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CONSTRUCTIONS OF NEW EULER SYSTEMS

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1. Introduction

Euler systems are systems of cohomology classes of geometric origin which control the Galois cohomology of Galois representations of automorphic representations. This report recalls briefly the history of the subject (section 2) before giving an overview of three important recent contributions to the construction of new Euler systems: section 3 presents Euler systems for Rankin–Selberg products of modular forms (Bertolini, Darmon, and Rotger, 2015a,b; Darmon and Rotger, 2014, 2017; Lei, Loeffler, and Zerbes, 2014; Kings, Loeffler, and Zerbes, 2020), section 4 the interpolation of zeta elements over universal deformation rings (Nakamura, 2023) and section 5 the use of bipartite Euler systems to prove important cases of the Beilinson–Bloch–Kato conjecture for Rankin–Selberg motives (Liu, Tian, Xiao, Zhang, and Zhu, 2022).

1.1. Axiomatic definitions

The first works on Euler systems of V. Kolyvagin and K. Rubin proposed the following tentative axiomatic definitions for the known cases.

Let p be a prime number. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_E$. Let $\Sigma \supset \{p\}$ be a finite set of rational primes. For $K/\mathbb{Q}$ a sub-extension of $\bar{\mathbb{Q}}$, we write $G_{K,\Sigma}$ be the Galois group of the maximal extension of K unramified outside places of K above $\Sigma \cup \{\infty\}$. Now suppose $K/\mathbb{Q}$ is finite. Let (M, ρ) be an E-vector space with a continuous $G_{K,\Sigma}$-action. Because the $G_{K,\Sigma}$-action is continuous and $\mathcal{O}$, we may choose and let (T, ρ) an $\mathcal{O}_E$-lattice inside M which is $G_{K,\Sigma}$-stable. We denote by $M^*(1)$ the dual $G_{K,\Sigma}$-representation $\operatorname{Hom}_E(M,E)(1)$. Let $H^1_f(G_{K,\Sigma},T)$ be the first Bloch–Kato cohomology group of T (Bloch and Kato, 1990, section 3).

Let Ξ be the set

$$\Xi \overset{\text{def}}{=} \{m \geq 1 \mid \{\ell|m\} \cap \Sigma = \{p\}\}. \tag{1}$$

For $m \in \Xi$, we write S_m for the set of primes not in Σ and which divide m (the set Ξ should be understood as indexing partial Euler products; more mundanely its function is to avoid degenerate base cases).

Definition 1.1. — *Here $K = \mathbb{Q}$. A* cyclotomic Euler system *for T is a system*[1] *of classes*

$$\left\{\mathbf{z}(m) \in H^1_f\left(G_{\mathbb{Q}(\zeta_m),\Sigma},T\right)\right\}_{m\in\Xi} \tag{2}$$

satisfying the following compatibility relation for corestriction

$$\operatorname{Cor}_{\mathbb{Q}(\zeta_{m'})/\mathbb{Q}(\zeta_m)} \mathbf{z}(m') = \left(\prod_{\ell\in S_{m'}\setminus S_m} \det\left(1-\operatorname{Fr}(\ell)t|M^*(1)\right)_{t=\sigma(\ell)}\right)\cdot \mathbf{z}(m) \tag{3}$$

for all $m|m' \in \Xi$. Here, ζ_m is a primitive root of unity of order m, $\operatorname{Fr}(\ell)$ is the geometric Frobenius morphism in $G_{\mathbb{Q},\Sigma}$, $\sigma(\ell) \in \operatorname{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ is the geometric Frobenius morphism at ℓ and the action on $\mathbf{z}(m)$ in the righthand term of (3) is the natural action of $\mathcal{O}_E[\operatorname{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]$ on $H^1(G_{\mathbb{Q}(\zeta_m),\Sigma},T)$.

Notice in particular that the definition implies that

$$(\mathbf{z}(mp^n))_{n\geq 0} \in \varprojlim_n H^1\left(G_{\mathbb{Q}(\zeta_{mp^n}),\Sigma},T\right) \simeq H^1\left(G_{\mathbb{Q},\Sigma}, T\otimes_{\mathcal{O}_E}\mathbf{\Lambda}\right) \tag{4}$$

where $\mathbf{\Lambda}$ is the Iwasawa algebra attached to the extension of $\mathbb{Q}(\zeta_{mp^\infty})$.

Fundamental example. — Let M be the Tate motive $\mathbb{Q}(1)$. Fix a coherent system $\{\zeta_n \in \bar{\mathbb{Q}}|\forall(n,m)\in\mathbb{N}^2, \zeta_{nm}^m = \zeta_n\}_{n\geq 1}$ of primitive roots of unity. Put $\Xi = \{p\}$ (this choice is here to ensure that we are avoiding the degenerate case $1-\zeta_n$ for $n=1$ in the following). For all $n \in \Xi$, define the cyclotomic unit

$$\mathbf{z}(n) \overset{\text{def}}{=} (1-\zeta_n)\left(1-\zeta_n^{-1}\right) \in \mathbb{Z}[\zeta_n, 1/\Sigma]^\times \hookrightarrow H^1_f\left(G_{\mathbb{Q}(\zeta_n),\Sigma},\mathbb{Z}_p(1)\right).$$

Cyclotomic units satisfy the norm relation

$$N_{\mathbb{Q}(\zeta_{m'})/\mathbb{Q}(\zeta_m)}\left((1-\zeta_{m'})\left(1-\zeta_{m'}^{-1}\right)\right) = \left(\prod_{\ell\in S_{m'}\setminus S_m}(1-\sigma(\ell))\right)\left((1-\zeta_n)\left(1-\zeta_m^{-1}\right)\right)$$

1. Here and in the following, we call the underlying set of classes of an Euler system a *system* in order to emphasize the fact that they are close to being an inverse system for corestriction.

where as above $\sigma(\ell)$ is the geometric Frobenius morphism at ℓ in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. On the other hand, $M^*(1)$ is the trivial motive $\mathbb{Q}$ so $\det(1-\mathrm{Fr}(\ell)t|M^*(1)) = 1-t$. The classes $\mathbf{z}(m)$ thus satisfy the relation

$$\mathrm{Cor}_{\mathbb{Q}(\zeta_{m'})/\mathbb{Q}(\zeta_m)}\,\mathbf{z}(m') = \left(\prod_{\ell\in S_{m'}\setminus S_m} \det\left(1-\mathrm{Fr}(\ell)t|M^*(1)\right)_{t=\sigma(\ell)}\right)\cdot\mathbf{z}(m).$$

By definition, the system of classes

$$\left\{\mathbf{z}(m)\in H^1_f\left(G_{\mathbb{Q}(\zeta_m),\Sigma},\mathbb{Z}_p(1)\right)\right\}_{m\in\Xi} \tag{5}$$

is thus a cyclotomic Euler system.

For early accounts of this paradigmatic example of Euler systems and how it can be used to prove bounds on class groups in finite sub-extensions of $\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}$ and Iwasawa Main Conjectures, see (Perrin-Riou, 1990, 1998; Rubin, 2000) among many references.

Definition 1.2. — *Here $K/\mathbb{Q}$ is a quadratic imaginary extension and T is a self-dual Galois representation. Denote by $K(m)$ the ring-class field of conductor m (an abelian extension of K which is dihedral over $\mathbb{Q}$). An* anticyclotomic Euler system *for T is a system of classes*

$$\left\{\mathbf{z}(m)\in H^1_f\left(G_{K(m),\Sigma},T\right)\right\}_{m\in\Xi} \tag{6}$$

satisfying the following compatibility relation for corestriction

$$\mathrm{Cor}_{K(m\ell)/K(m)}\,\mathbf{z}(m\ell) = \Phi(\ell)\mathbf{z}(m) \tag{7}$$

whenever $\ell\nmid m$ is inert in K (note that this implies that $\ell\notin\Sigma$ unless $\ell=p$), where Φ is an expression linked with the Euler factor of T at ℓ (in important families of early examples, $\Phi(\ell)$ is the trace of $\mathrm{Fr}(\ell)$ *the geometric Frobenius morphism at ℓ acting on T).*

For early accounts of the typical example of anticyclotomic Euler systems - the Euler system of Heegner points - and how it can be used to prove bounds on Tate–Shafarevich groups, see (Gross, 1991; Perrin-Riou, 1990) among many references.

Because the system of zero classes obviously satisfies the requirements of definitions 1.1 and 1.2, an Euler system is an interesting object only if it contains a non-zero cohomology class, though we resist putting that requirement explicitly in the definitions because proving that an Euler system is non-zero has proven to be one of the hardest problems of the subject.

1.2. Euler systems in arithmetic geometry

The axiomatic definitions above makes no reference to arithmetic or geometry. Indeed, they raise two natural questions.

1. Why should anyone care about Euler systems?
2. Why should non-trivial Euler systems exist?

1.2.1. *What are Euler systems good for?* — The answer to the first question was provided by two groundbreaking insights of Kolyvagin (1990). His first contribution was to show that Euler systems gave rise to what has been called Kolyvagin systems after Mazur and Rubin (2004), that is to say new systems of cohomology classes whose local behavior forms a cascade of relations: typically, if $(\kappa(m\ell), \kappa(m))$ is a pair of classes in a Kolyvagin system, then the class $\kappa(m)$ is unramified at ℓ, the class $\kappa(m\ell)$ is ramified at ℓ and the localization of $\kappa(m\ell)$ at ℓ may be elucidated in terms of the localization of $\kappa(m)$. More precisely, we have the following definition (see Mazur and Rubin (2004, Definition 3.1.3) and Howard (2004b, Definition 1.2.3) for details).

Definition 1.3 (Kolyvagin (1990), Mazur and Rubin (2004), and Howard (2004b)) *A* cyclotomic Kolyvagin system *for T is a system of classes*

$$\left\{\kappa(m) \in H^1_{\mathcal{F}(m)}(G_{\mathbb{Q},\Sigma}, T/I_nT) \otimes G_m\right\}_{m\in\Xi,\textit{square-free}} \tag{8}$$

such that

1. $\mathrm{loc}_\ell\,\kappa(m)$ *is unramified if* $\ell \nmid m$.
2. $\mathrm{loc}_\ell\,\kappa(m\ell)$ *belongs to* $H^1_s(G_{\mathbb{Q}_\ell}, T/I_{m\ell}T) \otimes G_{m\ell}$ *and*

$$\phi^{fs}_\ell(\mathrm{loc}_\ell\,\kappa(m)) = \mathrm{loc}_\ell\,\kappa(m\ell) \in H^1_s(G_{\mathbb{Q}_\ell}, T/I_{m\ell}T) \otimes G_{m\ell}. \tag{9}$$

Note that because $m\ell$ is square-free, here alsi $\ell \nmid m$.

Here I_m is a suitable define ideal, G_m is the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$*, the subscripts indicate suitable Galois cohomology conditions and ϕ^{fs}_ℓ is a local comparison map between singular and unramified cohomology.*

An anticyclotomic Kolyvagin system *is defined in the same way for T self-dual, with $\mathbb{Q}$ replaced with a quadratic imaginary extension K, with m a square-free product of inert primes and with G_ℓ replaced with $\mathbb{F}^\times_\lambda/\mathbb{F}^\times_\ell$ for $\lambda \subset \mathcal{O}_K$ the only prime ideal over a rational prime ℓ inert in $\mathcal{O}_K$.*

Kolyvagin's second contribution was to show that such a cascade of local relations bounded and often determined fully the Bloch–Kato cohomology groups attached to T. Applying his method to the anticyclotomic system of Heegner points, he obtained the finiteness of the Tate–Shafarevich group of many elliptic curves (Kolyvagin, 1990), see section 2 for further details. Simultaneously, Rubin observed that the bounds obtained from Kolyvagin systems were sufficiently uniform in a cyclotomic deformation and would therefore go a long way towards proving Iwasawa Main Conjectures. In this way, he dervied a new proof of the classical Iwasawa Main Conjecture using the cyclotomic Euler system of cyclotomic units and of the Iwasawa Main Conjecture in the $\mathbb{Z}_p^2$-extension of a quadratic imaginary field using the Euler system of of elliptic units (Rubin, 1991), see section 2 again for further details.

Since then, Euler systems have been used to achieve a variety of remarkable results: proving properties of Mazur–Tate–Teitelbaum $\mathscr{L}$-invariants (Kato, Kurihara, and Tsuji, 1997), showing the finiteness of $\text{Ш}(E/\mathbb{Q})[p^\infty]$ even when E has large algebraic rank

(Perrin-Riou, 2003), obtaining parity resuls on the order of vanishing of L-functions (Nekovář, 2013), constructing new p-adic L-functions (Bertolini, Darmon, and Prasanna, 2013), constructing rational points on CM elliptic curves by p-adic methods (Burungale, Kobayashi, and Ota, 2024), etc... They have been an essential component of most of the proofs of Iwasawa Main Conjectures and Equivariant Tamagawa Number Conjectures known to date. This amply justifies the quest for new Euler systems.

In this survey, we henceforth concentrate on how to construct them.

1.2.2. *Why do Euler systems exist?* — The second question has been the object of considerable interest ever since the first seminal works on Euler systems by Thaine (1988), Kolyvagin (1990), Rubin (1991), Kolyvagin (1991b), Kolyvagin and Logachëv (1991), Flach (1992), and Nekovář (1992). For instance, Kato (1993b) made the following remark in his inimitable style.

> It seems to me that only known general method to discover such important elements is to open our mouths and wait for such elements to drop from the sky. However l do not know why these people [(2)] with small mouths can catch such elements so often.

An ambition of this report is to explain that multiple answers have emerged after developments of the last two decades, beyond waiting for such elements to drop from the sky.

As recalled in the brief historical section below, the chronologically first answer is due to Kato himself, who pointed out in Kato (1993a) that if the conjectures of Bloch–Kato on special values of L-functions of motives (Bloch and Kato, 1990) were true for motives with coefficients in $\mathbb{Q}[\mathrm{Gal}(K/\mathbb{Q})]$ and compatible with changes of extension (what has been called the equivariant refinement), then systems of classes satisfying the norm relations of definition 1.1 should exist in a very broad sense. Interestingly, these class do not live in $H^1_f(G_{\mathbb{Q}(\zeta_m),\Sigma}, T)$ in general but in the so-called p-adic fundamental line of the Galois representation M, that is to say the E-vector space of dimension 1

$$\mathrm{Det}_E^{-1} \mathrm{R}\,\Gamma_{\mathrm{et}}\,(\mathbb{Z}[\zeta_m, 1/p], M) \otimes_E \mathrm{Det}_E^{-1} M(-1)^+. \tag{10}$$

Here $(-)^+$ means the subspace invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, Det_A denotes the determinant functor from the category of perfect complexes of projective A-modules to the category of graded invertible A-modules (Knudsen and Mumford, 1976) and $\mathrm{R}\,\Gamma_{\mathrm{et}}\,(\mathbb{Z}[\zeta_m, 1/p], -)$ is the étale cohomology complex functor $\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Q}(\zeta_m), j_*-)$ j for the morphism $j\colon \mathrm{Spec}\,\mathbb{Q}(\zeta_m) \longrightarrow \mathrm{Spec}\,\mathbb{Z}[\zeta_m]$. In important early examples of Euler systems (cyclotomic units, elliptic units, Heegner points), it turns out that the étale cohomology complex $\mathrm{R}\,\Gamma_{\mathrm{et}}\,(\mathbb{Z}[\zeta_m, 1/p], M)$ is concentrated in degree 1 and that $H^1_{\mathrm{et}}\,(\mathbb{Z}[\zeta_m, 1/p], M)$ is of dimension 1 and thus isomorphic to $\mathrm{Det}_E^{-1} \mathrm{R}\,\Gamma_{\mathrm{et}}\,(\mathbb{Z}[\zeta_m, 1/p], M)$. In agreement with Kato's prediction, systems of classes therefore appear in the first cohomology group of T and thus there are Euler systems for T in the sense of the axiomatic definitions above.

2. That is to say S. Bloch and A. Beilinson.

At least two supplementary answers have appeared since then. One was suggested by the observation that an Euler system for a Galois representation (T, ρ) tends to extend to a large Galois deformation of the residual representation $(\overline{T}, \bar{\rho})$ (Howard (2007) and Nakamura (2023), see section 4 below). In particular, it is often possible to construct Euler systems for Galois representations in a situation where a direct link with algebraic cycles or related geometric objects would be very hard to establish by congruences with other Galois representations. A prototypical example of this phenomenon is the construction of Euler systems of CM points for nearly-ordinary eigencuspforms of any weight (Fouquet, 2013). Yet another answer came with the understanding that Euler systems reflected automorphic formulas for the special values of L-functions of automorphic objects in geometry. For instance, classical [3] examples of Euler systems (cyclotomic units, elliptic units, CM points, Kato's Euler system) corresponded respectively to Dirichlet's Class formula for cyclotomic extensions, Kronecker's limit formula for quadratic imaginary fields, the Gross–Zagier formula and its generalizations to totally real fields (Gross and Zagier, 1986; Zhang, 2001; Yuan, Zhang, and Zhang, 2013), and Shimura's and Beilinson's formula (Shimura, 1976; Beĭlinson, 1986) for the L-function of the product of an eigencuspform with two Eisenstein series. Since then, it has been understood that the triple product formula for the product of three eigencuspforms (Garrett, 1987; Gross and Kudla, 1992; Harris and Kudla, 1991) was a better framework to understand even classical Euler systems than systems of norm-compatible classes as in our previous axiomatic definitions (see Bertolini, Castella, Darmon, Dasgupta, Prasanna, and Rotger, 2014), and in fact that several other formulas of automorphic origins should have Euler systems counterparts. In section 5 below, we explain for instance how the global Gan–Gross–Prasad conjecture may be used to translate the non-vanishing of an automorphic L-function into the construction of some sort of Euler systems. Recently, it has been understood that doubling integral formulas and Piatetski-Shapiro's formula (Piatetski-Shapiro, 1997) also had Euler systems counterparts (Lemma, 2017; Loeffler, Skinner, and Zerbes, 2022). More generally, and as envisioned in a sense by Bloch and Beilinson even before Euler systems were considered (Beĭlinson (1986), what Kato called *opening our mouths and waiting for such elements to drop from the sky*), it seems to be gradually understood that Euler systems arise whenever there is a an inclusion of reductive groups $\mathbf{H} \hookrightarrow \mathbf{G}$ which translates into explicit formulas for L-functions.

It is very tempting to unify the two previous answers, either by considering p-adic L-functions in place of analytic L-functions in the approach of the last paragraph or in employing p-adic and deformation-theoretic methods to translate automorphic formulas into the geometric expression of the Euler systems. The most complete achievement known to this author in that direction is Colmez and Wang (2021), in which p-adic methods are used to express Kato's Euler system in the completed cohomology of the tower of modular curves as a a product of modular symbols, directly mimicking the Rankin–Selberg convolution formula (Shimura, 1976) in an arithmetic setting. As

3. For the purpose of this report, *classical* Euler systems are those constructed in the last century.

I comment in slightly more detail at the beginning of section 4, such translation of automorphic formulas using the geometry and cohomology properties of perfectoid Shimura varieties seems to me to be the correct way forward in the construction and study of Euler systems. Seeing the correct direction, however, is the easy part.

1.3. Acknowledgment

It is a pleasure to acknowledge innumerable discussions on the topic of Euler systems with Christophe Cornut, Francesco Lemma, Tadashi Ochiai and Xin Wan as well as the help I received in the preparation of this report from Raphaël Beuzart-Plessis, Murilo Corata Zanarella, Daniel Disegni and Joaquin Rodrigues Jacinto.

I remember vividly when I first heard the words *systèmes d'Euler*. In autumn 2003, I was advised by J-F. Mestre to learn about Euler systems and then go see Jan Nekovář, with the idea that he would be interested in supervising a master thesis and then a PhD. thesis on that topic. Everything I know about Euler systems, I have learned from trying to extract meaning from the sharp tidbits Jan Nekovář would occasionally impart in offhand remarks. If this text contains any insight on the topic, they came from him in this way. All errors remain of course mine, and Jan himself would be the first to list them but for his premature passing away.

2. Historical survey

This section briefly surveys important historical steps in the history of the construction of Euler systems in arithmetic geometry. Its aim being to sketch the historical development of the topic, mathematical arguments are sometimes exceedingly terse.

2.1. Early history

2.1.1. *The beginnings.* — As with every important subject, the history of Euler systems starts several decades before the object in question got a name. The insight that would become a century later the prototypical examples of Euler and Kolyvagin systems, namely that cyclotomic units and Gauss sums led to explicit factorization in group-rings, is due to Kummer (1850, 1855) and Stickelberger (1890). The question of how to determine the full structure of class groups of cyclotomic extensions using special systems of cyclotomic units is already explicitly raised in posthumous work of Herbrand (1932). Iwasawa (1964) observed that the Iwasawa Main Conjecture for towers of cyclotomic extensions could be stated in terms of special elements. Coleman (1979) established a direct link between norm-compatible systems of classes and p-adic L-functions and De Shalit (1987) noticed that elliptic units satisfied the norm-relations which now characterize Euler systems .

The history of the subject properly began when Thaine (1988) discovered that explicit local properties of factorizations in group-rings yielded explicit annihilators of class groups of real abelian extensions and when Rubin understood immediately that his idea could be enhanced to give a new proof of the Iwasawa Main Conjecture (thereby completely answering Herbrand's question) and a proof of the Iwasawa Main Conjecture for quadratic imaginary fields using the norm-relations satisfied by elliptic units (Rubin, 1991). Then, Kolyvagin coined the name *Euler system* and came to the striking realization that the methods of Kummer, Stickelberger, Thaine and Rubin could be pushed much further and with a much weaker starting point, effectively showing that it was enough to have a cascade of local compatibilities between cohomology classes to determine the structure of Selmer and Tate–Shafarevich group.

By showing that Heegner points form an Euler system, he proved the first theorem establishing the finiteness of the Tate–Shafarevich group for elliptic curves without complex multiplication, thereby immediately establishing Euler systems as objects of intense interest. Let more precisely $X_0(N)$ be the compact modular curve parametrizing cyclic N-isogeny between elliptic curves. Let E be a rational [(4)] elliptic curve of conductor N together with a modular parametrization $\pi\colon X_0(N) \longrightarrow E$.Let $K/\mathbb{Q}$ be a quadratic imaginary field such that every $\ell|N$ splits in K and let $\mathcal{N}$ be one of the two ideals of the ring of integers $\mathcal{O}_K$ of K defined by the factorization $N\mathcal{O}_K = \mathcal{N}\bar{\mathcal{N}}$. Let $x(1) \in X_0(N)(K^{ab})$ be the *Heegner point* of conductor 1, that is to say the point attached to the degree N isogeny $[\mathbb{C}/\mathcal{O}_K \longrightarrow \mathbb{C}/\mathcal{N}^{-1}]$ on the modular curve $X_0(N)$. Let $y(1) \in E(K)$ be the trace to K of $\pi(x(1))$. Finally, recall that the Tate–Shafarevich group $\text{Ш}(E/K)$ of E is the group

$$\text{Ш}(E/K) \stackrel{\text{def}}{=} \bigcap_v \operatorname{Ker}\left(H^1(G_K, E(\bar{K})) \longrightarrow H^1(G_{K_v}, E(\bar{K}_v))\right)$$

where the intersection is taken over all places of K.

Kolyvagin showed that $y(1)$ seen as a class in $H^1_f(G_{K,\Sigma}, T_pE)$ through the Kummer map is the first class of an Euler system and proved the following theorem.

Theorem 2.1 (Kolyvagin, 1990, 1991b). — *If $y(1) \notin E(K)_{\text{tors}}$, then both $[E(K) : \mathbb{Z} \cdot y(1)]$ and $|\text{Ш}(E/K)|$ are finite.*

2.1.2. *First axiomatization and Iwasawa theory.* — Soon after Rubin's and Kolyvagin's works, the theory of Euler systems took axiomatic form in Perrin-Riou, 1990, then in the three main articles Perrin-Riou, 1998; Kato, 1999a; Rubin, 2000, which all drew attention to the fact that the method of Euler systems could potentially work for general Galois representations (instead of Galois representations attached to characters or elliptic curves, as in examples known at the time) as well as their connection with Iwasawa theory. Simultaneously, Kato (1993a) observed that any motive M over $\mathbb{Q}$ should admit a cyclotomic Euler system, if a sufficiently general version of the Bloch–Kato conjectures was true. Here is a quick summary of the argument.

Let M be a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$ for simplicity of exposition and with good reduction outside Σ (by a motive M, we mean here a set $\{M_B, M_{\text{dR}}, \{M_{\text{et},p}\}_p\}$ of Betti realization, de Rham realization and p-adic étale realization for all prime p

4. At the time, this theorem was of course stated for *modular* rational elliptic curves.

related by comparison theorems and having a meromorphic L-function independent of the choice of the auxiliary prime ℓ - for concreteness, the reader may take the motive attached to an abelian variety or the motive attached to an eigencuspform in Scholl, 1990). Let $m|m'$ be two integers in Ξ. We consider the motives $M_m = M \times_{\mathbb{Q}} \mathbb{Q}(\zeta_m)$, $M_{m'} = M \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{m'})$ and their duals. Denote by G_m the Galois group of $\mathbb{Q}(\zeta_m)/\mathbb{Q}$. Denote by $a_n(M_m^*(1))$ the n-th coefficient of the L-function of $M_m^*(1)$.

The motive $M_m^*(1)$ with coefficients in $\mathbb{Q}[G_m]$ has a G_m-equivariant L-functions defined by

$$L_{\Sigma\cup S_m}^{G_m}(M^*(1),s) \overset{\text{def}}{=} \sum_{\sigma\in G_m} L_{\Sigma\cup S_m}(M_m^*(1),\sigma,s)\cdot\sigma^{-1} \tag{11}$$

with

$$L_{\Sigma\cup S_m}(M^*(1),\sigma,s) \overset{\text{def}}{=} \sum_{\chi_{\text{cyc}}(n)=\sigma^{-1}} a_n(M_m^*(1))\, n^{-s}. \tag{12}$$

The conjectures of Bloch and Kato, 1990 were reformulated in Fontaine, 1992; Kato, 1993a,b; Fontaine and Perrin-Riou, 1994 in terms of two motivic data: the fundamental line $\Delta(M\times\mathbb{Q}(\zeta_m))$, which is a $\mathbb{Q}[G_m]$-module projective of rank 1, and the zeta element $\mathbf{z}(M) \in \Delta(M\times\mathbb{Q}(\zeta_m))$. The fundamental line conjecturally comes with a canonical complex period isomorphism

$$\text{per}_{\mathbb{R}}\colon \Delta(M\times\mathbb{Q}(\zeta_m))\otimes_{\mathbb{Q}}\mathbb{R} \overset{\text{can}}{\xrightarrow{\sim}} \mathbb{R}[G_m] \ni L_{\Sigma\cup S_m}^{G_m}(M_m^*(1),0)\,. \tag{13}$$

Beilinson's conjecture predicts that

$$\mathbf{z}(M) \overset{\text{def}}{=} \text{per}_{\mathbb{R}}^{-1}\left(L_{\Sigma\cup S_m}^{G_m}(M^*(1),0)\right), \tag{14}$$

which *a priori* belongs to $\Delta(M\times\mathbb{Q}(\zeta_m))\otimes_{\mathbb{Q}}\mathbb{R}$, actually belongs to the rational subspace $\Delta(M\times\mathbb{Q}(\zeta_m))$. Define $\Delta(M\times\mathbb{Q}(\zeta_m))_{\mathbb{Q}_p}$ to be

$$\text{Det}_{\mathbb{Q}_p[G_m]}^{-1}\,\text{R}\,\Gamma_{\text{et}}\left(\mathbb{Z}[\zeta_m,\frac{1}{\Sigma\cup S_m}],M_{\text{et},p}\right)\otimes_{\mathbb{Q}_p[G_m]}\text{Det}_{\mathbb{Q}_p[G_m]}^{-1}\left(M_m(-1)^+\right) \tag{15}$$

The fundamental line conjecturally comes with a canonical p-adic period isomorphism

$$\text{per}_p\colon \Delta(M\times\mathbb{Q}(\zeta_m))\otimes_{\mathbb{Q}}\mathbb{Q}_p \overset{\text{can}}{\xrightarrow{\sim}} \Delta(M\times\mathbb{Q}(\zeta_m))_{\mathbb{Q}_p} \tag{16}$$

The trace map from $\mathbb{Z}[\zeta_{m'}]$ to $\mathbb{Z}[\zeta_m]$ sends

$$L_{\Sigma\cup S_{m'}}(M_{m'}^*(1),\sigma,s) \tag{17}$$

to

$$\prod_{\ell\in S_{m'}\setminus S_m}(\det(1-\text{Fr}(\ell)t|M_m^*(1))_{t=\ell^{-s}})\,L_{\Sigma\cup S_m}(M_m^*(1),\sigma,s) \tag{18}$$

If Beilinson's conjecture is true for $M\times\mathbb{Q}(\zeta_m)$ and $M\times\mathbb{Q}(\zeta_{m'})$, then writing explicitly the action of $\text{Fr}(\ell)$ on $M_m^*(1)$, we obtain

$$\text{Cor}_{\mathbb{Q}(\zeta_{m'})/\mathbb{Q}(\zeta_m)}\,\mathbf{z}(m') = \left(\prod_{\ell\in S_{m'}\setminus S_m}\det(1-\text{Fr}(\ell)t|M^*(1))_{t=\sigma(\ell)}\right)\cdot\mathbf{z}(m). \tag{19}$$

If the conjecture is true for all $m \in \Xi$, the system of zeta elements

$$\{\mathbf{z}(m) \in \Delta(M \times \mathbb{Q}(\zeta_m))\}_{m\in\Xi} \tag{20}$$

and its image

$$\left\{\mathbf{z}(m)_p \in \Delta(M \times \mathbb{Q}(\zeta_m))_{\mathbb{Q}_p}\right\}_{m\in\Xi} \tag{21}$$

through the p-adic period isomorphism therefore satisfy compatibility relations highly reminiscent of those defining an Euler system, though $\mathbf{z}(m)$ and $\mathbf{z}(m)_p$ live respectively in fundamental lines and in determinants of étale cohomology complexes rather than in $H^1(G_{\mathbb{Q}(\zeta_m),\Sigma}, M)$ as they should according to definition 1.1.

The first result known to this author using the idea of Kolyvagin's systems which follows Kolyvagin but is not a direct continuation of works preceding him is Flach, 1992. Let $E/\mathbb{Q}$ be an elliptic curve and let $V = \operatorname{Sym}^2 H^1_{\text{et}}(E \times \bar{\mathbb{Q}}, \mathbb{Q}_p)(2)$ be the symmetric square of the p-adic Galois representation attached to E. Following the original idea of Bloch and Beilinson, Flach uses the product of the modular curve $X_0(N)$ with itself and the algebraic cycles it induces in order to construct system of classes satisfying the local conditions typical of the first layer of Kolyvagin's construction and then to obtain a vanishing result for the first Bloch–Kato cohomology group of V.

2.2. Euler systems for eigencuspforms

Let $f \in S_k(\Gamma_0(N))$ be an eigencuspform of weight $k \geq 2$. In the 1990s, two groundbreaking works showed that the p-adic Galois representation $V(f)$ attached to f admitted Euler systems.

2.2.1. *The Euler system of CM cycles.* — The first to be published was Nekovář, 1992 which showeds that, just like eigencuspforms of weight 2 have an Euler system of Heegner points, $V(f)$ admitted an *anticyclotomic* Euler system arising from the images under the Abel–Jacobi map of Heegner or CM cycles on the Kuga-Sato variety whose cohomology contains $V(f)$ as a quotient (the idea of such CM cycles was introduced in Schoen, 1986). With that work would arise a problem that would recur in the history of Euler systems: Nekovář could show that a system of cycles gave rise to a system of cohomology classes

$$\left\{\mathbf{z}(m) \in H^1_f(G_{K(m),\Sigma}, T(f))\right\}_{m\in\Xi} \tag{22}$$

satisfying the norm relation of definition 1.2 for $\Phi(\ell) = \operatorname{tr}\operatorname{Fr}(\ell)$. From that, he could deduce that $H^1_f(G_{K,\Sigma}, V(f))$ is of dimension 1 as well as a bound on the p-part of an analogue of the Tate–Shafarevich group of $V(f)$ *provided* the first class $\mathbf{z}(1)$ of his Euler system was non-trivial.

Proving the non-triviality of $\mathbf{z}(1)$, however, proved to be a very hard problem. It is possible to link Heegner cycles to special values of p-adic L-functions (Nekovář, 1995; Bertolini, Darmon, and Prasanna, 2013) but the fact that these special values of p-adic L-functions are non-zero is itself a highly non-trivial result. A related question is whether the Euler system of Heegner points of Kolyvagin or of Heegner cycles of Nekovář is generically non-trivial, that is to say whether there exists a class $\mathbf{z}(m)$ which is non-zero. In the original case of Heegner points, Mazur (1984) conjectured that there existed non-zero classes $\mathbf{z}(mp^r)$ provided r be sufficiently large and Kolyvagin (1991a) conjectured that there are non-trivial $\mathbf{z}(m)$ with $m = \ell_1 \cdots \ell_r$ square-free and r sufficiently large. Both conjectures are now known to hold: the first by Cornut (2002), Vatsal (2002), and Cornut and Vatsal (2005), the second by Zhang (2014). Unfortunately, the generic non-vanishing of the Euler system of CM cycles is harder still, because even generically non-trivial cycles might have zero image through the Abel–Jacobi map. It was finally achieved 25 years after its construction in Burungale (2020, 2017).

2.2.2. *Kato's Euler system.* — The other work pertaining to Euler systems for eigen-cuspforms was the awe-inspiring work Kato (2004) constructing a *cyclotomic* Euler system

$$\{\mathbf{z}(m) \in H^1_f\left(G_{\mathbb{Q}(\zeta_m),\Sigma}, V(f)\right)\}_{m\in\Xi} \tag{23}$$

for $V(f)$. Here is a brief account of the construction (see Kato, 2004; Colmez, 2004 for details).

Let c, d, N and M be integers such that $c \wedge 6N = d \wedge 6M = 1$ [5]. We recall that the universal elliptic curve E over the affine modular curves $Y(N)$ of full-level N admits a modular unit ${}_c\theta_E \in \mathcal{O}\left(E\backslash E[c]\right)^\times$. The reader may get an idea of the construction and properties of ${}_c\theta_E$ by recalling that its evaluation at the complexe elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ (with $\Im\tau > 0$) and $z \in \mathbb{C}\backslash c^{-1}(\mathbb{Z} + +\tau\mathbb{Z})$ is given by

$${}_c\theta(\tau, z) = q^{\frac{c^2-1}{12}}(-t)^{\frac{c-c^2}{2}} \frac{\left(\prod\limits_{n=0}^{\infty}(1-q^n t)\prod\limits_{n=1}^{\infty}(1-q^n t^{-1})\right)^{c^2}}{\prod\limits_{n=0}^{\infty}(1-q^n t^c)\prod\limits_{n=1}^{\infty}(1-q^n t^{-c})}, q = e^{2\pi i\tau}, t = e^{2\pi i z}.$$

The construction of Kato's Euler system starts from a pair of modular units ${}_c\theta_E \in \mathcal{O}\left(E\backslash E[c]\right)^\times$, ${}_d\theta_E \in \mathcal{O}\left(E\backslash E[d]\right)^\times$ on the universal elliptic curves over the affine modular curves $Y(N)$ of full-level N and $Y(M)$ of full-level M respectively. Let

$${}_cg_{0,1/N} \overset{\text{def}}{=} \iota^*_{0,1/N}\left({}_c\theta_E\right),\ {}_dg_{1/M,0} \overset{\text{def}}{=} \iota^*_{1/M,0}\left({}_d\theta_E\right) \tag{24}$$

be respectively the pullbacks of ${}_c\theta_E$ to $\mathcal{O}(Y(N))^\times$ and ${}_d\theta_E$ to $\mathcal{O}(Y(M))^\times$ through the second and first distinguished section giving level-structure to $Y(M, N)$. Finally, let ${}_{c,d}\mathbf{z}_{M,N}$ be the Steinberg symbol

$${}_{c,d}\mathbf{z}_{M,N} \overset{\text{def}}{=} \{g_{1/M,0}, g_{0,1/N}\} \in K_2(Y(M,N)) \tag{25}$$

of the two Siegel units ${}_cg_{0,1/N}$ and ${}_dg_{1/M,0}$. Since $Y(M, N)$ is an affine curve, the Hochschild–Serre spectral sequence induces an isomorphism of functors

$$H^2_{\text{et}}(Y(M,N), -) \simeq H^1(G_{\mathbb{Q}}, H^1_{\text{et}}(Y(M,N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, -)) \tag{26}$$

5. Here $a \wedge b$ is the positive generator of the ideal (a, b).

Combining this fact with the Chern class map, with the projection coming from the covering

$$Y(M,N) \longrightarrow Y_1(N) \otimes \mathbb{Q}(\zeta_m)$$

for suitable choices of M and m, with Soulé twisting and projection to the Hecke-quotient attached to the eigencuspform f allows for the construction of the classes ${}_{c,d}\mathbf{z}(m) \in H^1_f(G_{\mathbb{Q}(\zeta_m),\Sigma}, V(f))$ of Kato's Euler system.

As in the discussion of the previous paragraph, the hardest part of Kato (2004) is to show that $\mathbf{z}(mp^r) \in H^1_f(G_{\mathbb{Q}(\zeta_{mp^r}),\Sigma}, V(f))$ is non-trivial provided r is large enough. In *loc. cit.*, this is achieved by relating

$$\exp^* \circ \operatorname{loc}_p \mathbf{z}(mp^r) \in D_{\mathrm{dR}}(V(f)) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{mp^r}) \tag{27}$$

to a non-zero, critical, special value of the L-function $L_{\{p\}}(M(f)^*(1), \chi, s-k)$ for χ a character of order p^r of $\mathrm{Gal}(\mathbb{Q}(\zeta_{mp^r})/\mathbb{Q})$ (Kato, 1999b), see also the Bourbaki report Colmez (2004).

Kato's construction would prove immensely influential. Indeed the statement that all cyclotomic Euler systems known before the 2020s are variations from this construction is not far from the truth.

2.3. After Kato

Since Kato (2004), progresses on the construction of Euler systems were manifold. Here is a too brief and too partial survey of a few directions.

2.3.1. *Euler systems with general coefficients.* — On the axiomatic front, B. Mazur, Rubin and B. Howard greatly improved the axiomatic underpinnings of the method of Euler systems by identifying precisely the axiomatic properties characterizing a Kolyvagin system (Mazur and Rubin, 2004; Howard, 2004b, 2006). These works (as well as Nekovář (2007), where an axiomatic framework is proposed with very weak assumptions) showed that the conclusion of the method of Euler system could often be obtained starting with system of classes loosely resembling Euler systems in the strict sense of the axiomatic definitions of the introduction. In a related direction, Ochiai (2005, 2006) showed that the method Euler systems could be used to deduce Iwasawa Main Conjectures not only for usual Iwasawa algebras attached to $\mathbb{Z}_p$-extensions but more generally for regular rings of large dimension (see also the independent Howard, 2004a).

As explained in section 4 below, Howard (2007) gave the first published [(6)] account known to this author of the interpolation of an Euler system to a p-adic family of automorphic forms by showing that the Euler system of Heegner points extended to Hida families of self-dual twists of eigencuspforms (see Hida (1986) for the Hida families of eigencuspforms and their Galois representations and Nekovář and Plater (2000) for

6. In the early 2000s, it was well-known to experts that Kato's Euler system extended to Hida families of eigencuspforms. This result is implicit in Fukaya (2003), but I don't know where it was first explicitly proved.

the definition of the relevant twisted self-dual Hida family). This was the first of a long series of such endeavors whose current culmination is the proof in Nakamura (2023) of the fact that Kato's Euler system extends to the universal deformation ring of a modular, absolutely irreducible residual representation

$$\bar{\rho}\colon G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p) \tag{28}$$

satisfying mild p-adic Hodge-theoretic technical hypotheses after restriction to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

2.3.2. *Bipartite Euler systems.* — In a breakthrough that is explained in section 5.1 below, Bertolini and Darmon, 2005 showed that elliptic curves admit a so-called bipartite Kolyvagin system of CM points (though not an Euler systems strictly speaking) even when $L(E/K,1) \neq 0$ (an hypothesis which precludes the existence of a non-trivial Euler system of Heegner points of Kolyvagin). This was the first of a series of work constructing Euler or Kolyvagin systems by various parametrization of Shimura varieties.

Aflalo and Nekovář (2010) observed that the Gross–Prasad conjecture could be used to bypasss delicate seemingly *ad hoc* computations and provided a conceptual explanation for the existence of Euler systems. This was exploited in the spectacular achievement of Y. Liu, Y. Tian, L. Xiao, W. Zhang and X. Zhu. Relying on most of the ideas presented in this survey, they used the Gan–Gross–Prasad to link Galois representations attached to Rankin–Selberg automorphic representations of $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$ with the algebraic geometry of unitary Shimura varieties, then showed that the bad reduction of these Shimura varieties provided system of diagonal classes which controlled Bloch–Kato cohomology groups (Liu, Tian, Xiao, Zhang, and Zhu, 2022). See section 5.2 for a an account of this impressive work.

2.3.3. *Diagonal cycles.* — Another breakthrough was achieved when it was understood in the series of articles Darmon and Rotger (2014), Bertolini, Darmon, and Rotger (2015a,b), and Darmon and Rotger (2017) that special algebraic cycles in the product of three Kuga–Sato varieties (the so-called Gross–Kudla–Schoen cycles) were linked to special values of L-functions and were sufficiently close to forming an Euler system to obtain bounds on Bloch–Kato cohomology groups (in fact, as alluded to in the introduction, Bertolini, Castella, Darmon, Dasgupta, Prasanna, and Rotger (2014) defended that all classical Euler systems were shadows of the Euler system of Gross–Kudla–Schoen cycles). One remarkable theorem obtained in this way is as follows.

Let $f \in S_2(\Gamma_0(N))$ be an eigencuspform with rational coefficients, let ρ_f be its p-adic Galois representation and let $E/\mathbb{Q}$ be the elliptic curve corresponding to f. Let $g \in S_1(\Gamma_1(N_g),\chi)$ and $h \in S_1(\Gamma_1(N_h),\chi^{-1})$ be two weight one eigencuspforms and write $\rho_g\colon G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_2(L)$ and $\rho_h\colon G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_2(L)$ be the Artin representations attached to them (Deligne and Serre, 1974). Let H be the Galois group of the extension cut out by the Artin representation

$$\rho \overset{\mathrm{def}}{=} \rho_g \otimes \rho_h\colon G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_4(L). \tag{29}$$

Let $E(H)^\rho$ be the ρ-isotypical component of the Galois-module $E(H)$, that is to say $\mathrm{Hom}_{G_{\mathbb{Q},\Sigma}}(\rho, E(H)\otimes_{\mathbb{Z}} L)$. Finally, assume that $N_f \wedge N_g N_h = 1$.

Theorem 2.2 (Darmon and Rotger, 2017). — *If* $L(\rho_f\otimes\rho_g\otimes\rho_h, 1) \neq 0$, *then* $E(H)^\rho = 0$.

As an example of the use of this theorem, note the following corollary. Let $K/\mathbb{Q}$ be a a quintic extension of type A_5 which is not totally real. Assume that primes dividing N are unramified in $\mathcal{O}_K$ and that $\mathrm{ord}_{s=1} L(E/K, s) = \mathrm{ord}_{s=1} L(E/\mathbb{Q}, s)$. Then, in agreement with the Birch and Swinnerton-Dyer Conjecture, $\mathrm{rank}_{\mathbb{Z}} E(K) = \mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q})$. In other words, if the L-function of E acquires no supplementary zero by passing to K, the elliptic curve E acquires no supplementary rational points over K.

As explained in section 3 below, A. Lei, D. Loeffler, G. Kings and S. Zerbes showed in a closely related development that the special cycles of Flach (1992) and Bertolini, Darmon, and Rotger (2015b) formed a genuine cyclotomic Euler system for the Galois representation $V(f)\otimes V(g)$, where f and g are eigencuspforms (Lei, Loeffler, and Zerbes, 2014; Kings, Loeffler, and Zerbes, 2017, 2020). Moreover, they were able to prove that this Euler system is non-trivial by relating it to Rankin–Selberg p-adic L-functions which are known not to vanish identically. See theorems 3.4 below for a summary of their results. Similar techniques yielded a very interesting array of various new Euler systems (for Hilbert modular surfaces Lei, Loeffler, and Zerbes, 2018, for automorphic representations of GSp_4 Loeffler, Skinner, and Zerbes, 2022...). At the time of writing of this historical survey, it remains typically hard to prove that these Euler systems are non-trivial.

2.3.4. *The future.* — Everything indicates that this report on new constructions of Euler systems comes too early and that the current decade will be marked by significant progress on that front, that we can but allude to.

Cornut has had a decade-long project of constructing Euler systems for special cycles on Shimura varieties relying on the inclusion of reductive groups

$$U(n-1,1) \hookrightarrow \mathrm{SO}(2n-2,2) \hookrightarrow \mathrm{SO}(2n-1,2). \tag{30}$$

C. Skinner and M. Vincentelli have very recently introduced a new strategy to construct Euler systems by pullback of Eisenstein cohomology classes from a large reductive group to a smaller one and have obtained in this way a provably *non-trivial* Euler system for $\mathrm{Ad}^0 T(f)$ for f an eigencuspform. D. Jetchev, Nekovář and Skinner have also announced a new way to present Kolyvagin's relations and obtained in this ways bounds in settings more general than the original one. All these new Euler systems as well as those described in this report may presumably be extended to p-adic families of automorphic forms, a property that is sure to have important consequences in Iwasawa theory.

An intriguing suggestion of Nekovář and Scholl is to consider a construction closely parallel to Kato's Euler system but which starts not with the universal elliptic curve

$$\begin{tikzcd} E \arrow[d] \\ Y(N) \end{tikzcd} \tag{31}$$

over the affine modular curve but with

$$\begin{tikzcd} \mathscr{A} \arrow[d] \\ \mathscr{Y}(N) \end{tikzcd} \tag{32}$$

where $Y(N)$ is a Hilbert modular variety over a totally real field F of degree r, $\mathscr{Y}(N)$ is the stack $[\Delta\backslash Y(N)]$ for a suitable finite index subgroup of totally positive units in $\mathcal{O}_F^\times$ of degree r and $\mathscr{A}$ is the stack of pointed r-dimensional abelian varieties with an $\mathcal{O}_F$-action. One can then mimick the definition of modular units ${}_c\theta \in \mathcal{O}(E\backslash E[c])^\times$ to obtain classes ${}_c\Theta \in H^{2r-1}_{\text{et}}(\mathscr{A}\backslash\mathscr{A}[c], \mathbb{Q}_p(r))$. As in the case of $F = \mathbb{Q}$, the cap product with $[\Delta]$ of the pushforward of ${}_c\Theta$ lies in $H^r_{\text{et}}(Y(N), \mathbb{Q}_p(r))$ and the cup-product of two such classes lives in $H^{2r}_{\text{et}}(Y(N), \mathbb{Q}_p(2r))$.

As mentioned in the brief outline of the construction of Kato's Euler system in section 2.2 above, when $F = \mathbb{Q}$, the Hochschild–Serre spectral sequence yields an isomorphism between $H^{2r}_{\text{et}}(Y(N), \mathbb{Q}_p(2r))$ and $H^1(G_{\mathbb{Q}}, H^1_{\text{et}}(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p(1)))$ which sends the cup-product ${}_c\Theta \cup_d \Theta$ to Kato's class ${}_{c,d}\mathbf{z}$. As the Galois representation $V(f)$ lives in $H^1_{\text{et}}(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p(1)))$ when f is in weight 2, we have constructed classes in the natural living place for an Euler system for $V(f)$. When $r > 1$, we obtain classes in

$$H^1(G_{\mathbb{Q}}, H^{2r-1}_{\text{et}}(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p(2r))). \tag{33}$$

However, the Galois representations attached to Hilbert eigencuspforms of parallel weight 2 can only in the middle-degree cohomology

$$H^r_{\text{et}}(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p(2r)) \tag{34}$$

of the Hilbert modular variety (Brylinski and Labesse, 1984) so that constructing classes in $H^1(G_{\mathbb{Q}}, H^{2r-1}_{\text{et}}(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p(2r)))$ seems to be uninteresting from the point of view of the arithmetic of Hilbert eigencuspforms.

In fact, taking into accounts the ideas of the end of section 1.2, this could have been guessed from the outset. Strictly speaking, the class $\mathbf{z}(m)$ of an Euler system is a class in $H^1_f(G_{\mathbb{Q}(\zeta_m),\Sigma}, T)$. Since $H^1_f(G_{\mathbb{Q}(\zeta_m),\Sigma}, T)$ is believed to be the image through the p-adic period isomorphism of motivic cohomology, it is natural to believe that $\mathbf{z}(m)$ comes from a class in some Ext^1 where extensions are computed in some putative category of motivic structures. For this class to have a chance to be non-zero and to be reasonably independent of arbitrary choices, it appears to be necessary that Ext^1 be of rank 1. This, according to Beilinson's conjecture, means that some L-function should vanish at order exactly one. Putting everything together, we arrive at the concluding remark of Bertolini, Castella, Darmon, Dasgupta, Prasanna, and Rotger, 2014 (also emphasized by Skinner) according to which the existence of a non-trivial Euler system is the arithmetic incarnation of an automorphic special value formula computing an L-function vanishing at order exactly one. When $r = 1$, we find ourselves precisely in this situation with respect to the Rankin–Selberg product formula for f and two Eisenstein series at $s = 2$, where the exact order of vanishing comes from the non-vanishing of $L(f, s)$ at $s = 0$ and the Γ-factors intervening in the functional equation. In a similar set-up with $r > 1$, Γ-factors would be responsible for a zero of order r. Hence, the relevant Ext group (if it makes sense) is believed to be of rank $r > 1$. There is thus little reason to expect an Euler system strictly speaking in that setting: the image of zeta elements through the p-adic period isomorphism would not live in the first cohomology group but in its r-th exterior power. In yet a third version of the same argument, notice that strictly speaking the Galois representation $V(f)$ does not appear in $H^r_{\text{et}}(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p(2r))$, it is its tensor induction $V(f)^{\otimes r} \otimes \mathbb{Z}[\text{Gal}(F'/F)]$ which does (here F'/F is a certain abelian extension depending on the level structure N).

Nevertheless, Nekovář and Scholl (2016) made the deep conjecture that there should exist a canonical supplementary *plectic* action of $\mathfrak{S}_r \ltimes G_F^r$ on $H^{2r}_{\text{et}}(Y(N), \mathbb{Q}_p(2r))$. Taking the Hochschild–Serre spectral sequence with respect to that plectic action (and not only the G_F-action) formally yields classes in

$$\bigwedge_{\mathbb{Q}_p[\text{Gal}(F'/F)]}^{r} H^1\left(G_{F'}, V(f)(2)\right). \tag{35}$$

Bringing in full circle the ideas of this historical survey, we notice that the space (35) is precisely where zeta elements attached to $V(f)$ and the abelian extension F'/F should live according to (14) and that it is also exactly what is believed to be the r-th exterior power of the motivic Ext group which corresponds to a zero of order r of an L-function according to Beilinson's conjecture.

This remarkable coming together of properties of motivic cohomology, of the cohomology of Shimura varieties and of arithmetic properties of special values of L-functions and zeta elements suggests that the future of Euler systems is indeed bright.

3. Rankin–Selberg products of modular forms

In this section, we describe how the original idea of Beilinson to push-forward Eisenstein motivic cohomology classes along the diagonal inclusion of the affine modular curve $Y_1(N)$ inside $Y_1(N) \times Y_1(N)$ yields a *non-trivial* cyclotomic Euler system for the Galois representation $V(f) \otimes V(g)$ attached to the Rankin–Selberg product of two eigencuspforms.

3.1. Rankin–Eisenstein classes

3.1.1. *Notations.* — Let A be a commutative ring and M be a free A-module. For $k \geq 0$, let $\mathrm{TSym}^k M$ be the submodule of $\mathfrak{S}_k$-invariants of $\bigotimes_{i=1}^{k} M$ with its natural ring structure. As usual, we denote by $Y_1(N)$ the affine modular curve of level $\Gamma_1(N)$ for $N \geq 4$ and we denote by $\pi\colon E \longrightarrow Y_1(N)$ the universal elliptic curve over $Y_1(N)$. Similar notations are used for other modular curves with different level-structures. Let

$$\Delta\colon Y(N) \hookrightarrow Y \times_{\operatorname{Spec}\mathbb{Z}[1/N]} Y(N) \tag{36}$$

be the diagonal embedding of the affine modular curve in two copies of itself and let $\pi_i\colon Y(N) \times Y(N) \longrightarrow Y(N)$ be the projection on the i-th copy.

As in Deligne (1969) and Scholl (1990), consider for $k \geq 0$ the k-th fibered product

$$E^k \overset{\text{def}}{=} E \times_Y \cdots \times_Y E \tag{37}$$

over an affine modular curve Y of the universal elliptic curve E with itself. There is a natural action of the group $\mu_2^k \rtimes \mathfrak{S}_k$ on E^k (the symmetric group permutes the factor and μ_2 acts by a sign on each factor). Let

$$\varepsilon_k\colon \mu_2^k \rtimes \mathfrak{S}_k \longrightarrow \mu_2 \tag{38}$$

be the character defined by

$$\varepsilon_k(\{\eta_1, \ldots, \eta_k\}, \sigma) = \varepsilon(\sigma)\prod_{i=1}^{k}\eta_i. \tag{39}$$

For X a regular scheme, the i-th motivic cohomology of X (Beĭlinson, 1984, Section 2.2) is the subspace

$$H^i_{\mathrm{mot}}(X, \mathbb{Q}(n)) \overset{\text{def}}{=} (K_{2n-i}(X) \otimes \mathbb{Q})^{(n)} \tag{40}$$

of K-theory of X of weight n under Adams operations. There are regulator maps from motivic cohomology to Betti, de Rham, étale p-adic and rigid syntomic cohomology. Let $H^i_{\mathrm{mot}}\left(Y, \mathrm{TSym}^k \mathcal{H}_{\mathbb{Q}}(n)\right)$ be

$$H^i_{\mathrm{mot}}\left(Y, \mathrm{TSym}^k \mathcal{H}_{\mathbb{Q}}(n)\right) \overset{\text{def}}{=} H^{i+k}_{\mathrm{mot}}\left(E^k, \mathbb{Q}(n+k)\right)(\varepsilon_k). \tag{41}$$

For ? a cohomology theory (? = mot is possible), let $\mathcal{F}_?$ be the trivial coefficient sheaf (for instance, $\mathcal{F}_B = \mathbb{Q}$, $\mathcal{F}_{\mathrm{et},p} = \mathbb{Q}_p$...). Let $\mathcal{H}_?^\vee$ be $R^1\pi_*\mathcal{F}_?$ and $\mathcal{H}_?$ be the dual sheaf $(R^1\pi_*\mathcal{F}_?)^\vee$. The image of $H^i_{\mathrm{mot}}\left(Y, \mathrm{TSym}^k \mathcal{H}_{\mathbb{Q}}(n)\right)$ through the regulator map to the cohomology theory ? is the usual cohomology group

$$H^i_?\left(Y, \mathrm{Sym}^k \mathcal{H}_?^\vee(n)\right) = H^i_?\left(Y, \mathrm{TSym}^k \mathcal{H}_?(n)\right). \tag{42}$$

Suppose k, k' are two positive integers. Let $\mathrm{TSym}^{[k,k']} \mathcal{H}_?$ be the sheaf on $Y \times Y$ defined by

$$\mathrm{TSym}^{[k,k']} \mathcal{H}_? \overset{\text{def}}{=} \pi_1^*\left(\mathrm{TSym}^k \mathcal{H}_?\right) \otimes \pi_2^*\left(\mathrm{TSym}^{k'} \mathcal{H}_?\right). \tag{43}$$

Then there is a push-forward Gysin map along the diagonal

$$\Delta_*\colon H^i_?\left(Y, \mathrm{TSym}^k\mathcal{H}_? \otimes \mathrm{TSym}^{k'}\mathcal{H}_?(n)\right) \longrightarrow H^{i+2}_?\left(Y, \mathrm{TSym}^{[k,k']}\mathcal{H}_?(n+1)\right) \tag{44}$$

The source of Δ_* is the target of a motivic Clebsch–Gordan map

$$CG^{[k,k',j]}_{\mathrm{mot}}\colon H^i_{\mathrm{mot}}(Y, \mathrm{TSym}^{k+k'-2j}\,\mathcal{H}_{\mathbb{Q}}(n)) \longrightarrow H^i_{\mathrm{mot}}(Y, \mathrm{TSym}^k\,\mathcal{H}_{\mathbb{Q}} \otimes \mathrm{TSym}^{k'}\,\mathcal{H}_{\mathbb{Q}}(n-j)). \tag{45}$$

3.1.2. *Definitions.* — Let $\iota\colon Y_1(N) \hookrightarrow X_1(N)$ be the compactification of the modular curve and $\jmath\colon X_1(N)\backslash Y_1(N) \hookrightarrow X_1(N)$ be the embedding of the boundary (that is to say the set of cusps). The residue map res_∞ of Huber and Kings (1999, section 1.1) and Kings (2015a, Definition 5.1.5) is the composition of the natural map from $H^1_{\mathrm{mot}}\left(Y_1(N), \mathrm{TSym}^k\,\mathcal{H}_{\mathbb{Q}}(1)\right)$ to $H^0_{\mathrm{mot}}(\partial Y, \iota^* R^1\jmath_*\,\mathrm{TSym}^k\,\mathcal{H}_{\mathbb{Q}}(1))$ with the complex Betti realization and identification of this latter cohomology group with $\mathbb{C}$.

Proposition-Definition 3.1 (Beĭlinson and Levin, 1994)

Let $N \geq 5$ and $b \in \mathbb{Z}/N\mathbb{Z}$ be a non-zero class. For all integers $k \geq 0$, there exists a non-zero Eisenstein motivic class

$$\mathrm{Eis}^k_{\mathrm{mot},b,N} \in H^1_{\mathrm{mot}}\left(Y_1(N), \mathrm{TSym}^k\,\mathcal{H}_{\mathbb{Q}}(1)\right) \tag{46}$$

satisfying the residue formula

$$\mathrm{res}_\infty\left(\mathrm{Eis}^k_{\mathrm{mot},b,N}\right) = -N^k\zeta(-1-k). \tag{47}$$

When $k=0$, $H^1_{\mathrm{mot}}\left(Y_1(N), \mathbb{Q}(1)\right) = \mathcal{O}(Y_1(N))^\times \otimes \mathbb{Q}$ and $\mathrm{Eis}^k_{\mathrm{mot},1,N}$ is the Siegel unit $g_{0,1/N}$ introduced in section 2.2.

Definition 3.1 (Kings, Loeffler, and Zerbes, 2020). — *Let $k \geq 0, k' \geq 0$ and $0 \leq j \leq$* $\min(k,k')$ *be integers. The* Rankin–Eisenstein motivic class *is the class*

$$\mathrm{Eis}^{[k,k',j]}_{\mathrm{mot},b,N} \in H^3\left(Y_1(N) \times_{\mathbb{Z}[1/N]} Y_1(N), \mathrm{TSym}^{[k,k']}\,\mathcal{H}_{\mathrm{mot}}(2-j)\right) \tag{48}$$

defined as

$$\mathrm{Eis}^{[k,k',j]}_{\mathrm{mot},b,N} \overset{\mathrm{def}}{=} \left(\Delta_* \circ CG^{[k,k',j]}_{\mathrm{mot}}\right)\left(\mathrm{Eis}^{k+k'-2j}_{\mathrm{mot},b,N}\right). \tag{49}$$

The p-adic étale cohomology group $H^3_{\mathrm{et}}\left(Y_1(N)^2 \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathrm{TSym}^{[k,k']}\,\mathcal{H}_{\mathrm{et}}(2-j)\right)$ vanishes since $Y_1(N)^2$ is an affine surface. Hence, just as in the original case of Kato, the Hochschild–Serre spectral sequence for the composition of functors $H^0(G_{\mathbb{Q}}, -)$ and $H^0_{\mathrm{et}}(- \times_{\mathbb{Q}} \bar{\mathbb{Q}}, -)$ induces an isomorphism which allows us to view the p-adic étale realization (over $\mathbb{Q}$) of $\mathrm{Eis}^{[k,k',j]}_{\mathrm{mot},b,N}$ as an element

$$\mathbf{z}([k,k',j,b,n]) \in H^1_{\mathrm{et}}\left(\mathbb{Z}[1/pN], V(f) \otimes V(g)(-j)\right). \tag{50}$$

In order to obtain an Euler system from these classes, observe that all constructions are compatible with change of level at p, which allows to consider arbitrary twisting.

As is briefly explained in section 2.2, the auxiliary m then comes from redoing this construction with the affine modular curve $Y(M,N)$ of level

$$K(M,N)=\prod_{\ell}K(M,N)_{\ell} \tag{51}$$

where

(52)
$$K(M,N)_{\ell}\overset{\text{def}}{=}\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}_{\ell})\mid a,b-1\equiv 0 \bmod \ell^{v_{\ell}(M)},\ c,d-1\equiv 0\bmod \ell^{v_{\ell}(N)}\right\}.$$

There is then a twisted trace map from $Y(M,N)$ to $Y_1(N)\times_{\mathbb{Q}}\mathbb{Q}(\zeta_m)$ which provides the supplementary abelian extension. In this way, one obtains the so-called Beilinson–Flach classes.

Definition 3.2 (Definition 6.1.2 of Kings, Loeffler, and Zerbes, 2017)
Let $\Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t_N\rangle)^{[j,j]}(2-j)$ be the Λ-adic sheaf attached to $\mathcal{H}_{\mathbb{Z}_p}$ by the construction of Kings, 2015b. The Beilinson–Flach *class*

$${}_cBF^{[j]}_{M,N,a}\in H^3_{\text{et}}\left(Y_1(N)^2\times_{\mathbb{Z}[1/Np]}\mathbb{Z}[\zeta_M],\Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t_N\rangle)^{[j,j]}(2-j)\right) \tag{53}$$

is the image of the motivic Rankin–Eisenstein class $\mathrm{Eis}^{[k,k',j]}_{\mathrm{mot},b,N}$ through a suitable projection from $H^3\left(Y_1(N)\times_{\mathbb{Z}[1/N]}Y_1(N),\mathrm{TSym}^{[k,k']}\mathcal{H}_{\mathrm{mot}}(2-j)\right)$ to

$$H^3_{\text{et}}\left(Y_1(N)^2\times_{\mathbb{Z}[1/Np]}\mathbb{Z}[\zeta_M],\Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t_N\rangle)^{[j,j]}(2-j)\right).$$

The next step is to extend this construction to Hida families. Let $f\in S_k(\Gamma_1(N_f))$ and $g\in S_{k'}(\Gamma_1(N_g))$ be p-ordinary eigencuspforms. Let $M(\mathbf{f})$ and $M(\mathbf{g})$ be Hida families containing f and g. By this, we mean as in Hida (1986) that there exists local ordinary p-adic Hida–Hecke algebras $\mathbb{T}^{\mathrm{ord}}_{Np^{\infty},\mathfrak{m}_f}$ and $\mathbb{T}^{\mathrm{ord}}_{Mp^{\infty},\mathfrak{m}_g}$ flat or relative dimension 1 over $\mathbb{Z}_p$ such that $\operatorname{Spec}\mathbb{T}^{\mathrm{ord}}_{Np^{\infty},\mathfrak{m}_f}(\bar{\mathbb{Q}}_p)$ (resp. $\operatorname{Spec}\mathbb{T}^{\mathrm{ord}}_{Mp^{\infty},\mathfrak{m}_g}(\bar{\mathbb{Q}}_p)$) contains a point corresponding to the system of eigenvalues of f (resp. g). Then $M(\mathbf{f})$ and $M(\mathbf{g})$ are irreducible, rank 2, p-ordinary Galois representations with coefficients in $\mathbb{T}^{\mathrm{ord}}_{Np^{\infty},\mathfrak{m}_f}$ and $\mathbb{T}^{\mathrm{ord}}_{Mp^{\infty},\mathfrak{m}_g}$ respectively characterized by the property that $\mathrm{tr}\,(\mathrm{Fr}(\ell))=T(\ell)$ for all $\ell\nmid NM$. Under small technical assumptions, the Galois representations $M(\mathbf{f})$ and $M(\mathbf{g})$ have geometric realizations

$$M(\mathbf{f})\simeq\varprojlim_r H^1_{\text{et}}\left(Y_1(Np^r),\mathbb{Z}_p(1)\right)_{\mathfrak{m}_f},\ \ M(\mathbf{g})\simeq\varprojlim_r H^1_{\text{et}}\left(Y_1(Mp^r),\mathbb{Z}_p(1)\right)_{\mathfrak{m}_g}. \tag{54}$$

This allows to realize the Rankin–Selberg Hida family $M(\mathbf{f})\otimes M(\mathbf{g})$ as a quotient

$$H^2_{\text{et}}\left(Y_1(Np)^2\times_{\mathbb{Q}}\bar{\mathbb{Q}},\Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t_N\rangle)^{\boxtimes 2}(2)\right)\longrightarrow M(\mathbf{f})\otimes M(\mathbf{g}) \tag{55}$$

where again $\Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t_N\rangle)^{\boxtimes 2}$ is a suitable Λ-adic sheaf, incorporating not only the Iwasawa action coming from the $\mathbb{Z}_p$-extension of $\mathbb{Q}$ but also the action of diamond operators on the towers of modular curves $\{Y_1(Np^r)\}_{r\geq 1},\{Y_1(Mp^r)\}_{r\geq 1}$.

Just as the Galois action on ${}_cBF^{[j]}_{M,N,a}$ is through a twist by $\mathbb{Q}(-j)$, the Galois action on $H^2_{\text{et}}\left(Y_1(Np)^2\times_{\mathbb{Q}}\bar{\mathbb{Q}},\Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t_N\rangle)^{\boxtimes 2}(2)\right)$ incorporates a twist which p-adically interpolates $\mathbb{Q}(-j)$ as j varies. We denote by $\Lambda(-\mathbf{j})$ the corresponding Galois representation.

Definition 3.3. — *The* Hida-cyclotomic Euler system

$$\left\{{}_c\mathbf{BF}^{\mathbf{f},\mathbf{g}}_m\in H^1_{\text{et}}\left(\mathbb{Z}\left[\frac{1}{N_fN_gmp},\zeta_m\right],M(\mathbf{f})\otimes M(\mathbf{g})\otimes\Lambda(-\mathbf{j})\right)\right\}_{m\in\Xi}\tag{56}$$

is the image of the compatible system of Beilinson–Flach classes through (55)*.*

3.2. Main theorem

We introduce the following data and notations. The $G_{\mathbb{Q}_p}$-representations $M(\mathbf{f})$ and $M(\mathbf{g})$ are ordinary, which means that for $M\in\{M(\mathbf{f}),M(\mathbf{g})\}$, there is a non-trivial filtration

$$0\longrightarrow F^+M\longrightarrow M\longrightarrow F^-M\longrightarrow 0.$$

and $M(\mathbf{f})\otimes M(\mathbf{g})$ are all ordinary. Write $F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))$ for the $G_{\mathbb{Q}_p}$-subrepresentation of $M(\mathbf{f})\otimes M(\mathbf{g})$

$$F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))\overset{\text{def}}{=}F^-M(\mathbf{f})\hat{\otimes}F^+M(\mathbf{g})\tag{57}$$

and $\mathbf{D}\left(F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))(-1-\mathbf{k}')\right)$ for

$$\left(F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))(-1-\mathbf{k}')\otimes_{\mathbb{Z}_p}\hat{\mathbb{Z}}_p^{\text{ur}}\right)^{G_{\mathbb{Q}_p}}.$$

The module $\mathbf{D}\left(F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))(-1-\mathbf{k}')\right)$ is endowed with a pairing $\langle\cdot,\cdot\rangle$ induced from the essential self-duality of $M(\mathbf{f})$ and $M(\mathbf{g})$.

Inside

$$\mathbf{D}\left(F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))(-1-\mathbf{k}')\right),$$

there is a class $\eta_{\mathbf{f}}\otimes\omega_{\mathbf{g}}$, where $\eta_{\mathbf{f}}$ and $\omega_{\mathbf{g}}$ are De Rham classes interpolating the de Rham classes attached to f and g in $D_{\text{dR}}(V(f))$ and $D_{\text{dR}}(V(g))$ (in fact $\eta_{\mathbf{f}}$ is only interpolated on the irreducible component of Spec $\mathbb{T}^{\text{ord}}_{Np^\infty,\mathfrak{m}_f}$ containing f but we ignore this subtlety). There is a dual exponential map

$$\text{Exp}^*\colon H^1(G_{\mathbb{Q}_p},F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))\hat{\otimes}\Lambda(-\mathbf{j}))\longrightarrow\mathbf{D}\left(F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))\right)\hat{\otimes}\Lambda(-\mathbf{j})$$

in which the disappearance of $-1-\mathbf{k}'$ indicates that the action of diamond operators on $\mathbf{D}\left(F^{-,+}(M(\mathbf{f})\otimes M(\mathbf{g}))\right)$ is through the weight character $1+\mathbf{k}'$. Finally we consider the Hida p-adic Rankin–Selberg L-function $L_p(\mathbf{f},\mathbf{g},1+\mathbf{j})$, where $\mathbf{j}$ is the cyclotomic character (Hida, 1988).

One of the main theorems of Lei, Loeffler, and Zerbes, 2014; Kings, Loeffler, and Zerbes, 2017, 2020 is the following.

Theorem 3.4. — *For some explicit non-zero product of Euler factors and correction terms* (Eul)*, there is an equality*

$$\left\langle \mathrm{Exp}^* \circ \mathrm{loc}_p \left({}_c\mathbf{BF}_1^{\mathbf{f},\mathbf{g}}\right), \eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}} \right\rangle = (\mathrm{Eul}) L_p(\mathbf{f}, \mathbf{g}, 1+\mathbf{j}) \tag{58}$$

As the p-adic L-function $L_p(\mathbf{f}, \mathbf{g}, 1+\mathbf{j})$ interpolates non-zero special value of complex Rankin–Selberg complex L-functions, (58) *implies in particular that the Euler system $\left\{{}_c\mathbf{BF}_m^{\mathbf{f},\mathbf{g}}\right\}$ is non-zero.*

The proof of theorem 3.4 is an interesting exercise in p-adic interpolation. Kings, Loeffler and Zerbes showed first that the image of the syntomic regulator map at crystalline points f_x and g_x with large weights k_x, k'_x of $M(\mathbf{f})$ and $M(\mathbf{g})$ respectively coincides with the special value of the Rankin–Selberg L-function $L(f_x, g_x, 1+j)$ for $0 \leq j \leq \min(k_x, k'_x)$ (Kings, Loeffler, and Zerbes, 2017). As both $\left\langle \mathrm{Exp}^* \circ \mathrm{loc}_p \left({}_c\mathbf{BF}_1^{\mathbf{f},\mathbf{g}}\right), \eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}} \right\rangle$ and $L_p(\mathbf{f}, \mathbf{g}, 1+\mathbf{j})$ are analytic function on $M(\mathbf{f}) \otimes M(\mathbf{g})$ and as the set of points (f_x, g_x) as above is Zariski-dense, this implies equality (58) of theorem 3.4. The last remark then follows as the p-adic Rankin–Selberg L-function is not trivial.

3.2.1. *An application.* — Here is an application of the Euler system of Beilinson–Flach classes.

Suppose that $f \in S_k(\Gamma_0(N))$ is crystalline and that g is a theta-series attached to a quadratic imaginary extension K. There is then a Greenberg–Iwasawa Rankin–Selberg p-adic L-function $\mathcal{L}_K^{\mathrm{Gr}}(f)$ which is an element of a certain three-variables Iwasawa algebra which interpolates certain special values of the Rankin–Selberg L-functions $L(f \otimes \xi, s)$ attached to certain characters ξ of the Galois group of the $\mathbb{Z}_p^2$-extension K_∞ of K (see Eischen and Wan (2016) for the construction and details). For simplicity of exposition, we assume that $L(f \otimes \xi, s)$ is a power-series in $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$ (in fact, it is in general needed to extend scalars to the ring of integers of a large unramified extension of $\mathbb{Q}_p$).The Greenberg–Iwasawa Rankin–Selberg p-adic L-function $\mathcal{L}_K^{\mathrm{Gr}}(f)$ appears as the constant term of a p-adic family of Eisenstein $\mathbf{E}$ series for $U(3,1)$. If $\mathcal{L}_K^{\mathrm{Gr}}(f)$ vanishes modulo $\mathcal{P}^n$ for $\mathcal{P}$ a prime ideal of the Iwasawa algebra, there is thus a congruence modulo $\mathcal{P}^n$ between $\mathbf{E}$ and a p-adic family of cuspidal representations of $U(3,1)$ and hence congruences between reducible Galois representations and irreducible Galois representations. Using the method of Ribet (1976), this constructs an extension of Galois representations which contributes to the Greenberg–Iwasawa Rankin–Selberg Selmer group $X_K^{\mathrm{Gr}}(f)$, which we assume again for simplicity to be a $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-module of finite type.

The existence of these extension classes is enough to prove the inclusion

$$\mathrm{char}_{\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]}\left(X_K^{\mathrm{Gr}}(f)\right) \subseteq (\mathcal{L}_K^{\mathrm{Gr}}(f)). \tag{59}$$

of ideals of $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$, where $\mathrm{char}_A M$ for a normal ring A denotes the characteristic ideal of the finite type A-module M, that is to say the ideal

$$\mathrm{char}_A M \overset{\mathrm{def}}{=} \prod_{\mathcal{P}} \mathcal{P}^{\ell_{A_\mathcal{P}}(M_\mathcal{P})}$$

where the product is taken over all prime ideals of height 1 of A if M is A-torsion and the zero ideal otherwise.

One may not readily deduce consequences in cyclotomic Iwasawa theory from the inclusion (59). The problem is that the Greenberg–Iwasawa Rankin–Selberg p-adic L-function $\mathcal{L}_K^{\mathrm{Gr}}(f)$ has a relation with the complex Rankin–Selberg L-function in a restricted range of interpolation. However, the divisibility (59) may be translated into a divisibility involving Beilinson–Flach classes using theorem 3.4 (and an extension of this theorem in Loeffler and Zerbes (2016) in order to allow f to be not necessarily ordinary at p). The crucial point is that the divisibility of Beilinson–Flach, in contrast to (59), holds over the full deformation space, so that it can be specialized to the trivial character of the anticyclotomic $\mathbb{Z}_p$-extension of K. In this way, one obtains the following.

Theorem 3.5 (Fouquet and Wan, 2022). — *Let $f \in S_k(\Gamma_0(N))$ be a p-crystalline eigencuspform. Suppose that $\bar{\rho}_f$ is absolutely irreducible and that its image has cardinality divisible by p. Further assume for simplicity that $\bar{\rho}|G_{\mathbb{Q}_p}$ is absolutely irreducible. Then Kato's cyclotomic Iwasawa Main Conjecture for modular motives (Kato, 2004, Conjecture 12.10) holds. Explicitly, there is an equality of characteristic ideals*

$$\mathrm{char}_\Lambda H^2_{\mathrm{et}}\left(\mathbb{Z}[1/p], T(f)\otimes\Lambda\right) = \mathrm{char}_\Lambda H^1_{\mathrm{et}}\left(\mathbb{Z}[1/p], T(f)\otimes\Lambda\right)/\Lambda\cdot\mathbf{z}(f)_{\mathrm{Iw}} \tag{60}$$

where $\mathbf{z}(f)_{\mathrm{Iw}}$ is the first class of Kato's Euler system. In addition, the Iwasawa Main Conjecture in the $\mathbb{Z}_p^2$-extension of K holds for f (in other words, the divisibility (iii) of Kings, Loeffler, and Zerbes (2020, Theorem 11.4.3) is actually an equality).

Notice that thanks to the Beilinson–Flach Euler systems, the *cyclotomic* Iwasawa Main Conjecture for the eigencuspform f is obtained from a divisibility over a certain subspace of $\operatorname{Spec}\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-extension which excludes the cyclotomic direction (see Fouquet and Wan (2022) for details).

4. p-adic interpolation

Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the $\mathbb{Z}_p$-extension of $\mathbb{Q}$. The axiomatic definition 1.1 implies that the classes forming a cyclotomic Euler systems for T extend to classes with values in $T\otimes\Lambda$ where $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$. As shown by equality (19) with $m' = mp^\infty$, this extension property also follows from general formulations of the conjectures of Bloch–Kato. Since Λ is essentially the universal deformation ring of residual characters of $G_{\mathbb{Q},\Sigma}$, this suggests that Euler systems might extend more generally to universal deformation rings.

As early as Kato (1993b), it was envisioned that zeta elements (that is to say Euler systems in fundamental lines) should extend to *any* p-adic families of motives. For his own Euler system, this belief was vindicated by Nakamura (2023). In this section, we first explain as warm-up how the Euler system of Heegner points extends to Hida families (following Howard (2007)), then we outline the definitive result of Nakamura (2023). Nakamura's proof and the alternative proof of Colmez–Wang rely on the full strength of the p-adic Local Langlands Correspondance (Colmez, 2010; Paškūnas, 2013) so appear to be restricted to GL_2 over $\mathbb{Q}_p$ at present. Nevertheless, it seems to this author that using completed cohomology (Breuil, 2004; Emerton, 2006) to translate automorphic formulas for special values of L-functions into p-adic properties that interpolate over deformations (as in the factorization of Kato's Euler system in Colmez and Wang (2021)) is obviously a very fruitful direction of research. In fact, I am convinced that as in Kato (2004, Section 10.8) and Colmez and Wang (2021, Chapitre 8), the key to this translation lies in the structure of perfectoid Shimura varieties (Scholze, 2015). Again, guessing the correct direction is the easy part.

4.1. Heegner points in Hida families (Howard, 2007)

Let $N\geq 1$ be an integer. Consider the tower $\{X_1(Np^r)\}_{r\geq 0}$ of modular curves. We fix $K/\mathbb{Q}$ a quadratic imaginary extension for which the ideal $N\mathcal{O}_K$ is divisible by an ideal $\mathcal{N}$ such that $\mathcal{O}_K/\mathcal{N}\simeq\mathbb{Z}/N\mathbb{Z}$. On each curve $X_1(Np^r)$, there is a large supply of CM points

$$\left\{x(g)=[z,g]\in\mathrm{GL}_2(\mathbb{Q})\backslash\left(\mathbb{C}-\mathbb{R}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})/K_1(Np^r)\right)|g\in\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})\right\}\tag{61}$$

where z is the unique point of $\mathbb{C}-\mathbb{R}$ of positive imaginary part fixed under the action of $K^\times$ seen as torus in $\mathrm{GL}_2(\mathbb{Q})$. Let $\mathfrak{c}$ be a product of distinct rational primes all inert in $\mathcal{O}_K$ and of a power of p. The *Heegner point* $x(\mathfrak{c},r)\in X_1(Np^r)$ of conductor $\mathfrak{c}$ is the CM point $x(g)$ with g defined by the following conditions.

1. If $\ell\nmid\mathfrak{c}p$, then g_ℓ is the identity.
2. If $\ell|\mathfrak{c}$ and $\ell\nmid p$, then $g_\ell=\begin{pmatrix}\ell&0\\0&1\end{pmatrix}$.
3. If $\ell=p$, then $g_p=\begin{pmatrix}p^{r+v_p(\mathfrak{c})}&0\\0&1\end{pmatrix}$.

When $\mathfrak{c}$ is prime to p, the Heegner point $x(\mathfrak{c},0)$ corresponds to the cyclic isogeny of complex tori $[\mathbb{C}/\mathcal{O}_{\mathfrak{c}}\longrightarrow\mathbb{C}/\left(\mathcal{O}_{\mathfrak{c}}\cap\mathcal{N}\right)^{-1}]$ where $\mathcal{O}_{\mathfrak{c}}$ is $\mathbb{Z}+\mathfrak{c}\mathcal{O}_K$. The system of Heegner points

$$\{x(\mathfrak{c}p^m,r)\in X_1(Np^r)(\bar{K})\}_{\mathfrak{c},m,r}\tag{62}$$

satisfies the following proposition.

Proposition 4.1. — *Let $\mathfrak{c}$ be a product of distinct rational primes all inert in $\mathcal{O}_K$ and of a power of p. Let $\ell\nmid\mathfrak{c}$ be a rational prime inert in $\mathcal{O}_K$. Denote by $T(\ell)$ and $U(p)$ be the usual Hecke operators at ℓ and p respectively. Then:*

1.

$$T(\ell)\cdot x(\mathfrak{c},r)=\sum_{\sigma\in\mathbb{F}_\lambda^\times/\mathbb{F}_\ell^\times}\sigma\cdot x(\mathfrak{c}\ell,r)\tag{63}$$

2. If $\pi_{r+1,r}$ denotes the natural projection from $X_1(Np^{r+1})$ to $X_1(Np^r)$, then

$$U(p)\cdot x(\mathfrak{c},r)=\pi_{r+1,r}\left(\sum_{\sigma\in\mathbb{Z}/p\mathbb{Z}}\sigma\cdot x(\mathfrak{c},r+1)\right) \tag{64}$$

In the above proposition, $\mathbb{F}_\lambda$ is the residual field of the only prime $\lambda\subset\mathcal{O}_K$ above ℓ in K. The group $\mathbb{F}_\lambda^\times/\mathbb{F}_\ell^\times$ acts on $x(\mathfrak{c}\ell,r)$ because it is isomorphic as a group with the Galois group of the field of definition of $x(\mathfrak{c}\ell,r)$ over the field of definition of $x(\mathfrak{c},r)$ and the group $\mathbb{Z}/p\mathbb{Z}$ acts on $x(\mathfrak{c},r+1)$ because it is isomorphic as a group to the Galois group of the field of definition of $x(\mathfrak{c},r+1)$ over the field of definition of $x(\mathfrak{c},r)$.

Let $\mathfrak{m}$ be a maximal ideal of the ordinary Hida–Hecke algebra $\mathbb{T}^{\mathrm{ord}}_{Np^\infty}$ acting on

$$\varprojlim_r H^1_{\mathrm{et}}\left(X_1(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Z}_p\right)^{\mathrm{ord}}. \tag{65}$$

Let Σ be a finite set of finite primes containing $\{\ell|Np\}$. Let $\mathbf{z}(\mathfrak{c}p^m,r)^{\mathrm{ord}}\in H^1\left(G_{K(\mathfrak{c}p^m),\Sigma},H^1_{\mathrm{et}}\left(X_1(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Z}_p\right)^{\mathrm{ord}}\right)$ be the image of the divisor $U(p)^{-r}\cdot x(\mathfrak{c}p^m)$ under the Kummer map. Statement 2 of proposition 4.1 implies that the $\mathbf{z}(\mathfrak{c}p^m,r)$ for varying r form a system

$$\mathbf{z}(\mathfrak{c}p^m)^{\mathrm{ord}}_{\mathfrak{m}}\overset{\mathrm{def}}{=}\varprojlim_r\mathbf{z}(\mathfrak{c}p^m,r)^{\mathrm{ord}}_{\mathfrak{m}}\in H^1\left(G_{K(\mathfrak{c}p^m),\Sigma},\varprojlim_r H^1_{\mathrm{et}}\left(X_1(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Z}_p\right)^{\dagger}_{\mathfrak{m}}\right). \tag{66}$$

Here, $(-)^\dagger$ indicates a self-dual twist which reflects the fact that there is a non-trivial cyclotomic Galois action on $\mathbf{z}(\mathfrak{c}p^m)^{\mathrm{ord}}_{\mathfrak{m}}$ which is compatible with the action of diamond correspondences on the modular curves. Let us put

$$\mathcal{T}^\dagger\overset{\mathrm{def}}{=}\varprojlim_r H^1_{\mathrm{et}}\left(X_1(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Z}_p\right)^{\dagger}_{\mathfrak{m}}. \tag{67}$$

Recall that Ξ is the set $\{m\geq 1\mid\{\ell|m\}\cap\Sigma=\{p\}\}$. If no $\ell|N$ ramifies in $\mathcal{O}_K$, then the system

$$\left\{\mathbf{z}(\mathfrak{c})^{\mathrm{ord}}\in H^1_f\left(G_{K(\mathfrak{c})},\mathcal{T}^\dagger\right)\right\}_{\mathfrak{c}\in\Xi} \tag{68}$$

is an anticyclotomic Euler system for $\mathcal{T}^\dagger$ which recovers by construction the Euler system of CM points after specialization at weight 2 classical points and hence is non-trivial by Cornut, 2002.

As we have in fact already seen in the case of Beilinson–Flach classes, the extension of the Euler system of CM points to the Hida family $\mathcal{T}^\dagger$ amounts to a compatibility with change of levels. This will be important in the next section as well.

4.2. Universal zeta elements for GL_2 (Nakamura, 2023)

4.2.1. *Statement.* — Let Σ be a finite set of primes containing p. Let

$$\bar{\rho}\colon G_{\mathbb{Q},\Sigma}\longrightarrow\mathrm{GL}_2(k)$$

be an odd Galois representation satisfying the following.

Assumption 4.1. — *1. If $p^* = (-1)^{(p-1)/2}p$, then $\bar{\rho}|G_{\mathbb{Q}(\sqrt{p^*})}$ is absolutely irreducible.*

2. If $\bar{\rho}|G_{\mathbb{Q}_p}$ is an extension

$$0 \longrightarrow \chi_1 \longrightarrow \bar{\rho}|G_{\mathbb{Q}_p} \longrightarrow \chi_2 \longrightarrow 0, \tag{69}$$

then $\chi_1^{-1}\chi_2 \notin \{1, \bar{\chi}_{\mathrm{cyc}}\}$.

Let (T_Σ, ρ_Σ) be the universal Galois deformation parametrizing deformations of $\bar{\rho}$ unramified outside Σ. Under assumption 4.1, it is known that T_Σ is a free module of rank 2 over a universal local Hecke algebra $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar{\rho}}}$ and that $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar{\rho}}}$ is a complete intersection ring flat of relative dimension 3 over $\mathbb{Z}_p$. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_{\mathrm{Iw}}$ be the Iwasawa-algebra $\mathcal{O}_{\mathrm{Iw}}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$, where we recall that $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. We say that $\lambda(f) \in \operatorname{Spec} \mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar{\rho}}}[1/p](\mathcal{O})$ is a classical point if $T_\Sigma \otimes_{\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar{\rho}}}, \lambda(f)} \mathcal{O}_{\mathrm{Iw}}$ is isomorphic to the classical cyclotomic deformation $T(f)_{\mathrm{Iw}} \overset{\mathrm{def}}{=} T(f) \otimes_{\mathcal{O}} \mathcal{O}_{\mathrm{Iw}}$ of the Galois representation $T(f)$ attached to an eigencuspform of weight $k \geq 2$ with eigenvalues in $\mathcal{O}$.

Theorem 4.2 (Nakamura, 2023). — *There exists a zeta morphism of $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar{\rho}}}$-modules*

$$\mathbf{z}_\Sigma \colon T_\Sigma(-1)^+ \longrightarrow H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma) \tag{70}$$

such that for every classical point $\lambda(f) \colon \mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar{\rho}}}[1/p] \longrightarrow \mathcal{O}$ the following diagram commutes

$$\begin{tikzcd} T_\Sigma(-1)^+ \arrow[r, "\mathbf{z}_\Sigma"] \arrow[d] & H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma) \arrow[d] \\ T(f)_{\mathrm{Iw}}(-1)^+ \arrow[r, "\mathbf{z}(f)_{\Sigma,\mathrm{Iw}}"] & H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T(f)_{\mathrm{Iw}}) \end{tikzcd} \tag{71}$$

where $\mathbf{z}(f)_{\Sigma,\mathrm{Iw}}$ is the zeta morphism of Kato (2004, Theorem 12.5 (1)).

Concretely, the module $T(f)_{\mathrm{Iw}}(-1)^+$ is free of rank 1 over $\mathcal{O}_{\mathrm{Iw}}$. The morphism $\mathbf{z}(f)_{\Sigma,\mathrm{Iw}}$ sends a generator e of this module to the first class of Kato's Euler system. Hence, the morphism $\mathbf{z}_\Sigma$ interpolates all Euler systems of all classical points of $\operatorname{Spec} \mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar{\rho}}}$.

4.2.2. *Outline of the proof of theorem 4.2.* — We fix N an integer[7] prime to p and put $\Sigma_0 = \{\ell | N\}$, $\Sigma = \Sigma_0 \cup \{p\}$. In section 2.2, we have seen how Siegel units yield elements

$${}_{c,d}\mathbf{z}_{Np^r,\mathrm{mot}} \in K_2(Y(Np^r)) \tag{72}$$

in the second K-theory group of the full modular curve $Y(Np^r)$. An important property of the classes ${}_{c,d}\mathbf{z}_{Np^r,\mathrm{mot}}$ for various N is their compatibility with the norm map

$$K_2(Y(N')) \longrightarrow K_2(Y(N)) \tag{73}$$

7. In order for the following statements to be always strictly true, it is necessary to assume $N \geq 3$. Since we are interested in the situation where Np^r is eventually very large, we gloss over this subtlety.

whenever $N|N'$ (Kato, 2004, Proposition 2.3). This is the counterpart in this setting of the second statement of proposition 4.1, with the important distinction that action of $U(p)$ intervene.

In particular, we may consider a universal K_2-class

$$(74)\qquad {}_{c,d}\mathbf{z}_{Np^\infty,\mathrm{mot}} \overset{\mathrm{def}}{=} (\varprojlim_r {}_{c,d}\mathbf{z}_{Np^r} \in K_2(Y(Np^r))) \in K_2(Y(Np^\infty)).$$

The image of ${}_{c,d}\mathbf{z}_{Np^\infty,\mathrm{mot}}$ through the Chern class map and the Hochschild–Serre isomorphism is a class

$$(75)\qquad {}_{c,d}\mathbf{z}_{Np^\infty,\mathrm{et}} \in H^1_{\mathrm{et}}\left(\mathbb{Z}[1/Np], \varprojlim_m \varprojlim_r H^1_{\mathrm{et}}\left(Y(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Z}/p^m\mathbb{Z}\right)\right).$$

Using again the norm compatibility, we take the direct limit with respect to N to obtain

$$(76)\qquad {}_{c,d}\mathbf{z}_{p^\infty,\mathrm{et}} \in \varinjlim_N H^1_{\mathrm{et}}\left(\mathbb{Z}[1/Np], \varprojlim_m \varprojlim_r H^1_{\mathrm{et}}\left(Y(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Z}/p^m\mathbb{Z}\right)\right).$$

The completed cohomology group

$$(77)\qquad \tilde{H}^1_{\mathrm{et}}\left(Y(N)\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Z}_p\right) \overset{\mathrm{def}}{=} \varprojlim_m \varprojlim_r H^1_{\mathrm{et}}\left(Y(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Z}/p^m\mathbb{Z}\right)$$

is naturally endowed with a $G_{\mathbb{Q},\Sigma}$-action and with a Hecke action of

$$(78)\qquad \mathbb{T}^\Sigma(N) \overset{\mathrm{def}}{=} \varprojlim_r \mathbb{T}^\Sigma(Np^r)$$

where $\mathbb{T}^\Sigma(Np^r)$ is the p-adic Hecke algebra generated by the usual Hecke operators $T(\ell), S(\ell)$ for $\ell \notin \Sigma$ acting on $H^1_{\mathrm{et}}\left(Y(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Z}/p^m\mathbb{Z}\right)$. We localize all these constructions at the maximal ideal $\mathfrak{m}_{\bar\rho} \in \operatorname{Spec}^{\max}\mathbb{T}^\Sigma(N)$ attached to $\bar\rho$ to obtain

$$(79)\qquad {}_{c,d}\mathbf{z}_{Np^\infty,\mathfrak{m}_{\bar\rho}} \in H^1_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], \tilde{H}^1_{\mathrm{et}}\left(Y(N)\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Z}_p\right)_{\mathfrak{m}_{\bar\rho}}\right).$$

We can further take the direct limit over all integers N with prime factors included in Σ to obtain

$$(80)\qquad \tilde{H}^1_{\mathrm{et}}(\mathbb{Z}_p) \overset{\mathrm{def}}{=} \varinjlim_N \varprojlim_m \varprojlim_r H^1_{\mathrm{et}}\left(Y(Np^r)\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Z}/p^m\mathbb{Z}\right)$$

and

$$(81)\qquad {}_{c,d}\mathbf{z}_{p^\infty,\mathfrak{m}_{\bar\rho}} \in H^1_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], \tilde{H}^1_{\mathrm{et}}(\mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}}\right).$$

The cohomology group $\tilde{H}^1_{\mathrm{et}}(\mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}}$ is endowed with an action of the universal Hecke algebra

$$(82)\qquad \mathbb{T}^\Sigma_{\mathfrak{m}_{\bar\rho}} \overset{\mathrm{def}}{=} \varprojlim_N \mathbb{T}^\Sigma(N)_{\mathfrak{m}_{\bar\rho}}.$$

As in Kato (2004, Theorem 12.4), we view ${}_{c,d}\mathbf{z}_{p^\infty,\mathfrak{m}_{\bar\rho}}$ has a morphism

$$(83)\qquad \mathbf{z}(p^\infty)_{\mathfrak{m}_{\bar\rho}}\colon \tilde{H}^1_{\mathrm{et}}(\mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}}(-1)^+ \longrightarrow H^1_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], \tilde{H}^1_{\mathrm{et}}(\mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}}\right).$$

Proposition 4.3. — *Under assumption 4.1, $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$ is a complete intersection ring flat of relative dimension 3 over $\mathbb{Z}_p$. Let ρ^u_Σ with coefficients in $R_\Sigma(\bar\rho)$ be the universal deformation ring of $\bar\rho$ parametrizing deformations unramified outside of Σ. Then $R_\Sigma(\bar\rho)$ is isomorphic to $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$ is isomorphic through the map which sends $\operatorname{tr}(\rho^u(\mathrm{Fr}(\ell)))$ to $T(\ell)$.*

Under assumption 4.1, let ρ_Σ be the universal deformation of $\bar\rho$. Let $\pi_p(\rho^*_\Sigma(1)|G_{\mathbb{Q}_p})$ be the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation attached to the $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$-dual $\rho^*_\Sigma(1)$ by the p-adic Local Langlands Correspondance (Colmez, 2010). For $\ell\in\Sigma_0$, let $\tilde\pi_\ell(\rho_\Sigma|G_{\mathbb{Q}_\ell})$ be the $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$-linear smooth contragredient representation of the $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representation $\pi(\rho^*_\Sigma(1)|G_{\mathbb{Q}_\ell})$ attached to $\rho^*_\Sigma(1)$ by the Local Langlands Correspondance in p-adic families (Emerton and Helm, 2014; Helm, 2016; Helm and Moss, 2018). M. Emerton has proposed a description $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}[G_{\mathbb{Q},\Sigma}\times\mathrm{GL}_2(\mathbb{Q}_p)\times\prod_{\ell\in\Sigma_0}\mathrm{GL}_2(\mathbb{Q}_\ell)]$-module structure of $\tilde H^1_{\mathrm{et}}(\mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}}$ (see Colmez and Wang (2021, Théorème 13.11) for a proof).

Proposition 4.4 (Colmez and Wang, 2021). — *Under assumption 4.1, there is an isomorphism*

$$\tilde H^1_{\mathrm{et}}(\mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}}\simeq\rho_\Sigma\otimes\pi_p(\rho^*_\Sigma(1)|G_{\mathbb{Q}_p})^*\hat\otimes\bigotimes_{\ell\in\Sigma^p}\tilde\pi_\ell(\rho^*_\Sigma(1)|G_{\mathbb{Q}_\ell})\tag{84}$$

of $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}[G_{\mathbb{Q},\Sigma}\times\mathrm{GL}_2(\mathbb{Q}_p)\times\prod_{\ell\in\Sigma_0}\mathrm{GL}_2(\mathbb{Q}_\ell)]$-modules.

Outline of the proof of theorem 4.2. — Using the isomorphism (84) of proposition 4.3, we reinterpret the morphism (83) as a morphism

$$\begin{array}{c}\rho_\Sigma(-1)^+\otimes\pi_p(\rho^*_\Sigma(1)|G_{\mathbb{Q}_p})^*\hat\otimes\bigotimes_{\ell\in\Sigma^p}\tilde\pi_\ell(\rho^*_\Sigma(1)|G_{\mathbb{Q}_\ell})\\ \big\downarrow\\ H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],\rho_\Sigma)\otimes\pi_p(\rho^*_\Sigma(1)|G_{\mathbb{Q}_p})^*\hat\otimes\bigotimes_{\ell\in\Sigma^p}\tilde\pi_\ell(\rho^*_\Sigma(1)|G_{\mathbb{Q}_\ell})\end{array}\tag{85}$$

Here the superscript $+$ denotes the free sub-$\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$-module of rank 1 of $\rho_\Sigma(-1)$ on which $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts trivially.

We apply the Bernstein–Zelevinsky derivative functor $(-)^{(2)}$ (Emerton and Helm (2014, Section 3)) to the morphism (85). As $\tilde\pi_\ell(\rho^*_\Sigma(1)|G_{\mathbb{Q}_\ell})$ is generic, its second derivative is a free $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$-module or rank 1. We may thus view (85) as a morphism

$$\begin{array}{c}\rho_\Sigma(-1)^+\otimes\pi_p(\rho^*_\Sigma(1)|G_{\mathbb{Q}_p})^*\\ \big\downarrow\\ H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],\rho_\Sigma)\otimes\pi_p(\rho^*_\Sigma(1)|G_{\mathbb{Q}_p})^*\end{array}\tag{86}$$

Using the description of the (dual of the) so-called Montréal functor (Colmez (2010, Section IV.2.3)) given in Paškūnas (2013), it is shown in Nakamura (2023, Appendix B) that there is an isomorphism

$$\mathrm{End}(\pi_p(\rho_\Sigma^*(1)|G_{\mathbb{Q}_p})^*) \simeq \mathbb{T}^\Sigma_{\mathfrak{m}_{\bar\rho}} \tag{87}$$

where endomorphisms are taken in the category of $\mathcal{O}[\mathrm{GL}_2(\mathbb{Q}_p)]$-modules [8]. There is thus a bijection between

$$\mathrm{Hom}_{\mathbb{T}^\Sigma_{\mathfrak{m}_{\bar\rho}}[\mathrm{GL}_2(\mathbb{Q}_p)]}\left(\rho_\Sigma(-1)^+ \otimes \pi_p(\rho_\Sigma^*(1)|G_{\mathbb{Q}_p})^*, H^1_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], \rho_\Sigma\right) \otimes \pi_p(\rho_\Sigma^*(1)|G_{\mathbb{Q}_p})^*\right) \tag{88}$$

and

$$\mathrm{Hom}_{\mathbb{T}^\Sigma_{\mathfrak{m}_{\bar\rho}}}\left(\rho_\Sigma(-1)^+, H^1_{\mathrm{et}}\left(\mathbb{Z}[1/\Sigma], \rho_\Sigma\right)\right). \tag{89}$$

We have obtained morphism (70)

$$\mathbf{z}_\Sigma : T_\Sigma(-1)^+ \longrightarrow H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma) \tag{90}$$

of theorem 4.2. The commutativity of diagram (71) then follows easily from the compatibility of the completed cohomology with taking invariants under the action of a given compact open subgroup giving the level-structure of a classical eigencuspform. □

5. Bipartite Euler systems

As explained in the introduction, an important conceptual feature of Kolyvagin systems is the fact that a system of local compatibilities between unramified and ramified cohomology classes yields bounds on Selmer groups of Galois representations. In this section, we first explain as a warm-up one concrete incarnation of this phenomenon in which the underlying mechanism has been particularly well-understood, then we give an outline of how far this idea has been pushed in recent works.

At the heart of the conjectures of Tate and Beilinson (Tate, 1965; Beĭlinson, 1984) is the idea that algebraic cycles should be related to special values of L-functions. In favorable cases, this core idea takes the following form. Inside the local at p cohomology modulo ℓ^n of a given Shimura variety X, one sometimes find the images of a large supply of algebraic cycles Z_i coming from auxiliary Shimura varieties X_i. The local properties of Z_i are described in geometric terms in terms of the bad reduction of X_i. The local information obtained on these classes in this way allows by self-duality to bound Selmer groups. An important insight of Liu, Tian, Xiao, Zhang, and Zhu (2022) is that the existence of such a favorable description in geometric term follows from the Gan–Gross–Prasad conjecture.

8. In Paškūnas (2013, 2015), this is shown only under the hypothesis $p \geq 5$. We use Paškūnas (2016, Proposition 2.13, 2.33) to extend the result to $p > 2$.

5.1. Anticyclotomic Iwasawa theory for elliptic curves (Bertolini and Darmon, 2005)

Bertolini and Darmon (2005) pioneered a new use of Euler systems in the anticyclotomic Iwasawa theory of elliptic curves that has proved of immense importance.

5.1.1. *Statement of the result.* — The setting is as follows. Let $E/\mathbb{Q}$ be a rational elliptic curve and let $f \in S_2$ be the eigencuspform attached to E by modularity. Let p be a prime of ordinary reduction for E. Let K be an imaginary quadratic field and let K_∞/K be the anticyclotomic $\mathbb{Z}_p$-extension of K. Write Λ for the anticyclotomic Iwasawa algebra $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$. There is then a p-adic L-function $\mathcal{L}_p(E/K) \in \Lambda$ interpolating special values $L(E/K, \chi, 1)$ of the Rankin–Selberg complex L-function $L(E/K, s)$ twisted by characters of $\mathrm{Gal}(K_\infty/K)$. Let N_0 be the conductor of E and let N be the integer defined by

$$N = \begin{cases} pN_0 & \text{if } E \text{ has good ordinary reduction at } p, \\ N_0 & \text{if } E \text{ has ordinary multiplicative reduction at } p. \end{cases} \tag{91}$$

The crucial assumption in Bertolini and Darmon (2005) is the following.

Assumption 5.1. — *Assume that N factors as $N = pN^+N^-$ where all primes dividing N^+ split in $\mathcal{O}_K$ and N^- is a product of an* odd *number of distinct primes inert in $\mathcal{O}_K$.*

Notice that this assumption is incompatible with the Heegner hypothesis. More precisely, under assumption 5.1, the generic sign of the functional equation of $\mathcal{L}_p(E/K)$ (that is to say the sign of the functional equation of $L(E/K, \chi, 1)$ for all χ except possibly finitely many) is $+1$. Moreover, it is known under that hypothesis that $\mathcal{L}_p(E/K)$ is a non-zero element of Λ (Vatsal, 2002; Cornut and Vatsal, 2004). It follows from the original work of Kolyvagin and Gross–Zagier that there can be no compatible system of Heegner points on E in the $\mathbb{Z}_p$-extension K_∞/K, a fact which seems to preclude the use of the Euler system of Heegner points in that setting to study the arithmetic ot T_pE. Nevertheless, we have the following result.

Let $\tilde{H}^2_f(K_\infty/K, T_pE)$ be the second cohomology group of the Nekovář–Selmer complex attached to Greenberg's ordinary condition at p (Nekovář, 2006). This is a compact Λ-module of finite type, which is also the Pontrjagin dual of the Λ-adic Selmer group of $E[p^\infty]$.

Theorem 5.1 (Bertolini and Darmon, 2005). — *Under assumption 5.1 and mild technical hypotheses on E and p, the Λ-module $\tilde{H}^2_f(K_\infty/K, T_pE)$ is torsion and there is a divisibility of ideals*

$$\mathrm{char}_\Lambda\, \tilde{H}^2_f(K_\infty/K, T_pE) \mid (\mathcal{L}_p(E/K))\,\Lambda. \tag{92}$$

In particular $E(K_\infty)$ is a finitely generated abelian group.

Among the technical requirements on E, K and p, one of them is that f be p-isolated, that is to say that there is no congruence between f and an eigencuspform of similar weight and level.

5.1.2. *Outline of the proof.* — The fundamental idea of the proof is to switch from the modular curve of level N to Shimura curves which satisfy the Heegner hypothesis and hence have Heegner points.

According to assumption 5.1, $N = pN^+N^-$ where N^- is a product of an odd number of distinct primes. Consider ℓ a prime number satisfying the following properties.

1. $\ell \nmid N$.
2. ℓ is inert in $\mathcal{O}_K$.
3. $\ell \not\equiv \pm 1 \bmod p$.
4. The crucial level-raising assumption: if a_ℓ denote the eigenvalue of f under the action of the Hecke operator $T(\ell)$, then p^n divides $\ell + 1 - a_\ell$ or $\ell + 1 + a_\ell$.

These assumptions imply that we may level-raise T_pE at ℓ modulo p^n, that is to say so that T_pE is an unramified $G_{\mathbb{Q}_\ell}$-module with semisimple Frobenius action which occurs in the Jacobian of a quaternionic Shimura curve ramified at ℓ.

Let $X^{(\ell)}$ be the Shimura curve $X(pN^+, N^-\ell)$ of level pN^+ attached to the indefinite quaternion algebra ramified at $N^-\ell$. By the Jacquet–Langlands correspondence, the Galois representation $T_pE \bmod p^n$ is a quotient $T_n^{(\ell)}$ of the Tate module of the Jacobian J^ℓ of the curve $X^{(\ell)}$. There are two reasons why the situation is now more favorable than it was *a priori* for T_pE/p^nT_pE seen as a quotient of the Tate module of the Jacobian of a modular curve.

First, there is an explicit description of the ℓ-special fiber $J^{(\ell)}_{\mathbb{F}_{\ell^2}}$ of $J^{(\ell)}$ (Drinfel'd, 1976; Boutot, 1997; Boutot and Carayol, 1991). Let Φ_ℓ be the group of connected components of $J^{(\ell)}_{\mathbb{F}_{\ell^2}}$. The local cohomology at ℓ of $T_n^{(\ell)}$, and hence of $T_pE \bmod p^n$, is described in terms of Φ_ℓ. More precisely, let $\mathbb{C}_\ell$ be the completion of an algebraic closure of $\mathbb{Q}_\ell$. Let $\hat{\mathcal{H}}_\ell$ be Drinfled's ℓ-adic upper half-plane viewed as a formal scheme over $\mathbb{Z}_\ell$. By definition, the generic fiber of $\hat{\mathcal{H}}_\ell$ is the rigid analytic space $\mathcal{H}_\ell$ over $\mathbb{Q}_\ell$ whose $\mathbb{C}_\ell$-valued points are

$$\mathcal{H}_\ell\left(\mathbb{C}_\ell\right) = \mathbb{P}^1\left(\mathbb{C}_\ell\right) - \mathbb{P}^1\left(\mathbb{Q}_\ell\right). \tag{93}$$

The special fiber H_ℓ of $\hat{\mathcal{H}}_\ell$ is an infinite sequence of projective lines intersecting at ordinary double points. The curve $X^{(\ell)}$ has a $\mathbb{Z}_\ell$-model $X^{(\ell)}_{\mathbb{Z}_\ell}$ whose formal completion $\hat{X}^{(\ell)}_{\mathbb{Z}_\ell}$ alongside its special fiber is isomorphic as formal scheme over $\mathbb{Z}_{\ell^2}$ to the quotient of $\hat{\mathcal{H}}_\ell$ by a suitable arithmetic subgroup $\Gamma_{\ell,1}$ of $\mathrm{PGL}_2(\mathbb{Q}_\ell)$ acting on it (quotient in the category of formal schemes). This provides a description of the so-called dual graph of $X^{(\ell)}_{\mathbb{F}_{\ell^2}}$. The dual graph is the finite graph whose set of vertices is the set of irreducible components of the geometric special fiber $X^{(\ell)}_{\mathbb{F}_{\ell^2}}$ and whose set of edges is the set of singular points of $X^{(\ell)}_{\mathbb{F}_{\ell^2}}$, two vertices being joined by an edge if and only if the two corresponding components intersect at the singular point corresponding to the edge. The previous isomorphism between $\hat{X}^{(\ell)}_{\mathbb{Z}_\ell}$ and a quotient of $\hat{\mathcal{H}}_\ell$ identifies the dual graph of $X^{(\ell)}_{\mathbb{F}_{\ell^2}}$ with the quotient of the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_\ell)$ by $\Gamma_{\ell,1}$. This allows for an explicit description of the dual graph and hence, by way of Grothendieck's monodromy pairing, of Φ_ℓ. This explicit description in turn is crucial in order to describe the image of special points in terms of the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_\ell)$. Because the generic sign of $\mathcal{L}(E/K)$ is $+1$, this p-adic Rankin–Selberg L-function is an inverse limit of elements in finite group algebras related to the structure of the Bruhat–Tits tree, and we thereby obtained a link between local properties of special points and the p-adic Rankin–Selberg L-function;

Second, $X^{(\ell)}$ being a Shimura curve, it comes equipped with a supply of CM-points. Among them, one can find a norm-compatible system of points in the extension K_∞/K which, after inclusion in the Jacobian and into cohomology, yields a class $\kappa_n(\ell) \in H^1(K_\infty, T_n^{(\ell)})$.

Seen as classes in $H^1(K_\infty, T_pE/p^nT_pE)$ through the Jacquet–Langlands correspondence, the classes $\kappa_n(\ell)$ for various ℓ satisfy strong local properties at ℓ, which have been axiomatized in Howard (2006) under the name bipartite Kolyvagin system. More precisely, they satisfy what Bertolini and Darmon (2005) calls the first and second explicit reciprocity laws.

Theorem 5.2 (First explicit law). — *The class $\kappa_n(\ell) \in H^1(K_\infty, T_pE \bmod p^n)$ is ramified at ℓ and the projection of $\mathrm{loc}_\ell\,\kappa_n(\ell)$ to ramified cohomology identifies it with $\mathcal{L}_p(E/K) \bmod p^n$.*

In order to make sense of this statement, we note that under the hypothesis of Bertolini and Darmon (2005), the ramified cohomology part of $H^1(K_{\infty,\ell}, T_pE \bmod p^n)$ is free of rank one over $\Lambda/p^n\Lambda$ so that it makes sense to compare the localization $\mathrm{loc}_\ell\,\kappa_n(\ell)$ with $\mathcal{L}_p(E/K) \bmod p^n$.

Theorem 5.3 (Second explicit law). — *The class $\kappa_n(\ell_1) \in H^1(K_\infty, T_pE \bmod p^n)$ is unramified at ℓ_2 and the projection of $\mathrm{loc}_{\ell_2}\,\kappa_n(\ell_1)$ to unramified cohomology identifies it with $\mathcal{L}_p(g/K) \bmod p^n$ where g is a weight 2 eigencuspform for the* definite *quaternion algebra of discriminant $N^-\ell_1\ell_2$ congruent to f and where $\mathcal{L}_p(g/K) \in \Lambda$ is the p-adic L-function attached to g.*

The proofs of theorems 5.2 and 5.3 are massively simplified by the fact that we compare elements which naturally live in modules free of rank 1, thanks to the p-isolated hypothesis (this will play an important role in the next section).

Theorems 5.2 and 5.3 are highly reminiscent of the local properties of Kolyvagin systems, and indeed Bertolini and Darmon (2005) and Howard (2006) show that they are enough to provide the divisibility statement (92) of theorem 5.1.

5.2. Rankin–Selberg motives (Liu, Tian, Xiao, Zhang, and Zhu, 2022)

Liu (2016) and Liu and Tian (2020) and especially Liu, Tian, Xiao, Zhang, and Zhu (2022) generalized the ideas of the previous subsection to an extraordinary degree and obtained in this way a remarkable contribution to the arithmetic of Rankin–Selberg L-functions and one of the spectacular progress towards the conjectures of Beilinson and Bloch–Kato on the order of vanishing of L-functions (Beĭlinson (1984, Conjecture 3.7a)), Bloch and Kato (1990, Conjecture 5.3), Fontaine (1992, "Conjecture" $C_r(M)$)).

5.2.1. *Setting and statement of results.* — Let $n \geq 1$ be an integer. Let F be a CM extension of a totally real field F^+. For technical reasons, we assume that $F^+ \neq \mathbb{Q}$. A complex representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ is said to be *relevant* (Liu, Tian, Xiao, Zhang, and Zhu, 2022, Definition 1.1.3) if it satisfies the following conditions.

1. Π is an irreducible cuspidal automorphic representation.
2. If $\tau \in \mathrm{Gal}(F/F^+)$ is the complex conjugation, then $\Pi^\vee \simeq \Pi \circ \tau$ (Π is conjugate self-dual).
3. Suppose w is an archimedean place. Let $\arg\colon \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ be the argument character defined by $\arg(z) = z/|z|$. Then Π_w is the irreducible principal series induced from the characters $(\arg^{1-n}, \arg^{3-n}, \ldots, \arg^{n-3}, \arg^{n-1})$.

If Π is a relevant representation of $\mathrm{GL}_n(\mathbb{A})$ and if ℓ is a rational prime, there is a continuous, conjugate self-dual up to a twist by $\chi_{\mathrm{cyc}}^{1-n}$, semisimple Galois representation

$$\rho_\Pi\colon G_F \longrightarrow \mathrm{GL}_n\left(\bar{\mathbb{Q}}_\ell\right) \tag{94}$$

which is characterized by the fact that the G_{F_w}-representation $\rho|G_{F_w}$ corresponds to the $GL_n(F_w)$-representation $\Pi_w \otimes |\det(\cdot)|_w^{(1-n)/2}$ through the Local Langlands Correspondance for any non-archimedean place w of F (Chenevier and Harris, 2013; Caraiani, 2012).

Write $\{n_0, n_1\} = \{n, n+1\}$ with n_0 even and n_1 odd. We fix two relevant representations Π_0 and Π_1 of $\mathrm{GL}_{n_0}(\mathbb{A}_F)$ and $\mathrm{GL}_{n_1}(\mathbb{A}_F)$ respectively, a prime ℓ and the ℓ-adic Galois representations ρ_1 and ρ_2 attached respectively to Π_0 and Π_1. The Galois representations ρ_0 and ρ_1 have *a priori* coefficients in $\bar{\mathbb{Q}}_\ell$ but they descend to Galois representations with coefficients in a finite extension $E_\lambda/\mathbb{Q}_\ell$ which we also fix. Henceforth, we view them as λ-adic representations.

The Rankin–Selberg L-function of $\Pi_0 \times \Pi_1$ at the center of symmetry of its functional equation is conjecturally linked to the arithmetic of the λ-adic Galois representations $\rho_0 \otimes \rho_1(n)$ by the following version of the Beilinson conjecture in the version of Bloch–Kato.

CONJECTURE 5.4. —

$$\mathrm{ord}_{s=1/2} L\left(\Pi_0 \times \Pi_1, s\right) = \dim_{E_\lambda} H^1_f\left(G_F, \rho_0 \otimes \rho_1(n)\right). \tag{95}$$

In particular, we certainly expect the following.

Conjecture 5.5. — *If* $L(\Pi_0 \times \Pi_1, 1/2) \neq 0$, *then* $H^1_f(G_F, \rho_0 \otimes \rho_1(n))$ *vanishes.*

Suppose $n = 1$. In that case, conjecture 5.5 follows from the work of Kolyvagin and Gross–Zagier (Kolyvagin, 1990; Gross and Zagier, 1986) and theorem 5.1 is an Iwasawa-theoretic refinement of this conjecture under the same hypothesis.

One of the main and most remarkable theorems of Liu, Tian, Xiao, Zhang, and Zhu, 2022 is thus an impressive generalization of Kolyvagin (1990) and Gross and Zagier (1986) from $n = 1$ to any $n \geq 2$.

Theorem 5.6 (Liu, Tian, Xiao, Zhang, and Zhu, 2022). — *For all admissible primes* λ, *conjecture 5.5 holds.*

Here admissible is a technical condition gathering numerous conditions playing different roles in the proof. An important effective corollary of this theorem is the following.

Corollary 5.7. — *Let* E *and* E' *be two rational elliptic curves. Let* A *and* A' *be the base change of* E *and* E' *to* F^+. *Suppose the following:*

1. *The extension* $F^+/\mathbb{Q}$ *is non-trivial and solvable.*
2. $\mathrm{End}\, A_{\bar{F}} = \mathrm{End}\, A'_{\bar{F}} = \mathbb{Z}$.
3. $A_{\bar{F}}$ *and* $A'_{\bar{F}}$ *are not isogenous to each other.*

If

$$L\left((\mathrm{Sym}^{n_0-1} A) \otimes (\mathrm{Sym}^{n_1-1} A'), n\right) \neq 0, \tag{96}$$

then

$$H^1_f\left(G_F, \mathrm{Sym}^{n_0-1} H^1_{\mathrm{et}}\left(A \times_{F^+} \bar{F}, \mathbb{Q}_\ell\right) \otimes_{\mathbb{Q}_\ell} \mathrm{Sym}^{n_1-1} H^1_{\mathrm{et}}\left(A' \times_{F^+} \bar{F}, \mathbb{Q}_\ell\right)(n)\right) = 0 \tag{97}$$

for all primes ℓ *except those inside a finite set which is effectively bounded.*

Note that under these assumptions, the symmetric powers of E and E' are modular by Newton and Thorne (2021a,b) and the same is true for the symmetric powers of A and A' by solvable base-change (Arthur and Clozel, 1989).

5.2.2. *General overview of the strategy.* — Thanks to the recent proof the Gan–Gross–Prasad conjecture for $U(n) \times U(n+1)$ by Beuzart-Plessis, Liu, Zhang, and Zhu (2021), the non-vanishing of $L(\Pi_0 \times \Pi_1, 1/2)$ implies that Π_0 and Π_1 arise as spaces of locally constant functions on unitary Shimura varieties, so that techniques in arithmetic geometry may be used to study their arithmetic. The fundamental idea of Liu, Tian, Xiao, Zhang, and Zhu (2022) is akin to that of Bertolini and Darmon (2005) as described in the previous section, that is to switch from the Shimura varieties attached to the unitary group of a *definite* hermitian lattice to the unitary groups of *indefinite* hermitian lattices by adding an auxiliary prime of ramification. The main difficulty in carrying over this rough strategy is that the comparison between classes of geometric origin at different primes (the so-called reciprocity laws, that is to say theorems 5.2 and 5.3 above) relies on explicit descriptions of the special fiber at some prime $\mathfrak{p}$ of Shimura varieties which by design have bad reduction at $\mathfrak{p}$. When the Shimura variety in question is a quaternionic Shimura curve, as is the case for $n = 1$, the results of Ceredník and Drinfeld on p-adic uniformization of Shimura curves provide such a description. In the general $n \geq 2$ case, the Shimura varieties arising are moduli schemes of unitary abelian schemes whose special fibers have special strata which are fibered projective spaces of Deligne–Lusztig varieties. The explicit description of these strata turns out to be extremely intricate and is a significant achievement of Liu, Tian, Xiao, Zhang, and Zhu (2022).

5.2.3. *Main objects.* — In this section, we introduce the main objects involved in the proof of theorem 5.6. The key objects to keep in mind are the following

1. Moduli schemes $\mathbf{M}_{\mathfrak{p}}, S^{\circ}_{\mathfrak{p}}$ and $S^{\bullet}_{\mathfrak{p}}$ which represent sets of equivalence classes of sextuples formed of two triplets of unitary abelian schemes with supplementary level-structure at $\mathfrak{p}$ an inert prime depending on the properties of the choice of the underlying hermitian space.
2. The stratification of the special fiber of $\mathbf{M}_{\mathfrak{p}}$ at p.
3. The so-called *basic correspondences*, which are morphisms relating the strata of the special fiber of $\mathbf{M}_{\mathfrak{p}}$ at p to $S^{\circ}_{\mathfrak{p}}$ and $S^{\bullet}_{\mathfrak{p}}$.
4. The compatibilities of the previous three items when passing from n to $n + 1$.

5.2.4. *Hermitian spaces, Shimura varieties, automorphic representations.* — We fix a finite set Σ of finite places of F. There is an abstract unitary Hecke algebra $\mathbb{T}^{\Sigma}$. If Π is a relevant representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and if for all $w \notin \Sigma$, the representation Π_w is unramified, then the Satake parameter of Π_w for $w \notin \Sigma$ is unitary and so there is a local complex Hecke-morphism

$$\phi_{\Pi,w} \colon \mathbb{T}_w \longrightarrow \mathbb{C} \tag{98}$$

and hence a complex Hecke-morphism

$$\phi_{\Pi} \colon \mathbb{T}^{\Sigma} \longrightarrow \mathbb{C} \tag{99}$$

obtained by the tensor product of the local Hecke-morphisms at all $w \notin \Sigma$. It follows from Shin and Templier (2014) that $\phi_{\Pi,w}$ and ϕ_{Π} have values in the unit ball $\mathcal{O}_{E_\lambda}$ of E_λ.

Now consider V a hermitian $\mathcal{O}_F$-module of rank n, that is to say a projective $\mathcal{O}_F$-module (more generally, a projective $\mathcal{O}_F \otimes_{\mathcal{O}_{F^+}} A$-module for A a suitable $\mathcal{O}_{F^+}$-algebra) of rank n together with a perfect hermitian pairing (which is normalized so as to be linear in the first variable). If V is a hermitian space over F, we denote by $V_\sharp$ the hermitian space $V \oplus \mathcal{O}_F \cdot e$ with e of norm 1. In the following, V is either *standard definite*, meaning it has signature $(n, 0)$ at every archimedean place of F^+, or *standard indefinite*, meaning it has signature $(n - 1, 1)$ at exactly one fixed archimedean place $\tau_{F^+,\infty}$ and $(n, 0)$ at all other archimedean places.

To a standard indefinite hermitian $\mathcal{O}_F$-module V, we attach the Shimura variety $\mathrm{Sh}(V, -)$ associated with the reductive group $\mathrm{Res}_{F/\mathbb{Q}}\, U(V)$. To a standard definite V, we attach the Shimura set $\mathrm{Sh}(V,-)$ which sends a neat compact open subgroup $K\subset U(V)(\mathbb{A}_{F^+}^{(\infty)})$ to the double quotient

$$\mathrm{Sh}(V,K)\overset{\mathrm{def}}{=}U(V)(F^+)\backslash U(V)(\mathbb{A}_{F^+}^{(\infty)})/K. \tag{100}$$

If π is a discrete automorphic representation of $U(V)(\mathbb{A}_{F^+})$ such that π^∞ occurs in either

$$\varinjlim_K \mathbb{C}[\mathrm{Sh}(V,K)] \tag{101}$$

or in

$$\varinjlim_K H^i_{\mathrm{et}}\left(\mathrm{Sh}(V,K)\times_F\bar{F},\bar{\mathbb{Q}}_\ell\right) \tag{102}$$

for some $i\in\mathbb{Z}$, then there is a base-changed representation $BC(\pi)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ unique up to isomorphism. If V is standard indefinite, the conjugate ℓ-adic Galois representation attached to $BC(\pi)$ appears with multiplicity one in

$$\varinjlim_K H^{n-1}_{\mathrm{et}}\left(\mathrm{Sh}(V,K)\times_F\bar{F},\bar{\mathbb{Q}}_\ell\right). \tag{103}$$

5.2.5. *Moduli schemes.* — Next, we introduce several moduli problem functors parametrizing certain CM abelian varieties. The fundamental data are as follows.

1. Unitary abelian schemes over S, that is to say pairs (A,λ) where A is an S-abelian scheme with a supplementary action of $\mathcal{O}_F$ and $\lambda\colon A\longrightarrow A^\vee$ is a quasi-polarization. The quasi-polarization is said to be p-principal if λ and λ^{-1} become isogenies after multiplication by an element of $\mathbb{Z}_{(p)}^\times$. Fix $\tau\colon F\hookrightarrow\mathbb{C}$ and denote by $\mathbb{Q}_p^\tau\supset\mathbb{Z}_p^\tau$ the composition of $\mathbb{Q}_p$ with $\tau(F)$ and its unit ball. If S is a scheme over $\mathbb{Z}_p^\tau$ (as will usually be the case in the following), then the $\mathcal{O}_S$-bilinear pairing

$$H_1^{\mathrm{dR}}(A/S)_\tau\times H_1^{\mathrm{dR}}(A/S)_{\bar\tau}\longrightarrow\mathcal{O}_S \tag{104}$$

induced by the quasi-polarization λ is perfect if (and only if) λ is p-principal (here, the subscripts τ and $\bar\tau$ indicate respectively the maximal submodule on which $\mathcal{O}_F$ acts through its inclusion in $\mathbb{Z}_p^\tau$ and through its inclusion in $\mathbb{Z}_p^{\bar\tau}$).

2. A CM-type Φ containing a preferred embedding $\tau_\infty\colon F\hookrightarrow\mathbb{C}$ extending $\tau_{F^+,\infty}$, that is to say a formal sum

$$\sum_{\tau\colon F\hookrightarrow\mathbb{C}} r_\tau\tau \tag{105}$$

with coefficients $r_\tau\in\mathbb{N}$ satisfying $r_\tau+r_{\bar\tau}=1$ for all τ and $r_{\tau_\infty}=1$. More generally, we consider generalized CM-types of rank n, which are formal sums as above but this time with $r_\tau+r_{\bar\tau}=n$.

3. A rational skew-hermitian space W_0 over $\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ of rank one and CM type Φ.

4. A subtorus $T_0 \subset \left(\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\,\mathbb{G}\right) \otimes \mathbb{Z}[1/(\operatorname{disc} F)]$ satisfying

$$T_0(R) = \{a \in \mathcal{O}_F \otimes_{\mathbb{Z}} R | N_{F/F^+}a \in R^\times\} \tag{106}$$

for every $\mathbb{Z}[1/(\operatorname{disc} F)]$-algebra R.

5. A compact open subgroup K_0^p of $T_0(\mathbb{A}^{(p\infty)})$.

We then consider the moduli problem $\mathbf{T}_p^1(W_0, K_0^p)$ classifying set of equivalence classes of triples $(A_0, \lambda_0, \eta_0^p)$ over $\mathbb{Z}_{(p)}$ where (A_0, λ_0) is a unitary abelian scheme of signature type Φ with a p-principal quasi-polarization λ_0 and which is endowed with a K_0^p-level structure η_0^p defined using W_0. Here, equivalence is understood up to prime-to-p, $\mathcal{O}_F$-linear quasi-isogeny on A_0 and up to units in $\mathbb{Z}_{(p)}^\times$ on λ_0. Let $\mathbf{T}_p(W_0, K_0^p)$ be a certain closed sub-scheme of $\mathbf{T}_p^1(W_0, K_0^p)$ (see definitions 3.5.1-3.5.5 in Liu, Tian, Xiao, Zhang, and Zhu (2022) for details). Let $\mathbf{T}_{\mathfrak{p}}$ be a base-change of $\mathbf{T}_p(W_0, K_0^p)$ to a certain finite extension of $\mathbb{Z}_p$.

Let V be a standard hermitian space of rank n. A crucial object of Liu, Tian, Xiao, Zhang, and Zhu (2022) is the moduli problem functor $\mathbf{M}_{\mathfrak{p}}(V, -)$. If K^p is a sufficiently small compact open subgroup of $U(V)(\mathbb{A}_{F^+}^{(p\infty)})$, $\mathbf{M}_{\mathfrak{p}}(V, K^p)$ is an enrichment of $\mathbf{T}_p(W_0, K_0^p)$ obtained by classifying equivalence classes of sextuples $(A_0, \lambda_0, \eta_0^p, A, \lambda, \eta^p)$ where (A, λ, η^p) is an additional triple satisfying the following properties. If V is standard indefinite, A is a unitary abelian scheme with generalized signature type $n\Phi - \tau_\infty + \bar{\tau}_\infty$, with level structure η^p depending on V outside p and such that λ is p-principal. Note that in this case, the symbol $\mathfrak{p}$ in $\mathbf{M}_{\mathfrak{p}}$ is just a notation, as no such $\mathfrak{p}$ intervenes in the definition. If V is standard definite, we fix a well-chosen inert prime $\mathfrak{p}$ in $\operatorname{Spec} \mathcal{O}_{F^+}$ ($\mathfrak{p}$ is an unramified prime of degree one over $\mathbb{Q}$ which is inert in F). Then A is again a unitary abelian scheme with generalized signature type $n\Phi - \tau_\infty + \bar{\tau}_\infty$, with level structure η^p depending on V outside p but this time we require that $\ker \lambda[p^\infty]$ be of rank p^2 and contained in $A[\mathfrak{p}]$.

In both cases, there is an obvious forgetful functor

$$\mathbf{M}_{\mathfrak{p}}(V, -) \longrightarrow \mathbf{T}_{\mathfrak{p}}(V, -) \tag{107}$$

which is represented by a quasi-projective scheme of relative dimension $n-1$ which is smooth if V is standard indefinite and strictly semistable if V is standard definite. The moduli problem $\mathbf{M}_{\mathfrak{p}}(V, -)$ plays in the setting of that article the role of $X^{(\ell)}$ compared to the modular curve $X_0(N)$ in Bertolini and Darmon (2005).

Next, we define an auxiliary moduli problem functor $\mathbf{S}_{\mathfrak{p}}(V, -)$ which is constructed in a dual way compared to $\mathbf{M}_{\mathfrak{p}}(V, -)$. Like $\mathbf{M}_{\mathfrak{p}}(V, K^p)$, $\mathbf{S}_{\mathfrak{p}}(V, -)$ classifies sextuples

$$(A_0, \lambda_0, \eta_0^p, A^\star, \lambda^\star, \eta^{p\star}) \tag{108}$$

with $(A_0, \lambda_0, \eta_0^p) \in \mathbf{T}_{\mathfrak{p}}$. Both when V is standard definite and when V is standard indefinite, $\eta^{p\star}$ is a level structure outside p which depends on V. The pair $(A^\star, \lambda^\star)$ depends on whether V is indefinite or definite in a symmetric way as for $\mathbf{M}_{\mathfrak{p}}$ (in particular $\mathbf{S}_{\mathfrak{p}}$ does not depend on the choice of $\mathfrak{p}$ when V is standard definite). If V is standard indefinite, $A^\star$ is a unitary abelian scheme of generalized signature type $n\Phi$

and $\ker \lambda^\star[p^\infty]$ is trivial if n is odd and of rank p^2 contained in $A^\star[\mathfrak{p}]$ if n is even. If V is standard definite, $A^\star$ is a unitary abelian scheme of generalized signature type $n\Phi$ and $\lambda^\star$ is p-principal. Just as in the case of $\mathbf{M}_\mathfrak{p}$, there is an obvious forgetful functor

$$\mathbf{S}_\mathfrak{p}(V,-) \longrightarrow \mathbf{T}_\mathfrak{p}(V,-) \tag{109}$$

which is represented by a finite and étale scheme.

Henceforth, we assume that V is standard definite and fix a choice of compact open subgroup $K^p \subset U(V)(\mathbb{A}_{F^+}^{(p\infty)})$. In that case, there is a standard indefinite hermitian module V' such that the generic fiber of $\mathbf{M}_\mathfrak{p}(V,K^p)$ is the fiber product of a certain unitary Shimura variety $\mathrm{Sh}(V',K'^p)$ with the generic fiber of $\mathbf{T}_\mathfrak{p}$ and the étale cohomology of the special fiber of $\mathbf{M}_\mathfrak{p}(V,K^p)$ is computed in terms of the étale cohomology of $\mathrm{Sh}(V',K'^p)$. See Liu, Tian, Xiao, Zhang, and Zhu (2022, Definition 5.2.6 and lemma 5.2.7) for details.

The description of the special fiber is considerably more involved and is part of the technical heart of the paper. Let $(\mathcal{A},\lambda)$ be the universal abelian scheme over $\mathbf{M}_\mathfrak{p}(V,K^p)$ corresponding to the second abelian variety of the sextuple and let $\mathcal{A}^\vee$ be the dual abelian scheme. We then have a Hodge exact sequence

$$0 \longrightarrow \omega_{\mathcal{A}^\vee,\tau_\infty} \longrightarrow H_1^{\mathrm{dR}}(\mathcal{A})_{\tau_\infty} \longrightarrow \mathrm{Lie}_{\mathcal{A},\tau_\infty} \longrightarrow 0 \tag{110}$$

as well as perfect bilinear pairing between $H_1^{\mathrm{dR}}(\mathcal{A})_{\tau_\infty}$ and $H_1^{\mathrm{dR}}(\mathcal{A})_{\bar\tau_\infty}$ since $\mathcal{A}$ admits the p-principal quasi-polarization λ by definition of $\mathbf{M}_\mathfrak{p}$ (here the subscripts τ_∞ and $\bar\tau_\infty$ indicate as above the maximal submodule on which $\mathcal{O}_F$ acts through its morphism to the ring of integers of the composition of $\tau(F)$ with $\mathbb{Q}_p$). Let $M_\mathfrak{p}(V,K^p)$ be the special fiber of $\mathbf{M}_\mathfrak{p}(V,K^p)$. Then $M_\mathfrak{p}(V,K^p)$ is the union of two strata which meet at a third one, according to the following description.

1. The *balloon stratum* $M_\mathfrak{p}^\circ(V,K^p)$ is the locus of $M_\mathfrak{p}(V,K^p)$ on which $\omega_{\mathcal{A}^\vee,\tau_\infty}$ coincides with the right orthogonal complement of $H_1^{\mathrm{dR}}(\mathcal{A})_{\bar\tau_\infty}$.
2. The *ground stratum* $M_\mathfrak{p}^\bullet(V,K^p)$ is the the locus of $M_\mathfrak{p}(V,K^p)$ on which the right orthogonal complement of $H_1^{\mathrm{dR}}(\mathcal{A})_{\tau_\infty}$ is a sub line-bundle of $\omega_{\mathcal{A}^\vee,\bar\tau_\infty}$.
3. The *link stratum* $M_\mathfrak{p}^\dagger(V,K^p)$ is the intersection of $M_\mathfrak{p}^\circ(V,K^p)$ and $M_\mathfrak{p}^\bullet(V,K^p)$.

Then $M_\mathfrak{p}(V,K^p)$ is the union of the ballon and ground strata, these two strata are closed subschemes smooth over the special fiber $T_\mathfrak{p}$ of $\mathbf{T}_\mathfrak{p}$ and Liu, Tian, Xiao, Zhang, and Zhu (2022, Theorem 5.2.5) provide a description in terms of Lie-algebra property of their tangent sheaf relative to $T_\mathfrak{p}$. For instance, there is a canonical isomorphism between the relative tangent sheaf of the balloon stratum $M_\mathfrak{p}^\circ(V,K^p)$ and the sheaf of morphisms from $\omega_{\mathcal{A}^\vee,\bar\tau_\infty}$ to $\mathrm{Lie}_{\mathcal{A},\bar\tau_\infty}$. Moreover, the balloon stratum is essentially completely described by what the authors call the basic correspondence which identifies the balloon stratum as a fibration over the zero-dimensional special fiber $S_\mathfrak{p}^\circ(V,K^p)$ of $\mathbf{S}_\mathfrak{p}(V,K^p)$. Each fiber is a certain projective space and the intersection of each fiber with the link stratum is a certain Deligne–Lusztig variety attached to this projective space. See Liu, Tian, Xiao, Zhang, and Zhu (2022, Theorem 5.3.4) for details. If this author understands correctly, this description is the origin of the terminology: the balloon stratum is decomposed as fibers (balloons) linked to the ground by Deligne–Lusztig varieties.

In agreement with this intuitive picture, there is a similar description of the ground stratum (more precisely of the basic locus of the ground stratum) in terms of a basic correspondence with yet another slightly different moduli problem $S_{\mathfrak{p}}^{\bullet}$, in which $A^{\bullet}$ is a unitary abelian scheme of generalized signature type $n\Phi$ and $\ker \lambda^{\star}[p^{\infty}]$ is trivial if n is odd and of rank p^2 contained in $A^{\bullet}[\mathfrak{p}]$ if n is even, and a description of the link stratum in terms of a basic correspondence with a moduli problem combining $S_{\mathfrak{p}}^{\circ}$ and $S_{\mathfrak{p}}^{\bullet}$.

5.2.6. *Product of varieties and algebraic cycles.* — We introduce the following objects.

1. Two moduli problems $\mathbf{M}_{n_0}$ and $\mathbf{M}_{n_1}$ with $\mathbf{M}_{n_i} \stackrel{\text{def}}{=} \mathbf{M}_{\mathfrak{p}}(V_{n_i}, -)$ attached to a Hermitian module of rank n_i. We assume that these two moduli problems are compatible in the sense that a choice of compact open subgroups in $U(V_{n_i})(\mathbb{A}_{F^+}^{(\infty)})$ determines a choice of compact open subgroup in $U(V_{n_i})(\mathbb{A}_{F^+}^{(\infty)})$ (see Liu, Tian, Xiao, Zhang, and Zhu (2022, Definition 3.1.11 (3)) for details).
2. $\mathbf{P}$ is the fiber product over $\mathbf{T}_{\mathfrak{p}}$ of $\mathbf{M}_{n_0}$ and $\mathbf{M}_{n_1}$.
3. $P^{\circ,\circ}$ is the closed subscheme of the special fiber of $\mathbf{P}$ defined by $M_{n_0}^{\circ} \times_{T_{\mathfrak{p}}} M_{n_1}^{\circ}$ and similarly for $P^{\circ,\bullet}, P^{\circ,\dagger}$...
4. $\mathbf{Q}$ is the blow-up of $\mathbf{P}$ along the subscheme $P^{\circ,\circ}$. As with $\mathbf{P}$, consider also $Q^{?_1,?_2}$ with $?_i \in \{\circ, \bullet, \dagger\}$. Liu, Tian, Xiao, Zhang, and Zhu (2022, Lemma 5.11.3) describes completely the reduction graph [9] of the special fiber Q of $\mathbf{Q}$ in terms of the nine closed subschemes $Q^{?_1,?_2}$.

5.2.7. *Projection to ϕ_{Π_i}-eigenspaces.* — All these general constructions have Hecke actions and admit projection to Hecke-eigenspaces corresponding to the morphisms ϕ_{Π_0} and ϕ_{Π_1}. The cohomology of the strata of the Shimura varieties introduced in subsections 5.2.3 to 5.2.6 tends to vanish outside of very specific degrees after localization at suitable maximal ideals of the Hecke algebra (this is where the condition *admissible* occurs). This implies the freeness of local cohomology groups measuring the bad reduction of our Shimura varieties of interest by the weight spectral sequence (Liu, Tian, Xiao, Zhang, and Zhu, 2022, Theorem 6.3.4).

5.2.8. *Passing from n to $n+1$.* — Recall that after equation (109), we have assumed that V is standard definite. Let n be its rank. The next important technical achievement of Liu, Tian, Xiao, Zhang, and Zhu (2022) is to describe the functorial behavior of all the objects of the preceding paragraphs (the Shimura variety $\mathrm{Sh}(V', -)$, the moduli scheme $\mathbf{M}_{\mathfrak{p}}(V, -)$, compact open subgroup providing a level structure at p, the special fiber $M_{\mathfrak{p}}(V, -)$ of $\mathbf{M}_{\mathfrak{p}}(V, -)$ its balloon, ground and link strata, the descriptions of these strata by the basic correspondence in terms of auxiliary moduli problems, the

9. Here, by *reduction graph* of Q, we mean a complete description of the unions of the strata of increasing codimension of Q - this is would be the dual graph if Q were a curve.

Deligne–Lusztig varieties appearing...) when rank is increased by 1, that is to say when V is replaced by $V_\sharp$.

An insight of the technical complexity of the problem may be gained by noticing that one of the basic results in Liu, Tian, Xiao, Zhang, and Zhu (2022) is the construction of a commutative diagram involving in total 22 different strata of 10 different Shimura varieties (Liu, Tian, Xiao, Zhang, and Zhu, 2022, p. 239).

5.2.9. *Proof of the main theorem.* —

5.2.10. *The main geometric input.* — With these tools at hand, one can finally state and prove the reciprocity laws which generalize theorems 5.2 and 5.3 to this setting.

Let L be either a finite ring in which p is invertible, or the ring of integers of a finite extension of $\mathbb{Q}_\ell$, or a finite extension of $\mathbb{Q}_\ell$. We recall that we are considering compatible moduli problems $\mathbf{M}_{n_0}$ and $\mathbf{M}_{n_1}$ with $\{n_0, n_1\} = \{n, n+1\}$.

Theorem 5.8 (Liu, Tian, Xiao, Zhang, and Zhu (2022) Theorem 5.11.5)

For f the tensor product of two L-valued functions with finite support on $\mathrm{Sh}(V_{n_0}, K_{n_0})$ *and* $\mathrm{Sh}(V_{n_1}, K_{n_1})$*, the elements of L*

$$\int_{\bar{P}^{\bullet,\bullet}}^{\mathcal{I}} \mathrm{cl}\left(P^{\bullet}_{\triangle}\right) \cup \left(\mathrm{inc}_!^{\bullet,\dagger}(T^{\bullet;\circ}_{n_0,\mathfrak{p}} \otimes I^{\circ}_{n_1,\mathfrak{p}} f) + (p+1)^2\, \mathrm{inc}_!^{\bullet,\bullet}(T^{\bullet;\circ}_{n_0,\mathfrak{p}} \otimes T^{\bullet;\circ}_{n_1,\mathfrak{p}} f)\right) \tag{111}$$

and

$$\sum_{s \in \mathrm{Sh}\left(V_n, K_n^p K_{n,p}\right)} \left(I^{\circ}_{n_0,\mathfrak{p}} \otimes T^{\circ}_{n_1,\mathfrak{p}} f\right)(s, \mathrm{sh}^{\circ}_{\uparrow}(s)) \tag{112}$$

are equal.

Here

1. $\mathbf{P}$ is the fiber product over $\mathbf{T}_\mathfrak{p}$ of $\mathbf{M}_{n_0}$ and $\mathbf{M}_{n_1}$.
2. $\bar{P}^{\bullet,\bullet}$ is the the fiber product over $T_\mathfrak{p}$ of the ground strata of M_{n_0} and M_{n_1} base-changed to an algebraic closure of $\mathbb{F}_p$.
3. $P^{\bullet}_{\triangle}$ is the graph of the application $m_\uparrow \colon M^{\bullet}_n \longrightarrow M^{\bullet}_{n+1}$ induced from the enhancement of V (of rank n) to $V_\sharp$. It is a closed sub-scheme of $P^{\bullet,\bullet}$.
4. $\mathrm{cl}(P^{\bullet}_{\triangle})$ is the class of $P^{\bullet}_{\triangle}$ in $H^{2n}(\bar{P}^{\bullet,\bullet}, L(n))$.
5. $\mathrm{inc}_!{}^{\bullet,\dagger}$ and $\mathrm{inc}_!{}^{\bullet,\bullet}$ are maps from the L-module of L-valued functions with finite support on Shimura varieties to equivariant cohomology of $\bar{P}^{\bullet,\bullet}$. More precisely, $\mathrm{inc}_!{}^{\bullet,\dagger}$ is a map

$$\mathrm{inc}_!{}^{\bullet,\dagger} \colon L[\mathrm{Sh}\left(V_{n_0}, K^p K_p^{\bullet}\right)] \otimes_L L[\mathrm{Sh}\left(V_{n_1}, K^p K_p^{\circ}\right)] \longrightarrow H_c^{2(n-1)}\left(\bar{P}^{\bullet,\bullet}, L(n-1)\right) \tag{113}$$

and $\mathrm{inc}_!{}^{\bullet,\bullet}$ is a map

$$\mathrm{inc}_!{}^{\bullet,\bullet} \colon L[\mathrm{Sh}\left(V_{n_0}, K^p K_p^{\bullet}\right)] \otimes_L L[\mathrm{Sh}\left(V_{n_1}, K^p K_p^{\bullet}\right)] \longrightarrow H_c^{2(n-1)}\left(\bar{P}^{\bullet,\bullet}, L(n-1)\right) \tag{114}$$

6. $T^{\bullet;\circ}_{n_i,\mathfrak{p}}$ is the characteristic function of $K^{\bullet}_{\mathfrak{p}} K^{\circ}_{\mathfrak{p}}$ in $\mathbb{Z}[K^{\bullet}_{\mathfrak{p}} \backslash U(V_{n_i})(F^+_{\mathfrak{p}})/K^{\circ}_{\mathfrak{p}}]$.
7. $T^{\circ;\bullet}_{n_i,\mathfrak{p}}$ is the characteristic function of $K^{\circ}_{\mathfrak{p}} K^{\bullet}_{\mathfrak{p}}$ in $\mathbb{Z}[K^{\circ}_{\mathfrak{p}} \backslash U(V_{n_i})(F^+_{\mathfrak{p}})/K^{\bullet}_{\mathfrak{p}}]$.

8. $I^{\circ}_{n_i,\mathfrak{p}} \in \mathbb{Z}[K^{\circ}_{\mathfrak{p}}\backslash U(V)(F^+_{\mathfrak{p}})/K^{\circ}_{\mathfrak{p}}]$ is the intertwining operator $T^{\circ,\bullet}_{n_i,\mathfrak{p}} \circ T^{\bullet,\circ}_{n_i,\mathfrak{p}}$.
9. $T^{\circ}_{n_1,\mathfrak{p}}$ is a double coset $K^{\circ}_{\mathfrak{p}}\varpi^t K^{\circ}_{\mathfrak{p}}$ with $t \in \mathbb{Z}^n$ well-chosen.
10. $\mathrm{sh}^{\circ}_{\uparrow}\colon \mathrm{Sh}(V_n, K_n) \longrightarrow \mathrm{Sh}(V_{n+1}, K_{n+1})$ is the morphism of Shimura varieties induced by the inclusion $V_n \hookrightarrow V_{n+1}$.
11. Finally, $\int^{\mathcal{I}}_{\bar{P}^{\bullet,\bullet}}$ is a trace map defined as follows. Denote by $H^{2n}_{\mathcal{I}}\left(\bar{P}^{\bullet,\bullet}, L(n)\right)$ the maximal L-submodule of $H^{2n}\left(\bar{P}^{\bullet,\bullet}, L(n)\right)$ on which $T_0(\mathbb{A}^{(p\infty)})/T_0(\mathbb{Z}_{(p)})K^p_0$ acts trivially (recall that $\mathbf{T}_p(W_0, K^p_0)$ has a natural action of $T_0(\mathbb{A}^{(p\infty)})/T_0(\mathbb{Z}_{(p)})K^p_0$). Let $\mathcal{Y} = \{Y\}$ be a set of representatives of $T_0(\mathbb{A}^{(p\infty)})/T_0(\mathbb{Z}_{(p)})K^p_0$-orbits of the set of connected components of $\bar{P}^{\bullet,\bullet}$. Then

$$\int^{\mathcal{I}}_{\bar{P}^{\bullet,\bullet}} : H^{2n}_{\mathcal{I}}\left(\bar{P}^{\bullet,\bullet}, L(n)\right) \longrightarrow L \tag{115}$$

is the inclusion of $H^{2n}_{\mathcal{I}}\left(\bar{P}^{\bullet,\bullet}, L(n)\right)$ inside $H^{2n}_{\mathcal{I}}\left(\bar{P}^{\bullet,\bullet}, L(n)\right)$ composed with

$$H^{2n}\left(\bar{P}^{\bullet,\bullet}, L(n)\right) \longrightarrow \sum_{Y\in\mathcal{Y}} H^{2n}\left(Y, L(n)\right) \xrightarrow{\Sigma\,\mathrm{tr}_Y} L. \tag{116}$$

5.2.11. *The Euler system argument.* — Theorem 5.8 allows for a characterization of the image under the Abel–Jacobi map of the algebraic cycle $\mathbf{Q}$ in terms of Hecke operators (Liu, Tian, Xiao, Zhang, and Zhu, 2022, Theorem 7.2.8, Corollary 7.2.9 and Theorem 7.3.4) which is a close counterpart of the explicit reciprocity laws of Bertolini and Darmon (2005).

The first and central part of the argument relies on the Gan–Gross–Prasad conjecture for $\Pi_0 \times \Pi_1$ and provides a crucial link between the automorphic input provided by the automorphic representations of interest and the geometric input given by theorem 5.8. More precisely, the proof of the Gan–Gross–Prasad conjecture (Beuzart-Plessis, Liu, Zhang, and Zhu, 2021) provides a function $f \in \mathcal{O}_{E_\lambda}[\mathrm{Sh}(V_{n_0}, K_{n_0})][\ker \phi_{\Pi_0}] \otimes_{\mathcal{O}_{E_\lambda}} \mathcal{O}_{E_\lambda}[\mathrm{Sh}(V_{n_1}, K_{n_1})][\ker \phi_{\Pi_1}]$ such that

$$\sum_{s\in \mathrm{Sh}(V_n,K_n)} f(s, \mathrm{sh}_{\uparrow}(s)) \neq 0. \tag{117}$$

The function f is paired against the image $AJ(\mathbf{Q})$ of $\mathbf{Q}$ under the Abel–Jacobi map. Since we may choose ℓ and the auxiliary prime p so that the supplementary terms in theorem 5.8 are all invertible, this theorem links the exponent of the localization at p of $AJ(\mathbf{Q})$ in $H^1_{\mathrm{sing}}(G_{\mathbb{Q}_{p^2}}, H^{n_0-1}_{\mathcal{I}}(\bar{M}_{\mathfrak{p}}(V), R\psi\mathcal{O}_{E_\lambda})_{\mathfrak{m}_0})$ with geometric data. Here

$$H^1_{\mathrm{sing}}(G_{\mathbb{Q}_{p^2}}, H^{n_0-1}_{\mathcal{I}}(\bar{M}_{\mathfrak{p}}(V), R\psi\mathcal{O}_{E_\lambda})_{\mathfrak{m}_0}) \tag{118}$$

is the singular part of the Galois cohomology of the étale cohomology of $\bar{M}_{\mathfrak{p}}$ localized at the maximal ideal attached to the Galois representation ρ_0 of dimension n_0.

As in section 5.1.2, the argument relies on a description in terms of bad reduction of $\mathrm{Sh}(V_{n_0})$ of the module

$$H^1_{\mathrm{sing}}\left(G_{\mathbb{Q}_{p^2}}, H^{n_0-1}_{\mathcal{I}}(\bar{M}_{\mathfrak{p}}(V), R\psi\mathcal{O}_{E_\lambda})_{\mathfrak{m}_0}\right). \tag{119}$$

In section 5.1.2, this was achieved thanks to the hypothesis that f be p-isolated. In Liu, Tian, Xiao, Zhang, and Zhu (2022), the need of that strong supplementary assumption is bypassed using a Galois deformation theoretic argument in which the freeness of some Hecke-modules follows from an $R = T$ theorem for self-dual Galois representations.

To summarize, what Liu, Tian, Xiao, Zhang, and Zhu (2022) achieves is the construction of a bipartite Kolyvagin system of diagonal algebraic cycles in unitary Shimura varieties realizing the Galois representations attached to Rankin–Selberg products of relevant automorphic representations. This Kolyvagin system is provably non-trivial because one may compute intersections of the cycles of interest in terms of test functions, which are themselves known to be non-zero thanks to the link the Gan–Gross–Prasad conjecture establishes between their existence and the non-vanishing of the Rankin–Selberg product L-function.

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