in D$; in particular, such $\theta\colon E \to E \otimes \omega_C(D)$ factor as $\theta\colon E \to E \otimes \omega_C$ and forgetting the flag gives the morphism $\pi\colon \widetilde{\mathcal{M}}_{n,d} \to \mathcal{M}_{n,d}$ which is an iterated flag bundle.

The following proposition collects various results concerning the cohomology of the moduli spaces appearing in this diagram.

Proposition 4.3. — *For n and d coprime, the following statements hold.*

i) *The cohomology of $\mathcal{M}_{n,d}^{D\text{-par}}$ is pure,*
ii) *The restriction maps ι_0^* and ι_μ^* on cohomology are isomorphisms,*
iii) *The restriction map ι_0^* respects the perverse filtrations,*
iv) *The restriction map j^* is injective and thus $H^*_{\mathrm{pure}}(\mathcal{M}_{n,d}^{D\text{-par,ell}}) = H^*(\mathcal{M}_{n,d}^{D\text{-par}})$. Moreover j^* respects the perverse filtrations.*

Proof. — For i), one shows there is a semi-projective $\mathbb{G}_m$-action scaling the Higgs field analogously to Proposition 1.7. For ii), one uses that χ_D is smooth and equivariant for the scaling $\mathbb{G}_m$-action (see HMMS, Proposition 8.14 and the proof of HMMS, Proposition 8.18) and thus restricting to a fibre is a cohomological equivalence as in Hausel, Letellier, and Rodriguez-Villegas (2011, Theorem 7.2.1). For iii), the projectivisation of $\mathcal{M}_{n,d}^{D\text{-par}}$ with respect to the scaling $\mathbb{G}_m$-action is used and the D-parabolic Hitchin map is extended to a map to a weighted projective space as in Hausel (1998). Since

$\mathcal{M}_{n,d}^{D\text{-par}}$ has pure cohomology groups, its cohomology can be described as a quotient of the cohomology of its compactification by the image of a class L pulled back from a hyperplane class on the weighted projective space. By (HMMS, Theorem 8.5), on the cohomology of the compactified moduli space, the perverse filtration is given (up to a shift depending on the cohomological degree) by the canonical weight filtration induced by the nilpotent operator L, which is computed using kernels and images of powers of L. The interaction of the kernels of powers of L with the extension of ι_0 to the compactification is then described in order to prove ι_0^* (or strictly speaking its inverse) respects the perverse filtrations (see HMMS, Proposition 8.18 for details).

For iv), we know the isomorphism ι_μ^* factors as

$$\iota_\mu^*\colon H^*(\mathcal{M}_{n,d}^{D\text{-par}}) \xrightarrow{j^*} H^*(\mathcal{M}_{n,d}^{D\text{-par,ell}}) \longrightarrow H^*(\chi_D^{-1}(\mu))$$

and thus j^* is injective. The last part of iv) follows by (HMMS, Theorem 8.5). □

In the parabolic setting, there are additional tautological classes coming from the flags. Let $\mathcal{L}_{i,j} \to \mathfrak{M}_{n,d}^{D\text{-par}}$ be the line bundle whose fibre is the j-th subquotient in the flag over E_{x_i} and let

$$y_{i,j} = c_1(L_{i,j}) \in H^2(\mathfrak{M}_{n,d}^{D\text{-par}}). \tag{18}$$

We let $H^*_{\text{taut}}(\mathfrak{M}_{n,d}^{D\text{-par}})$ denote the subring generated by the usual tautological classes $f(\gamma)$ for a symmetric function f and class $\gamma \in H^*(C)$, and the additional parabolic tautological classes $y_{i,j}$. By a variant of Markman's argument (Markman, 2002), one should be able to prove that the Künneth components of the diagonal in $\mathfrak{M}_{n,d}^{D\text{-par,ell}} \times \overline{\mathfrak{M}_{n,d}^{D\text{-par}}}$ are generated by tautological classes and thus the pure cohomology is tautological:

$$H^*_{\text{taut}}(\mathfrak{M}_{n,d}^{D\text{-par,ell}}) = H^*_{\text{pure}}(\mathfrak{M}_{n,d}^{D\text{-par,ell}}).$$

In the parabolic case, one can define Hecke modifications at parabolic points as actual morphisms between semistable D-parabolic moduli spaces (for different choices of parabolic stability parameters). Since the elliptic locus consists of D-parabolic Higgs bundles which are stable for all stability parameters, we get Hecke morphisms between them. More precisely, for each parabolic point x_i and $1 \leq j \leq n$, we have a Hecke morphism

$$\mathcal{H}_{i,j}\colon \mathfrak{M}_{n,d}^{D\text{-par,ell}} \to \mathfrak{M}_{n,d-1}^{D\text{-par,ell}}, \quad (E, \theta, E^\bullet_{x_i}) \mapsto (E', \theta', E'^\bullet_{x_i})$$

where $E' \subset E$ is the kernel of the projection $E \to E_{x_i} \to L_{i,j} = \mathcal{O}_{x_i}$ onto the j-th eigenspace of $\mathrm{res}_{x_i}(\theta)$. Thinking of the parabolic structure as a filtration of sheaves, this operation cyclically permutes the filtration as described by Boden and Yokogawa (1999, §5). We denote the corresponding parabolic Hecke operator on cohomology by

$$X_{i,j} = \mathcal{H}^*_{i,j}\colon H^*(\mathfrak{M}_{n,d-1}^{D\text{-par,ell}}) \to H^*(\mathfrak{M}_{n,d}^{D\text{-par,ell}}). \tag{19}$$

Theorem 4.4. — *Assume n and d are coprime and D is an effective reduced divisor. Consider the following statements.*

$(1)_D$ *$P = C$ holds for $H^*_{\text{taut}}(\mathcal{M}_{n,d}^{D\text{-par,ell}})$.*

(2)$_D$ $\quad$ *$P = C$ holds for $H^*_{\mathrm{taut}}(\mathcal{M}^{D\text{-par}}_{n,d})$.*
(3)$_D$ $\quad$ *$P = C$ holds for $H^*_{\mathrm{taut}}(\mathcal{M}^{D\text{-spar}}_{n,d})$.*
(4) $\quad$ *$P = C$ holds for $H^*_{\mathrm{taut}}(\mathcal{M}_{n,d}) = H^*(\mathcal{M}_{n,d})$.*

Then $(1)_D \implies (2)_D \implies (3)_D$ *and for $D = x$, we have* $(3)_x \implies (4)$.

Proof. — The first three implications follow from Proposition 4.3.

If $D = x$, we sketch the final implication following (HMMS, Theorem 8.20). The relative dimension of π is the dimension of the flag variety GL_n/B, which is $\binom{n}{2}$ and coincides with the codimension of the closed immersion i. Moreover, (up to sign) the Euler classes of the relative tangent bundle of $\widetilde{M}_{n,d} \to \mathcal{M}_{n,d}$ and the normal bundle for $i\colon \widetilde{M}_{n,d} \hookrightarrow \mathcal{M}^{x\text{-spar}}_{n,d}$ are

$$\Delta := \prod_{1 \le j < k \le n} (y_j - y_k)$$

where $y_j := c_1(L_j)$ are the first Chern classes of the tautological line bundles coming from the flag at x as in Eq. (18). For the maps

$$A := \pi_* i^* \colon H^*(\mathcal{M}^{x\text{-spar}}_{n,d}) \rightleftarrows H^{*-2\binom{n}{2}}(\mathcal{M}_{n,d}) : B := i_* \pi^*$$

which both respect the perverse filtrations, we have

$$AB = \pi_* i^* i_* \pi^* = \pm \pi_* \Delta \pi^* = \pm n!$$

from which we deduce B is injective and the claim follows. $\square$

Remark 4.5. — A similar reduction is performed in the third $P = W$ proof (Maulik, Shen, and Yin, 2023, §5.4.3), where they work with the stronger statement $C \subset P$ on the full cohomology as opposed to the tautological (or pure) cohomology.

Hence it suffices to show $P = C$ on the elliptic parabolic moduli space $\mathcal{M}^{x\text{-par},\mathrm{ell}}_{n,d}$.

4.4. Step (B): Lie algebra actions on cohomology

Let S be a smooth surface and assume we have substacks $\mathfrak{M}_{S,\alpha} \subset \mathfrak{Coh}_{S,\alpha}$ satisfying the assumptions of §4.2.1 in order to define Hecke operators $T_n(\xi)$ and tautological operators $\psi_k(\gamma)$ indexed by classes $\xi, \gamma \in H = H^*_{\mathrm{pure}}(S)$. In this section, we will describe the key results of Mellit, Minets, Schiffmann, and Vasserot (2023), which are also proved in (HMMS).

4.4.1. *Action of the Hecke operators on the tautological cohomology.* — From the definition of Hecke correspondence given in Diagram (16), we see that over $\mathcal{Z}_\alpha \times S$ there is a tautological exact sequence

$$0 \to \pi^*_{2,S}\mathcal{F}_{\alpha-\delta} \to \pi^*_{3,S}\mathcal{F}_\alpha \to \pi^*_{1,S}\mathcal{F}_\delta \to 0$$

where $\pi_{i,S} := \pi_i \times \mathrm{Id}_S$. By combining this with the projection formula, they compute the commutator of the Hecke operator $T_n(\xi) := T(\xi u^n)$ for $\xi u^n \in H^*(\mathfrak{M}_{S,\delta}) \cong H^*(S)[u]$ with the tautological operator $p_k(\gamma)$ for $\gamma \in H^*(S)$ to be

$$[p_k(\gamma), T_n(\xi)] = T\left(\frac{u^k - (u-t_1)^k - (u-t_2)^k + (u-t_1+t_2)^k}{t_1t_2}\gamma\xi u^n\right), \tag{20}$$

where $[p_k(\gamma), T_n(\xi)] := p_k^{\alpha}(\gamma)T_n(\xi) - (-1)^{\deg(\gamma)\deg(\xi)}T_n(\xi)p_k^{\alpha-\delta}(\gamma)$ is viewed as an operator from $H^*(\mathfrak{M}_{S,\alpha-\delta})$ to $H^*(\mathfrak{M}_{S,\alpha})$, and t_1 and t_2 denote the Chern roots of TS.

By induction, to describe the Hecke action on $\bigoplus_\alpha H^*_{\mathrm{taut}}(\mathfrak{M}_{S,\alpha})$ it suffices to compute $T_n(\xi)(1)$. This is achieved via classical intersection theory: by (HMMS, Eq. (3.2)), we have

$$T_n(\xi)(1) = h_{n+1-\mathrm{rk}\,\mathcal{F}_\alpha}(\xi), \tag{21}$$

where h_k is the complete homogeneous symmetric function of degree k.

4.4.2. *Lifting to the Fock space.* — In (HMMS), the authors study the action of $T_n(\xi)$ by lifting it to a universal model Λ_S for the tautological cohomology. The Fock space Λ_S is a polynomial ring (over $\mathbb{C}$) generated by $p_k(\gamma)$ for $\gamma \in H$ modulo the relations

$$p_k(\gamma + \gamma') = p_k(\gamma) + p_k(\gamma') \quad \text{and} \quad p_k(\lambda\gamma) = \lambda p_k(\gamma) \quad \text{for } \lambda \in \mathbb{C}.$$

This is a graded super-commutative ring where $p_k(\gamma)$ has degree $2k - 4 + i$ for $\gamma \in H^i$. For any $f \in \Lambda_S$ of positive degree, one can define $f(\gamma)$ by expressing f in terms of the p_k's. To lift the Hecke operator to Λ_S, we need two homomorphisms

$$R\colon \Lambda_S \rightleftarrows \Lambda_S \otimes H[u] : Q$$

where in view of Eq. (21), $Q(f \otimes \xi u^n) := f h_n(\xi)$ and, in view of Eq. (20),

$$R(p_k(\gamma)) := p_k(\gamma) - \frac{u^k - (u-t_1)^k - (u-t_2)^k + (u-t_1+t_2)^k}{t_1t_2}\gamma.$$

Proposition 4.6 (HMMS, Eq. (3.3)). — *For $\xi \in H$ and $n \geq 0$, the operator $T_n(\xi)$: $H^*(\mathfrak{M}_{S,\alpha-\delta}) \to H^*(\mathfrak{M}_{S,\alpha})$ lifts to the Fock space in the sense that we have a commutative diagram*

$$\begin{tikzcd}
\Lambda_S \otimes H[u] \arrow[r, "\xi u^{n-\mathrm{rk}\,\mathcal{F}_\alpha+1}"] & \Lambda_S \otimes H[u] \arrow[d, "Q"] \\
\Lambda_S \arrow[u, "R"] \arrow[r] \arrow[d, two heads] & \Lambda_S \arrow[d, two heads] \\
H^*(\mathfrak{M}_{S,\alpha-\delta}) \arrow[r, "T_n(\xi)"] & H^*(\mathfrak{M}_{S,\alpha}).
\end{tikzcd}$$

Proof. — This follows from Eq. (20) and (21) and the construction of R and Q. $\square$

To deal with the shift appearing in the top arrow of the above commutative diagram and in Eq. (21), the authors in (HMMS) define Hecke operators $T_n(\xi)\colon \Lambda_S \to \Lambda_S$ to induce $T_{n-\mathrm{rk}\,\mathcal{F}_\alpha+1}(\xi)$ on the cohomology groups of the stacks $\mathfrak{M}_{S,\alpha}$ (see HMMS, Remark 5.3). They similarly define tautological operators $\psi_k(\gamma)\colon \Lambda_S \to \Lambda_S$ as in Eq. (17).

One of the central ingredients due to Mellit, Minets, Schiffmann, and Vasserot (2023), whose proof is also sketched in (HMMS, §4), is the commutator relations between the Hecke and tautological operators on Λ_S, as well as quadratic and cubic relations given in (HMMS, Theorem 4.1). Let us just state the commutator relation which, after comparing with the much more complicated commutator relation in Eq. (20), explains why replacing the tautological classes $p_k(\gamma)$ with the tautological ψ-classes $\psi_k(\gamma)$ is desirable: we have

$$[\psi_k(\gamma), T_n(\xi)] = kT_{n+k-1}(\gamma\xi). \tag{22}$$

In (HMMS, §5), the authors fix a basis Π of H and define a *surface W-algebra* which is generated by operators $T_n(\pi)$ and $\psi_n(\pi)$ for $\pi \in \Pi$ and $n \geq 0$ modulo the relations in (HMMS, Theorem 4.1) and the additional relation $[\psi_m(\xi), \psi_n(\gamma)] = 0$. As an immediate corollary, they obtain a representation of this surface W-algebra on the cohomology $\oplus_{i\in\mathbb{Z}} H^*(\mathfrak{M}_{S,\alpha+i\delta})$ (see HMMS, Corollary 5.2). These results culminate in the following theorem.

THEOREM 4.7 (Mellit, Minets, Schiffmann, and Vasserot, 2023; (HMMS, Theorem 5.5)) *Assume both $s_2 = 0 \in H^4(S)$ and $s_1\Delta = 0 \in H^6(S\times S)$. Then the surface W-algebra is isomorphic to the universal enveloping algebra of the Lie algebra with basis $D_{m,n}(\pi)$ where $m, n \in \mathbb{N}$ and π is an element of a fixed basis Π of H and Lie bracket*

$$[D_{m,n}(\xi), D_{m',n'}(\xi')] = (nm' - mn')D_{m+m',n+n'-1}(\xi\xi').$$

The relation between the generators of this algebra and the previous operators is given by $D_{0,n}(\xi) = \psi_n(\xi)$ and $D_{1,n}(\xi) = T_n(\xi)$. Moreover, there is a representation of this algebra on $\oplus_{i\in\mathbb{Z}} H^*(\mathfrak{M}_{S,\alpha+i\delta})$ extending the given action of the tautological operators $\psi_n(\xi)$ and Hecke operators $T_n(\xi)$.

The first index m records the shift in the δ direction: $D_{m,n}(\xi)\colon H^*(\mathfrak{M}_{S,\alpha}) \to H^*(\mathfrak{M}_{S,\alpha+m\delta})$ and the second index corresponds to a shift in cohomological degrees. We see that the shift in these indices under the commutator is not quite symmetric: the m-index is the sum of the m-indices, but the n-index is the sum of the n-indices minus 1. This will be dealt with by a degeneration procedure in the next section that also reduces the m-index by 1.

In the Higgs case (see §4.2.3), the classes s_2 and $s_1\Delta$ vanish for degree reasons. More generally, without these vanishing assumptions, analogous relations are proved modulo a filtration in (HMMS, §5.2).

4.5. Step (C): Periodicity, cohomological χ-independence and degeneration

Whilst the tautological operators act on the cohomology of individual moduli stacks, the Hecke operators increase the Chern character by δ. However, by twisting by a line bundle, these moduli stacks are periodic (for example, in the Higgs case, $\mathfrak{M}_{n,d}$ up to isomorphism only depends on d modulo n, as tensoring by a degree one line bundle induces $\mathfrak{M}_{n,d} \simeq \mathfrak{M}_{n,d+n}$). To consider the cohomology of just one moduli stack (or space) at a time, we will identify the cohomology of different moduli stacks (to give a cohomological χ-independence result) using a specific Hecke correspondence.

Proposition 4.8. — *Fix $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and assume that there is a line bundle L on S such that $-\otimes L\colon \mathfrak{M}_{S,\alpha} \to \mathfrak{M}_{S,\alpha\,\mathrm{ch}(L)}$ is an isomorphism and $c_1(L)\alpha_1 > 0$. Then $T_0(c_1(L))$ induces an isomorphism $H^*_{\mathrm{taut}}(\mathfrak{M}_{S,\alpha}) \cong H^*_{\mathrm{taut}}(\mathfrak{M}_{S,\alpha+\delta})$.*

Proof. — Let us outline the proof and refer to (HMMS, Proposition 6.2) for details. First one shows $T_0(c_1(L))$ is surjective inductively using the fact that $c_1(L)\alpha_1 \neq 0$ and the description of the action of Hecke operators provided by (HMMS, Theorem 5.8). By iterating the surjection $T_0(c_1(L))$, one obtains

$$\dim H^i_{\mathrm{taut}}(\mathfrak{M}_{S,\alpha}) \geq \dim H^i_{\mathrm{taut}}(\mathfrak{M}_{S,\alpha+\delta}) \geq \cdots \geq \dim H^i_{\mathrm{taut}}(\mathfrak{M}_{S,\alpha+\delta c_1(L)\alpha_1});$$

however the first and last dimensions are equal via the isomorphism coming from tensoring with L. Hence, these inequalities are equalities and $T_0(c_1(L))$ is an isomorphism. □

In the D-twisted Higgs case, we have $S = \mathrm{Tot}(\omega_C(D))$ and we take $\mathfrak{M}_{S,\alpha}$ to be the elliptic loci in the stack of Higgs bundles (viewed as sheaves on S via the spectral correspondence).

Let us write $\eta := c_1(L)$, so $T_0(\eta)$ can be used to identify the tautological cohomology of different stacks. If $X := T_0(\eta)/\eta\alpha_1$, then the operators $X^{-m}D_{m,n}$ act on the cohomology of $\mathfrak{M}_{S,\alpha}$. Moreover the sequence of operators $X^{-m}D_{m,n}$ depends polynomially on m and the non-zero coefficients are nilpotent if $n = 0$ (HMMS, Proposition 6.3).

At this point a degeneration is used to obtain new operators $\widetilde{D}_{m,n}$, which also depend polynomially on m, with better symmetry properties (in particular they will enable us to find the desired actions of $\mathcal{H}_2$ and $\mathfrak{sl}_2$). The authors view this degeneration as analogous to degenerating a spherical trigonometric Cherednik algebra to a rational Cherednik algebra. The operators $\widetilde{D}_{m,n}$ are defined via the following generating series:

$$D_{m,n}(\xi) := \widetilde{X}^m \sum_i \frac{m^i}{i!} \widetilde{D}_{i,n}(\xi),$$

where $\widetilde{X} := Xe^{\theta/\eta\alpha_1}$ and θ is the linear term in the polynomial expansion of $X^{-k}D_{k,0}(\eta)$, which is nilpotent and so $e^{\pm k\theta}$ are polynomials in k. The operators $\widetilde{D}_{m,n}$ depend polynomially on m and, by construction, the linear term in the polynomial expansion of $\widetilde{X}^{-k}D_{k,0}(\eta)$ vanishes. The commutator of these operators is computed in (HMMS, Proposition 6.4) and now exhibits more natural symmetry between m and n. If one assumes s_2 and $s_1\Delta$ vanish as in Theorem 4.7, then these identities hold on the nose rather than modulo a certain filtration and in the Higgs case they simplify further as we will see below. Consequently, the algebra $\widetilde{W}$ generated by the operators $\widetilde{D}_{m,n}$ turns out to be generated by $\psi_2(1)$ and $\widetilde{D}_{k,0}(\xi)$ (see HMMS, Corollary 6.5).

4.6. Step (D): Reduced operators on the cohomology of moduli spaces

In the situations we are interested in (e.g. the Higgs case), the moduli stacks $\mathfrak{M}_{S,\alpha}$ will be trivial $\mathbb{G}_m$-gerbes over their good moduli spaces $\mathcal{M}_{S,\alpha}$. Consequently

$$H^*(\mathfrak{M}_{S,\alpha}) \simeq H^*(\mathcal{M}_{S,\alpha}) \otimes H^*(B\mathbb{G}_m) \simeq H^*(\mathcal{M}_{S,\alpha})[u].$$

We would like to remove this tensor factor of $H^*(B\mathbb{G}_m)$ to pass from the stack to the moduli space in a canonical way. To do this we express the cohomology of $\mathfrak{M}_{S,\alpha}$ as a polynomial ring over a so-called reduced cohomology ring with variable y given by a normalised tautological operator.

As above, we continue to let $\eta := c_1(L)$ and we write $n := \eta\alpha_1 > 0$ (which will equal the rank in the Higgs case). From the above commutator computations, one obtains

$$[\widetilde{D}_{0,1}(\eta), \widetilde{D}_{1,0}(1)] = n$$

where $\widetilde{D}_{0,1}(\eta) = \psi_1(\eta)$. Hence for $y := \psi_1(\eta)/n$ and $\partial_y := -\widetilde{D}_{1,0}(1)$, we have $[\partial_y, y] = 1$. Since ∂_y has cohomological degree -2, it is locally nilpotent and moreover, y provides a slice of the associated $\mathbb{G}_a$-action. Thus if $H^*(\mathfrak{M}_{S,\alpha})_{\mathrm{red}} := \ker(\partial_y)$, we obtain a canonical decomposition

$$H^*(\mathfrak{M}_{S,\alpha}) \cong H^*(\mathfrak{M}_{S,\alpha})_{\mathrm{red}}[y].$$

Note that we can view $H^*(\mathfrak{M}_{S,\alpha})_{\mathrm{red}}$ as a subspace given by $\ker(\partial_y)$ and as a quotient space given by the cokernel of y, just as we can view a slice of a $\mathbb{G}_a$-action as a subset or a quotient.

Since Ad_{∂_y} acts locally nilpotently on $\widetilde{W}$, repeating this argument we can write

$$\widetilde{W} = \widetilde{W}'[y]$$

and then check that Ad_y acts locally nilpotently on $\widetilde{W}'$, so that

$$\widetilde{W} = \widetilde{W}'[y] = W_{\mathrm{red}}[y, \partial_y],$$

where W_{red} consists of operators commuting with y and ∂_y. In particular, the action of the algebra W_{red} preserves $H^*(\mathfrak{M}_{S,\alpha})_{\mathrm{red}}$. For $f \in H^*(\mathfrak{M}_{S,\alpha})$ (resp. $f \in \widetilde{W}$), we let $f_{\mathrm{red}} \in H^*(\mathfrak{M}_{S,\alpha})_{\mathrm{red}}$ (resp. $f_{\mathrm{red}} \in W_{\mathrm{red}}$) denote the constant term in its polynomial expansion. Consequently we get operators $\widetilde{D}_{m,n}(\xi)_{\mathrm{red}}$ on $H^*_{\mathrm{taut}}(\mathfrak{M}_{S,\alpha})_{\mathrm{red}}$, whose commmutator relations are similar to, but more complicated than, those in Theorem 4.7 due to the appearance of additional terms; see (HMMS, Proposition 7.3).

A first candidate for the $\mathfrak{sl}_2$-triple $\langle \mathbf{e}, \mathbf{f}, \mathbf{h} \rangle$ is obtained from these operators, where $\mathbf{e}$ comes from a tautological operator and $\mathbf{f}$ comes from a Hecke transformation. We next explain this in further detail in the Higgs case, where the relations simplify.

4.7. Step (E): Finding an $\mathfrak{sl}_2$-triple in the Higgs setting to prove $P = W$

As in §4.3, we will now consider elliptic loci in moduli stacks (and spaces) of D-twisted Higgs bundles on C, which we denote by $\mathfrak{M}_{n,d}^{D,\mathrm{ell}}$ (and $\mathcal{M}_{n,d}^{D,\mathrm{ell}}$). We have

$$H^*_{\mathrm{taut}}(\mathfrak{M}_{n,d}^{D,\mathrm{ell}})_{\mathrm{red}} \simeq H^*_{\mathrm{taut}}(\mathcal{M}_{n,d}^{D,\mathrm{ell}})$$

as the reduced tautological cohomology of the stack consists of polynomials in the tautological classes which are invariant under twisting the universal sheaf by a line bundle on C. Under this identification the classes $\psi_k(\xi)_{\mathrm{red}} = \widetilde{D}_{0,k}(\xi)_{\mathrm{red}}(1)$ generate the tautological cohomology of the moduli space $\mathcal{M}_{n,d}^{D,\mathrm{ell}}$. We fix a basis $\Pi = \{1, \gamma_1, \ldots, \gamma_{2g}, \omega\}$ of $H = H^*(C)$ and we take $\eta := \omega$ so that $\eta\alpha_1 = n$.

Before we turn our attention to the $\mathfrak{sl}_2$-triples needed for the proof of the $P = W$ conjecture, let us quickly explain how to produce the $\mathcal{H}_2$-action in the title of (HMMS). To simplify the relations, the idea is to introduce a formal variable x and 'undo' the operation of reduction. This gives operators $\widetilde{D}_{m,n}(\xi)_{\mathrm{unred}}$ which act on

$$H^*_{\mathrm{taut}}(\mathfrak{M}_{n,d}^{D,\mathrm{ell}})[x] \simeq H^*_{\mathrm{taut}}(\mathcal{M}_{n,d}^{D,\mathrm{ell}})[x, y]$$

and by restricting to the subalgebra formed by elements $\widetilde{D}_{m,n}(1)_{\mathrm{unred}}$ one obtains an action of $\mathcal{H}_2$ as described by (HMMS, Corollary 7.4).

The commutator relations in (HMMS, Proposition 7.3) have two important consequences for the proof of $P = W$ provided one chooses d appropriately as mentioned in (HMMS, §7.3) so that $\widetilde{D}_{0,2}(1)_{\mathrm{red}}$ vanishes: first one obtains an *original* $\mathfrak{sl}_2$-triple $\langle \mathbf{e}_{\mathrm{orig}}, \mathbf{f}_{\mathrm{orig}}, \mathbf{h}_{\mathrm{orig}} \rangle$ acting on $H^*(\mathcal{M}_{n,d}^{D,\mathrm{ell}})$, and second the tautological classes $\psi_k(\xi)_{\mathrm{red}}$ are eigenvectors of weight k with respect to $\mathbf{h}_{\mathrm{orig}}$. This $\mathfrak{sl}_2$-triple is defined by

$$\mathbf{e}_{\mathrm{orig}} := \frac{1}{2}\widetilde{D}_{0,2}(1)_{\mathrm{red}} \quad \text{and} \quad \mathbf{f}_{\mathrm{orig}} := \frac{1}{2}\widetilde{D}_{2,0}(1)_{\mathrm{red}} \quad \text{and} \quad \mathbf{h}_{\mathrm{orig}} := -\widetilde{D}_{1,1}(1)_{\mathrm{red}}$$

where $\mathbf{e}_{\mathrm{orig}}$ is constructed from the tautological operator $\psi_2(1)_{\mathrm{red}} = \widetilde{D}_{0,2}(1)_{\mathrm{red}}$ and $\mathbf{f}_{\mathrm{orig}}$ is constructed from a Hecke operator. Then the tautological classes satisfy

$$[\mathbf{h}_{\mathrm{orig}}, \psi_k(\xi)_{\mathrm{red}}] = k\psi_k(\xi)_{\mathrm{red}}. \tag{23}$$

The following result combines the general result (HMMS, Theorem 6.9) with the Higgs-specific result (HMMS, Proposition 7.7).

Proposition 4.9. — *The original $\mathfrak{sl}_2$-triple $\langle \mathbf{e}_{\mathrm{orig}}, \mathbf{f}_{\mathrm{orig}}, \mathbf{h}_{\mathrm{orig}} \rangle$ acts on $H^*_{\mathrm{taut}}(\mathcal{M}_{n,d}^{D,\mathrm{ell}})$. Let Q_i denote the sum of the $\mathbf{h}_{\mathrm{orig}}$-eigenspaces with eigenvalues $\leq i$; then the following statements hold.*

i) *The tautological classes satisfy $\psi_k(\xi)_{\mathrm{red}} Q_i \subset Q_{i+k}$.*
ii) *The Hecke operators satisfy $\widetilde{D}_{k,0}(\xi)_{\mathrm{red}} Q_i \subset Q_i$.*
iii) *On $\mathrm{Gr}^Q H^*(\mathcal{M}_{n,d}^{D,\mathrm{ell}})$ the operator $\mathbf{e}_{\mathrm{orig}}$ coincides with $\frac{1}{2}\psi_2(1)_{\mathrm{red}}$ and satisfies the Lefschetz property: $\frac{1}{2}\psi_2(1)^i_{\mathrm{red}} \colon Q_{-i}/Q_{-i-1} \to Q_i/Q_{i-1}$ is an isomorphism for all i.*

We would like to conclude that the eigenspace filtration Q_i is (up to a shift in indices) the perverse filtration for the Hitchin fibration; this would be the case if $\mathbf{e}_{\mathrm{orig}}$ came from a relatively ample class as in the Relative Hard Lefschetz Theorem (see Remark 1.12). To prove this is the case, we need to modify our $\mathfrak{sl}_2$-triple as follows.

THEOREM 4.10 (HMMS, Propositions 8.11 - 8.13). — *There is an $\mathfrak{sl}_2$-triple $\langle \mathbf{e}, \mathbf{f}, \mathbf{h}\rangle$ acting on $H^*_{\mathrm{taut}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d})$ where $\mathbf{e}$ comes from a relatively ample class and $\mathbf{h}$ induces the same filtration as $\mathbf{h}_{\mathrm{orig}}$. In particular, this filtration coincides with the perverse filtration for the Hitchin map (modulo a shift appearing in the definition of the perverse filtration). Moreover, $P^h_k H^*_{\mathrm{taut}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d})$ is the span of products of tautological classes $\prod_i \psi_{k_i}(\xi_i)_{\mathrm{red}}$ where $\sum_i k_i \leq k$ and thus $P = C = W$ holds on $H^*_{\mathrm{taut}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d})$.*

Proof. — If $\alpha \in H^2_{\mathrm{taut}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d})$ is a relatively ample class (with respect to the Hitchin map), then a small perturbation $\mathbf{e} := \alpha + \lambda\psi_2(1)_{\mathrm{red}}$ remains ample and this is part of an $\mathfrak{sl}_2$-triple $\langle \mathbf{e}, \mathbf{f}, \mathbf{h}\rangle$ acting on $H^*_{\mathrm{taut}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d})$ such that the $\mathbf{h}$-filtration coincides with the $\mathbf{h}_{\mathrm{orig}}$-filtration Q provided λ is chosen appropriately (see HMMS, Corollary 7.10). Moreover, the operators $\mathbf{e}, \mathbf{f}$ and $\mathbf{h}$ change the perversity by $2, -2$ and 0 respectively by (HMMS, Proposition 8.11), which guarantees that they induce an $\mathfrak{sl}_2$-triple on the associated graded vector space for the perverse filtration.

By the Relative Hard Lefschetz Theorem, the ample class $\mathbf{e}$ is part of an $\mathfrak{sl}_2$-triple $\langle \mathbf{e}, \mathbf{f}', \mathbf{h}'\rangle$ such that on the associated graded vector space for the perverse filtration, the $\mathbf{h}'$-graded pieces are the graded pieces for the perverse filtration (see Remark 1.12). On this associated graded object, one can show that $\mathbf{h}$ and $\mathbf{h}'$ coincide by showing they commute and any eigenvalue of their difference is zero (HMMS, Proposition 8.12).

To conclude on the (ungraded) cohomology that the canonical eigenspace filtration Q coming from $\mathbf{h}$ (or equivalently $\mathbf{h}_{\mathrm{orig}}$) coincides with the perverse filtration P (up to a shift in indices by the relative dimension r_h of the Hitchin fibration), it suffices to show that one filtration is contained in the other. Indeed, as the Lefschetz operator satisfies the Lefschetz property with respect to both filtrations, a containment suffices to conclude equality as in the proof of Proposition 2.3 (v). For this, one can inductively show $P_{k-r_h} \subset Q_k$ by noting that $\mathbf{h}$ preserves the perversity and for $\beta \in P_{k-r_h}$, the operator $\mathbf{h}$ acts by multiplication by k on the class of β in the associated graded object for the perverse filtration, so we must have $\mathbf{h}(\beta) = k\beta + \beta'$ for some $\beta' \in P_{k-r_h-1}$.

Finally, Eq. (23) describes the $\mathbf{h}_{\mathrm{orig}}$-weights on the ψ-classes and from this one deduces that $P^h_k H^*_{\mathrm{taut}}(\mathcal{M}^{D,\mathrm{ell}}_{n,d})$ is the span of products $\prod_i \psi_{k_i}(\xi_i)_{\mathrm{red}}$ where $\sum_i k_i \leq k$. □

This does not quite suffice to prove the $P = W$ conjecture, as one needs a version of this result for the tautological cohomology of the elliptic locus in the moduli space of *D-parabolic* Higgs bundles, where $D = x$ is a single point (see Theorem 4.4). Fortunately this is possible, but requires extending the ring $H := H^*(C)$ to a ring H^D_n by introducing additional generators $p_{i,j}$ of degree 2 for each $x_i \in D$ and $1 \leq j \leq n$ modulo the relations

$$\sum_{j=1}^{n} p_{i,j} = \omega, \quad p_{i,j} p_{i',j'} = 0 \quad \text{and} \quad p_{i,j} H^{>0} = 0.$$

Since D-parabolic Higgs bundles on C can also be considered as Higgs bundles on the stacky curve given by taking the n-th root stack of C along D, we can view H_n^D as the orbifold cohomology of this stacky curve. We define additional tautological operators and Hecke operators by

$$\psi_k(p_{i,j}) = y_{i,j}^n \quad \text{and} \quad T_n(p_{i,j}) := y_{i,j}^n X_{i,j},$$

where $y_{i,j}$ and $X_{i,j}$ are defined in Equations (18) and (19). As in the non-parabolic case, the action of the Hecke transformation T on the tautological cohomology can be described by the commutator relations between ψ and T and the values of T on 1. The previous result can be upgraded to the moduli space $\mathcal{M}_{n,d}^{D\text{-par,ell}}$ where H is replaced by H_n^D (see HMMS, Theorem 7.6). In particular, Theorem 4.10 holds when $\mathcal{M}_{n,d}^{\text{ell}}$ is replaced by $\mathcal{M}_{n,d}^{D\text{-par,ell}}$ and $H = H^*(C)$ is replaced by H_n^D. Consequently, we can conclude the $P = W$ conjecture holds using Theorem 4.4.

REFERENCES

Jarod Alper (2013). "Good moduli spaces for Artin stacks", *Ann. Inst. Fourier (Grenoble)* **63** (6), pp. 2349–2402.

Dima Arinkin (2013). "Autoduality of compactified Jacobians for curves with plane singularities", *J. Algebraic Geom.* **22** (2), pp. 363–388.

Michael F. Atiyah and Raoul Bott (1983). "The Yang-Mills equations over Riemann surfaces", *Philos. Trans. R. Soc. Lond., Ser. A* **308**, pp. 523–615.

Arnaud Beauville (1995). "Sur la cohomologie de certains espaces de modules de fibrés vectoriels", in: *Geometry and analysis (Bombay, 1992),* Tata Inst. Fund. Res., Bombay, pp. 37–40.

Arnaud Beauville, Mudumbai Seshachalu Narasimhan, and Sundararaman Ramanan (1989). "Spectral curves and the generalised theta divisor", *J. Reine Angew. Math.* **398**, pp. 169–179.

Alexander A. Beĭlinson, Joseph Bernstein, and Pierre Deligne (1982). "Faisceaux pervers", in: *Analysis and topology on singular spaces, I (Luminy, 1981),* Astérisque, vol. 100, Soc. Math. France, Paris, pp. 5–171.

Andrzej Białynicki-Birula (1973). "Some theorems on actions of algebraic groups", *Ann. of Math. (2)* **98**, pp. 480–497.

Olivier Biquard (1997). "Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse)", *Ann. Sci. École Norm. Sup. (4)* **30** (1), pp. 41–96.

Olivier Biquard and Philip Boalch (2004). "Wild non-abelian Hodge theory on curves", *Compos. Math.* **140** (1), pp. 179–204.

Hans U. Boden and Kôji Yokogawa (1999). “Rationality of moduli spaces of parabolic bundles”, *J. London Math. Soc. (2)* **59** (2), pp. 461–478.

Barbara Bolognese, Alex Küronya, and Martin Ulirsch (2023). “$P = W$ phenomena on abelian varieties”, arXiv: 2303.03734.

Pierre-Henri Chaudouard and Gérard Laumon (2016). “Un théorème du support pour la fibration de Hitchin”, *Ann. Inst. Fourier (Grenoble)* **66** (2), pp. 711–727.

Kevin Corlette (1988). “Flat G-bundles with canonical metrics”, *J. Differential Geom.* **28** (3), pp. 361–382.

Ben Davison (2023). “Nonabelian Hodge theory for stacks and a stacky P = W conjecture”, *Adv. Math.* **415**, Paper No. 108889, 61.

Ben Davison, Lucien Hennecart, and Sebastian Schlegel Mejia (2023). “BPS Lie algebras for totally negative 2-Calabi-Yau categories and nonabelian Hodge theory for stacks”, arXiv: 2212.07668.

Mark A. de Cataldo (2017a). “A support theorem for the Hitchin fibration: the case of SL_n”, *Compos. Math.* **153** (6), pp. 1316–1347.

——— (2017b). “Perverse sheaves and the topology of algebraic varieties”, in: *Geometry of moduli spaces and representation theory,* IAS/Park City Math. Ser. vol. 24, Amer. Math. Soc., Providence, RI, pp. 1–58.

Mark A. de Cataldo, Tamás Hausel, and Luca Migliorini (2012). “Topology of Hitchin systems and Hodge theory of character varieties: the case A_1”, *Ann. of Math. (2)* **175** (3), pp. 1329–1407.

Mark A. de Cataldo, Jochen Heinloth, and Luca Migliorini (2021). “A support theorem for the Hitchin fibration: the case of GL_n and K_C”, *J. Reine Angew. Math.* **780**, pp. 41–77.

Mark A. de Cataldo and Davesh Maulik (2020). “The perverse filtration for the Hitchin fibration is locally constant”, *Pure Appl. Math. Q.* **16** (5), pp. 1441–1464.

Mark A. de Cataldo, Davesh Maulik, and Junliang Shen (2022a). “Hitchin fibrations, abelian surfaces, and the $P = W$ conjecture”, *J. Amer. Math. Soc.* **35** (3), pp. 911–953.

——— (2022b). “On the $P = W$ conjecture for SL_n”, *Selecta Math. (N.S.)* **28** (5), Paper No. 90, 21 pp.

Mark A. de Cataldo, Davesh Maulik, Junliang Shen, and Siqing Zhang (2021). “Cohomology of the moduli of Higgs bundles via positive characteristic”, arXiv: 2105.03043.

Mark A. de Cataldo and Luca Migliorini (2009). “The decomposition theorem, perverse sheaves and the topology of algebraic maps”, *Bull. Amer. Math. Soc. (N.S.)* **46** (4), pp. 535–633.

——— (2010). “The perverse filtration and the Lefschetz hyperplane theorem”, *Ann. of Math. (2)* **171** (3), pp. 2089–2113.

Ron Donagi and Tony Pantev (2012). “Langlands duality for Hitchin systems”, *Invent. Math.* **189** (3), pp. 653–735.

Simon K. Donaldson (1983). "A new proof of a theorem of Narasimhan and Seshadri", *J. Differential Geom.* **18** (2), pp. 269–277.

——— (1987). "Twisted harmonic maps and the self-duality equations", *Proc. London Math. Soc. (3)* **55** (1), pp. 127–131.

Geir Ellingsrud and Arild Strømme (1993). "Towards the Chow ring of the Hilbert scheme of $\mathbf{P}^2$", *J. Reine Angew. Math.* **441**, pp. 33–44.

Gerd Faltings (1993). "Stable G-bundles and projective connections", *J. Algebraic Geom.* **2** (3), pp. 507–568.

——— (2005). "A p-adic Simpson correspondence", *Adv. Math.* **198** (2), pp. 847–862.

Camilla Felisetti and Mirko Mauri (2022). "P = W conjectures for character varieties with symplectic resolution", *J. Éc. polytech. Math.* **9**, pp. 853–905.

John Fogarty (1973). "Fixed point schemes", *Amer. J. Math.* **95**, pp. 35–51.

William M. Goldman and Eugene Z. Xia (2008). "Rank one Higgs bundles and representations of fundamental groups of Riemann surfaces", *Mem. Amer. Math. Soc.* **193** (904), pp. viii+69.

Michael Groechenig (2014). "Hilbert schemes as moduli of Higgs bundles and local systems", *Int. Math. Res. Not. IMRN* (23), pp. 6523–6575.

Michael Groechenig, Dimitri Wyss, and Paul Ziegler (2020). "Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration", *Invent. Math.* **221** (2), pp. 505–596.

Ian Grojnowski (1996). "Instantons and affine algebras. I. The Hilbert scheme and vertex operators", *Math. Res. Lett.* **3** (2), pp. 275–291.

Thomas C. Hales (2012). "The fundamental lemma and the Hitchin fibration [after Ngô Bao Châu]", Séminaire Bourbaki, Exposé No. 1035, vol. 2010/2011; in: Astérisque **348**, pp. 233–263.

Andrew Harder, Zhiyuan Li, Junliang Shen, and Qizheng Yin (2021). "$P = W$ for Lagrangian fibrations and degenerations of hyper-Kähler manifolds", *Forum Math. Sigma* **9**, Paper No. e50, 6 pp.

Tamás Hausel (1998). "Compactification of moduli of Higgs bundles", *J. Reine Angew. Math.* **503**, pp. 169–192.

Tamás Hausel, Emmanuel Letellier, and Fernando Rodriguez-Villegas (2011). "Arithmetic harmonic analysis on character and quiver varieties", *Duke Math. J.* **160** (2), pp. 323–400.

Tamás Hausel, Anton Mellit, Alexandre Minets, and Olivier Schiffmann (2022). "$P = W$ via H_2", arXiv: 2209.05429 (heretofore cited as: HMMS).

Tamás Hausel and C. Pauly (2012). "Prym varieties of spectral covers", *Geom. Topol.* **16** (3), pp. 1609–1638.

Tamás Hausel and Fernando Rodriguez-Villegas (2008). "Mixed Hodge polynomials of character varieties", *Invent. Math.* **174** (3). With an appendix by N. M. Katz, pp. 555–624.

Tamás Hausel and Michael Thaddeus (2003). “Mirror symmetry, Langlands duality, and the Hitchin system”, *Invent. Math.* **153** (1), pp. 197–229.

——— (2004). “Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles”, *Proc. London Math. Soc. (3)* **88** (3), pp. 632–658.

Jochen Heinloth (2010). “Lectures on the moduli stack of vector bundles on a curve”, in: *Affine flag manifolds and principal bundles.* Trends Math. Birkhäuser/Springer Basel AG, Basel, pp. 123–153.

Nigel Hitchin (1987a). “Stable bundles and integrable systems”, *Duke Math. J.* **54** (1), pp. 91–114.

Nigel J. Hitchin (1987b). “The self-duality equations on a Riemann surface”, *Proc. London Math. Soc. (3)* **55** (1), pp. 59–126.

Anton Kapustin and Edward Witten (2007). “Electric-magnetic duality and the geometric Langlands program”, *Commun. Number Theory Phys.* **1** (1), pp. 1–236.

Ludmil Katzarkov, Alexander Noll, Pranav Pandit, and Carlos Simpson (2015). “Harmonic maps to buildings and singular perturbation theory”, *Comm. Math. Phys.* **336** (2), pp. 853–903.

Shun-ichi Kimura and Angelo Vistoli (1996). “Chow rings of infinite symmetric products”, *Duke Math. J.* **85** (2), pp. 411–430.

Tasuki Kinjo and Naoki Koseki (2021). “Cohomological χ-independence for Higgs bundles and Gopakumar-Vafa invariants”, arXiv: 2112.10053.

Guitang Lan, Mao Sheng, and Kang Zuo (2019). “Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups”, *J. Eur. Math. Soc. (JEMS)* **21** (10), pp. 3053–3112.

Stacy G. Langton (1975). “Valuative criteria for families of vector bundles on algebraic varieties”, *Ann. of Math. (2)* **101**, pp. 88–110.

Sean Lawton and Adam S. Sikora (2017). “Varieties of characters”, *Algebr. Represent. Theory* **20** (5), pp. 1133–1141.

Joseph Le Potier (1991). “Fibrés de Higgs et systèmes locaux”, Séminaire Bourbaki, Exposé No. 737, Vol. 1990/91; in: Astérisque **201-203**, pp. 221–268.

Wei-Ping Li, Zhenbo Qin, and Weiqiang Wang (2002). “Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces”, *Math. Ann.* **324** (1), pp. 105–133.

Alexander Lubotzky and Andy R. Magid (1985). “Varieties of representations of finitely generated groups”, *Mem. Amer. Math. Soc.* **58** (336), pp. xi+117.

Eyal Markman (2002). “Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces”, *J. Reine Angew. Math.* **544**, pp. 61–82.

Davesh Maulik and Junliang Shen (2021). “Endoscopic decompositions and the Hausel-Thaddeus conjecture”, *Forum Math. Pi* **9**, Paper No. e8, 49 pp.

——— (2022). “The $P = W$ conjecture for GL_n”, arXiv: 2209.02568 (heretofore cited as: MS).

Davesh Maulik, Junliang Shen, and Qizheng Yin (2023). “Perverse filtrations and Fourier transforms”, arXiv: 2308.13160.

Mirko Mauri, Enrica Mazzon, and Matthew Stevenson (2022). “On the geometric $\mathrm{P} = \mathrm{W}$ conjecture”, *Selecta Math. (N.S.)* **28** (3), Paper No. 65, 45 pp.

Vikram Bhagvandas Mehta and Conjeeveram Srirangachari Seshadri (1980). “Moduli of vector bundles on curves with parabolic structures”, *Math. Ann.* **248** (3), pp. 205–239.

Anton Mellit (2019). “Cell decompositions of character varieties”, arXiv: 1905.10685.

Anton Mellit, Alexandre Minets, Olivier Schiffmann, and Éric Vasserot (2023). “Coherent sheaves on surfaces, COHAs and deformed $W_{1+\infty}$-algebras”. arXiv: 2311.13415.

Takuro Mochizuki (2009). “Kobayashi–Hitchin correspondence for tame harmonic bundles. II”, *Geom. Topol.* **13** (1), pp. 359–455.

——— (2011). “Wild harmonic bundles and wild pure twistor D-modules”, *Astérisque* (340), x+607 pp.

David Mumford, John Fogarty, and Frances Kirwan (1994). *Geometric invariant theory*. Third Edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), vol. 34; Springer-Verlag, Berlin; xiv+292 pp.

Hiraku Nakajima (1997). “Heisenberg algebra and Hilbert schemes of points on projective surfaces”, *Ann. of Math. (2)* **145** (2), pp. 379–388.

Mudumbai Seshachalu Narasimhan and Conjeeveram Srirangachari Seshadri (1965). “Stable and unitary vector bundles on a compact Riemann surface”, *Ann. of Math. (2)* **82**, pp. 540–567.

Andrei Neguţ (2019). “Shuffle algebras associated to surfaces”, *Selecta Math. (N.S.)* **25** (3), Paper No. 36, 57 pp.

Peter E. Newstead (1978). *Introduction to moduli problems and orbit spaces*. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, Vol. 51, Narosa Publishing House, New Delhi, vi+183 pp.

Bao Châu Ngô (2006). “Fibration de Hitchin et endoscopie”, *Invent. Math.* **164** (2), pp. 399–453.

——— (2010). “Le lemme fondamental pour les algèbres de Lie”, *Publ. Math. Inst. Hautes Études Sci.* (111), pp. 1–169.

Nitin Nitsure (1991). “Moduli space of semistable pairs on a curve”, *Proc. London Math. Soc. (3)* **62** (2), pp. 275–300.

Arthur Ogus and Vitaly Vologodsky (2007). “Nonabelian Hodge theory in characteristic p”, *Publ. Math. Inst. Hautes Études Sci.* (106), pp. 1–138.

Alexander Polishchuk (2007). “Lie symmetries of the Chow group of a Jacobian and the tautological subring”, *J. Algebraic Geom.* **16** (3), pp. 459–476.

Sundararaman Ramanan (1973). “The moduli spaces of vector bundles over an algebraic curve”, *Math. Ann.* **200**, pp. 69–84.

Daniel Schaub (1998). “Courbes spectrales et compactifications de Jacobiennes”, *Math. Z.* **227** (2), pp. 295–312.

Conjeeveram Srirangachari Seshadri (1967). “Space of unitary vector bundles on a compact Riemann surface”, *Ann. of Math. (2)* **85**, pp. 303–336.

Junliang Shen and Qizheng Yin (2022). “Topology of Lagrangian fibrations and Hodge theory of hyper-Kähler manifolds”, *Duke Math. J.* **171** (1). With Appendix B by C. Voisin, pp. 209–241.

Junliang Shen and Zili Zhang (2021). “Perverse filtrations, Hilbert schemes, and the $P = W$ conjecture for parabolic Higgs bundles”, *Algebr. Geom.* **8** (4), pp. 465–489.

Vivek Shende (2017). “The weights of the tautological classes of character varieties”, *Int. Math. Res. Not. IMRN* (22), pp. 6832–6840.

Carlos Simpson (1988). “Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization”, *J. Amer. Math. Soc.* **1** (4), pp. 867–918.

——— (1990). “Harmonic bundles on noncompact curves”, *J. Amer. Math. Soc.* **3** (3), pp. 713–770.

——— (1992). “Higgs bundles and local systems”, *Inst. Hautes Études Sci. Publ. Math.* (75), pp. 5–95.

——— (1994a). “Moduli of representations of the fundamental group of a smooth projective variety. I”, *Inst. Hautes Études Sci. Publ. Math.* (79), pp. 47–129.

——— (1994b). “Moduli of representations of the fundamental group of a smooth projective variety. II”, *Inst. Hautes Études Sci. Publ. Math.* (80), pp. 5–79.

——— (1997). “The Hodge filtration on nonabelian cohomology”, in: *Algebraic geometry—Santa Cruz 1995.* Proc. Sympos. Pure Math. Vol. 62, Amer. Math. Soc., Providence, RI, pp. 217–281.

Geordie Williamson (2017). “The Hodge theory of the decomposition theorem”, Séminaire Bourbaki, Exposé No. 1115, Vol. 2015/2016; in: Astérisque **390**, pp. 335–367.

Zhiwei Yun (2011). “Global Springer theory”, *Adv. Math.* **228** (1), pp. 266–328.

——— (2012). “Langlands duality and global Springer theory”, *Compos. Math.* **148** (3), pp. 835–867.

Zili Zhang (2021). “The $P = W$ identity for cluster varieties”, *Math. Res. Lett.* **28** (3), pp. 925–944.

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