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UNIFORMITY IN DIOPHANTINE GEOMETRY

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Introduction

Mordell (1922/23) concluded his study of polynomial equations of degrees three and four in rational numbers by saying "in conclusion, I might note that the preceding works suggest to me the truth the following statements, concerning indeterminate equations, none of which, however, I can prove." The last of the statements was that "the same theorem" (namely that there are only finitely many rational solutions) "holds for any homogeneous equation of genus greater than unity, say, $f(x, y, z) = 0$." Mordell's conjecture as such was proven with the celebrated work of Faltings (1983).

The problem of effectively bounding the number of solutions by a function of purely geometric data has remained elusive over the past four decades. This exposé focuses on some recent spectacular progress giving such bounds and also resolving related questions.

What we might mean by effective bounds depends on the situation. When considering Mordell's conjecture as a statement about rational points on curves, one might hope to bound the number of such rational points by a constant depending only on the genus of the curve and the degree over the rational numbers of the number field in which one seeks the solutions. Alternatively, we might hope to find bounds on the sizes as measured, say by height, of the points on such a curve depending on such data as the heights of the coefficient of the defining equations and the genus of the curve. The problem in the first presentation remains open in general. Effective proofs of Mordell's conjecture due to Bombieri (1990) and Vojta (1992) give methods for bounding the number of rational points on a curve from some arithmetic information about the curve. The kind of uniform bounds that we discuss in this exposé align with and improve these bounds by eliminating the dependence on the arithmetic data.

Using the theorem of Mordell (1922/23) and Weil (1929) that the group of rational points on an abelian variety is finitely generated as an abelian group, Lang reformulated the Mordell conjecture in a more geometric form which then naturally generalizes to a statement about higher dimensional varieties.

Start with a finitely generated field K of characteristic zero and C a smooth projective curve of genus at least two over K. Fixing any base point on C (which we may assume to be K-rational, for otherwise $C(K) = \varnothing$ which is finite) we have an embedding of C as a closed algebraic subvariety of its Jacobian, $\mathrm{Jac}(C)$, which is an abelian variety defined over K. Let $\Gamma := \mathrm{Jac}(C)(K)$ be the finitely generated group of K-rational points on the Jacobian. Since the embedding induces a bijection on rational points, we may realize $C(K)$ as the intersection $C(\mathbb{C}) \cap \Gamma$. After making a few other observations, one sees that the Mordell conjecture is equivalent to the assertion that for any abelian variety A over the complex numbers, finitely generated group $\Gamma \leq A(\mathbb{C})$, and algebraic curve $C \subseteq A$ of genus at least two, $\Gamma \cap C(\mathbb{C})$ is finite. This formulation of the Mordell conjecture suggests the Mordell–Lang conjecture which asserts that for any abelian variety A defined over the complex numbers, finite *rank* subgroup $\Gamma \leq A(\mathbb{C})$ (by which we mean that $\mathrm{rk}\,\Gamma := \dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q} < \infty$), and closed subvariety $X \subseteq A$, the intersection $X(\mathbb{C}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.

The formulation of the Mordell–Lang conjecture suggests geometric approaches to its proof based on such methods as the geometry of numbers and the complex analytic geometry of the presentation of an abelian variety as a complex torus, that is, as the quotient of a finite dimensional complex vector space by a lattice. Historically, some partial results towards Mordell's conjecture used this kind of geometric reasoning, notably with the proof of Mordell's conjecture over function fields by Manin (1963) and the gap principle of Mumford (1965a), but the original proofs of the full theorem are fundamentally arithmetic in nature making use of Arakelov theory.

The main theorem of Gao, Ge, and Kühne (2021) takes the following form.

Theorem 0.1. — *There is a function $c(g,d)$ depending on two natural number arguments so that for any abelian variety A over the complex numbers given with an appropriately chosen very ample line bundle L, algebraic subvariety X, and finite rank subgroup $\Gamma \leq A(\mathbb{C})$, there is a finite (possibly empty) sequence of connected algebraic subgroups $B_1, \ldots, B_m \leq A$ and points $\gamma_1, \ldots, \gamma_m$ so that*

$$X(\mathbb{C}) \cap \Gamma = \bigcup_{i=1}^{m} (\gamma_i + B_i(\mathbb{C})) \cap \Gamma$$

where

$$m \leq c(\dim A, \deg_L(X))^{1+\mathrm{rk}\,\Gamma}$$

and $\deg_L(B_i) \leq c(\dim A, \deg_L(X))$ *for each* i.

In the special case that X is a curve of genus greater than one embedded in its Jacobian, there are no translates of positive dimensional algebraic subgroups of A contained in X so that Theorem 0.1 asserts simply that the number of points in the intersection $X(\mathbb{C}) \cap \Gamma$ is at most $c^{1+\mathrm{rk}\,\Gamma}$ where c depends only on the degree of X as subvariety of A. This uniform result for curves may be obtained by combining the main theorem of Dimitrov, Gao, and Habegger (2021) with the results of Kühne (2021) and answers positively a problem raised by Mazur (1986).

While the proof of Theorem 0.1 does take into account some of the geometric methods anticipated by Lang's formulation of the Mordell–Lang conjecture, refined height estimates and equidistribution theorems are at its heart. The method used yields uniform versions of the Bogomolov conjecture about points of small height on subvarieties of abelian varieties. We will delay the statement of this theorem until after we have reviewed some of the theory of heights in Section 1.2.

A crucial new ingredient in these proofs is the study of "non-degenerate varieties" which may be defined in terms of differential geometric properties of the Betti map. We delay a description of this condition to Section 3.

An alternative approach to the uniform Mordell–Lang and Bogomolov conjectures is given by Yuan (2021). Yuan's method is based on a refined theory of adèlic line bundles developed by Yuan and Zhang (2021) and circumvents the analysis of non-degenerate varieties.

The main body of this exposé is organized as follows. We begin in Section 1 by recalling some of the theory of abelian varieties and of height functions. We recall some of the earlier work on these problems in Section 2. In particular, in Subsection 2.1 we recall the methods of Mumford and Vojta which were then refined and extended by Rémond for analyzing points which are nearly parallel relative. In Subsection 2.2 we survey some of the work on points of small height, discussing specifically the Manin–Mumford and Bogomolov conjectures. Section 3 is devoted to Betti maps and their interaction with algebraic subvarieties of abelian varieties. We enter into some of the technical details of the uniform Mordell–Lang conjecture in Section 4 where we outline the proof of the crucial new height inequality of Dimitrov, Gao, and Habegger (2021). We explain how points of small height are to be analyzed in Section 5. Finally, in Section 6 the new height inequalities and equidistribution theorems are combined to deduce the new uniform diophantine geometric theorems.

1. Basic properties of the arithmetic and geometry of abelian varieties

1.1. Abelian varieties

Abelian varieties lie at the core of the theorems we are considering. This subject is classical and well exposed in several excellent textbooks and course notes, including Lang (1983), Lange (2023), Milne (2008), and Mumford (2008). In this section we recall some of the basic theory.

By definition, an abelian variety A over a field K is a connected, projective algebraic group. It is a consequence of this definition that the group structure on A is commutative so that the adjective "abelian" which was chosen to honor Abel's work on abelian integrals remains consistent with our common practice of calling commutative groups "abelian".

When $K = \mathbb{C}$ is the field of complex numbers, then since A is projective, the group $A(\mathbb{C})$ is a compact, commutative, complex Lie group. As such, $A(\mathbb{C})$ fits into an exact sequence of complex Lie groups

$$0 \longrightarrow \Lambda_A \longrightarrow T_0A(\mathbb{C}) \xrightarrow{\exp_A} A(\mathbb{C}) \longrightarrow 0$$

where T_0A is the tangent space of A at the identity element 0, $\exp_A$ is the Lie exponential map of A, and $\Lambda_A := \ker \exp_A$ is a lattice in $T_0A(\mathbb{C})$. Fixing a basis of $T_0(\mathbb{C})$ we may regard it as $\mathbb{C}^g$ where $g = \dim A$ and $A(\mathbb{C})$ as the quotient $\mathbb{C}^g/\Lambda_A$, a complex torus.

Passing to universal covers, it is easy to see that the data of a map of complex tori $\psi\colon A(\mathbb{C}) \to B(\mathbb{C})$ between complex abelian varieties is equivalent to that of a linear map $\widetilde{\psi}\colon T_0A(\mathbb{C}) \to T_0B(\mathbb{C})$ which takes Λ_A to Λ_B. In this way, the space of complex tori of dimensions g may be parameterized by the space of lattices in $\mathbb{C}^g$ up to the action of $\mathrm{GL}_g(\mathbb{C})$. Since not every complex torus is (the analytification of) an abelian variety, we must restrict the space of lattices in order to describe moduli spaces of abelian varieties.

Let $\mathfrak{h}_g$ the g^{th} Siegel upper halfspace consisting of symmetric $g \times g$ complex matrices whose imaginary parts are positive definite. For a sequence $D = \langle d_1, \ldots, d_g\rangle$ of positive integers with $d_i | d_{i+1}$ for $1 \le i < g$ we overload our notation writing $D = \mathrm{diag}(d_1, \ldots, d_g)$ for the diagonal matrix with d_i in the (i,i)-entry for $1 \le i \le g$.

The algebraic group $\mathrm{Sp}_{2g,D}$ is defined by

$$\mathrm{Sp}_{2g,D} := \{g \in \mathrm{GL}_{2g} : g \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} g^{\intercal} = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}\} \ .$$

Writing elements of $\mathrm{Sp}_{2g}(\mathbb{R})$ as 2×2 matrices of $g \times g$ real matrices, we have a transitive action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{h}_g$ via the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1} \ .$$

Via this action, we may see $\mathfrak{h}_g$ as the quotient of Sp_{2g} by the unitary group.

Form the semidirect product $\mathbb{R}^{2g} \rtimes \mathrm{Sp}_{2g,D}(\mathbb{R})$ via the usual action of GL_{2g} on $2g$-dimensional space. This group $\mathbb{R}^{2g} \rtimes \mathrm{Sp}_{2g,D}(\mathbb{R})$ acts on $\mathbb{C}^g \times \mathfrak{h}_g$ via the rule

$$(r_1, \ldots, r_{2g}, M) \cdot (z, \tau) := (z + (r_1, \ldots, r_g)D + (r_{g+1}, \ldots, r_{2g})\tau, M \cdot \tau) \ .$$

Given $\tau \in \mathfrak{h}_g$, we form the lattice $\Lambda_{\tau,D} := D\mathbb{Z}^g + \tau\mathbb{Z}^g$. The complex torus $\mathbb{C}^g/\Lambda_{D,\tau}$ is an abelian variety with a polarization of type D. Moreover, (the analytification of) every complex abelian variety arises in this way.

The coarse moduli space of g-dimensional abelian varieties with a polarization of type D may be realized as $\mathrm{Sp}_{2g,D}(\mathbb{Z})\backslash\mathfrak{h}_g$. Taking appropriate arithmetic subgroups $\Gamma \le \mathrm{Sp}_{2g,D}(\mathbb{Z})$ the quotient $\mathcal{A}_{g,D,\Gamma} = \Gamma\backslash\mathfrak{h}_g$ may be seen as a quasiprojective variety over the algebraic numbers giving a fine moduli space for polarized g-dimensional abelian varieties with some level structure and the universal abelian variety $\mathfrak{A}_{g,D,\Gamma} \to \mathcal{A}_{g,D,\Gamma}$ may be realized as the quotient $(\mathbb{Z}^{2g} \rtimes \Gamma)\backslash(\mathbb{C}^g \times \mathfrak{h}_g)$. For example, if $N \ge 3$ and Γ is the kernel of the reduction map from $\mathrm{Sp}_{2g,D}(\mathbb{Z}) \to \mathrm{Sp}_{2g,D}(\mathbb{Z}/N\mathbb{Z})$ then $\mathcal{A}_{g,D,\Gamma} = \mathcal{A}_{g,D,N}$ is the moduli space of abelian varieties with a polarization of type D and full level N structure.

For almost the problems we consider, the specific choice of polarization type and level structure is irrelevant. Indeed, even when one must keep track of these data in the course of a proof, one can usually reduce to the case of principally polarized abelian varietes. We shall simply write $\mathfrak{A}_g \to \mathcal{A}_g$ for some suitable choice of a universal abelian variety of dimension g.

Given an algebraically closed field k of characteristic zero, K an algebraically closed extension of k, and A an abelian variety over K, the K/k-trace of A, $\operatorname{Tr}_{K/k} A$, is an abelian variety over k given together with an embedding $\rho\colon (\operatorname{Tr}_{K/k} A)_K \hookrightarrow A$ of its base change to K in A. We write $A^{K/k}$ for the image of ρ. The map ρ is universal for maps from abelian varieties over k to A in the sense that if B is an abelian variety defined over k and $\psi\colon B_K \to A$ is a map of algebraic groups from the base change of B to K to A then there is a unique map $\widetilde{\psi}\colon B \to \operatorname{Tr}_{K/k} A$ for which $\psi = \rho \circ \widetilde{\psi}_K$.

1.2. Heights

Heights give a precise sense to the arithmetic size of points on algebraic varieties. For accounts of the theory of heights see Bombieri and Gubler (2006) or Lang (1995).

We mostly restrict our attention to heights of $\mathbb{Q}^{\mathrm{alg}}$-valued points of algebraic variety. However, we should note that theories of heights make sense for points valued in other fields, for example, in algebraic extensions of function fields and that this more general theory makes an appearance with the function field Bogomolov conjecture.

Consider a number field K. By a normalized place v on K we mean an absolute value $|\ |_v$ which comes by pullback from an embedding of K into $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$. Let us write K_v for the completion of K with respect to this absolute value. We define $d_v := [K_v : \mathbb{R}]$ if $|\ |_v$ is not ultrametric and $d_v := [K_v : \mathbb{Q}_p]$ if K_v is a finite extension of the p-adic numbers.

For a natural number N, the logarithmic Weil height on $\mathbb{P}^N(K)$, $h_{\mathbb{P}^N,K}\colon \mathbb{P}^N(\mathbb{Q}^{\mathrm{alg}}) \to [0,\infty)$, is defined by

$$h_{\mathbb{P}^N,K}([a_0 : \ldots : a_N]) := \frac{1}{[K:\mathbb{Q}]} \sum_v d_v \log \max\{|a_0|_v, \ldots, |a_N|_v\}$$

where the sum is taken over all normalized places on K. If $K \leq L$ is an extension of number fields, then the restriction of $h_{\mathbb{P}^N,L}$ to $\mathbb{P}^N(K)$ is equal to $h_{\mathbb{P}^N,K}$. Thus, expressing $\mathbb{Q}^{\mathrm{alg}}$ as a direct limit of number fields, we have a well-defined height function $h_{\mathbb{P}^N}\colon \mathbb{P}^N(\mathbb{Q}^{\mathrm{alg}}) \to [0,\infty)$.

For $X \subseteq \mathbb{P}^N$ an embedded projective variety, we may restrict the height function $h_{\mathbb{P}^N}$ to $X(\mathbb{Q}^{\mathrm{alg}})$ obtaining $h_X\colon X(\mathbb{Q}^{\mathrm{alg}}) \to [0,\infty)$. In what follows, we will drop the subscript writing simply "h".

If $f\colon X \to Y$ is a morphism of degree d between embedded projective varieties, then there are constants C_1 and C_2 so that for all $P \in \mathbb{P}^N(\mathbb{Q}^{\mathrm{alg}})$ we have $dh(P) + C_1 \leq h(f(P)) \leq dh(P) + C_2$. That is, $h(f(P)) = dh(P) + O(1)$. In particular, if $d = 1$, then the difference between $h(f(P))$ and $h(P)$ is bounded.

Consequently, if L is any very ample line bundle on X and $\psi\colon X \hookrightarrow \mathbb{P}^N$ and $\phi\colon X \hookrightarrow \mathbb{P}^M$ are two projective embeddings obtained from L, then the difference between the height functions on X obtained as $h_\psi(P) := h(\psi(P))$ and $h_\phi(P) := h(\phi(P))$ is bounded. That is, for each very ample line bundle L on a projective variety X, we may associate a function $h_{X,L}$ on $X(\mathbb{Q}^{\mathrm{alg}})$ which is well-defined up to a bounded function by the rule that $h_{X,L}(P) = h_{\mathbb{P}^N}(\psi_L(P)) + O(1)$ where $\psi_L\colon X \hookrightarrow \mathbb{P}^N$ is a projective embedding associated to L.

The Weil height machine extends the association of height functions to general line bundles. That is, there is a function which associates to each pair (X, L) consisting of a projective variety X defined over $\mathbb{Q}^{\mathrm{alg}}$ and a line bundle L on X a height function $h_{X,L}\colon X(\mathbb{Q}^{\mathrm{alg}}) \to \mathbb{R}$, well-defined up to a bounded function, so strictly speaking an element of the quotient of the group of real valued functions on $X(\mathbb{Q}^{\mathrm{alg}})$ by the group of bounded real valued functions on that set. This association satisfies the following rules.

- $h_{\mathbb{P}^N,\mathcal{O}(1)} = h + O(1)$
- If $f\colon X \to Y$ is regular map of projective varieties, and L is a line bundle on Y, then $h_{X,f^*L} = h_{Y,L} \circ f + O(1)$. So, in particular, if L is is very ample and $\psi_L\colon X \hookrightarrow \mathbb{P}^N$ is a projective embedding for which $L = \psi_L^*\mathcal{O}(1)$, then $h_{X,L} = h \circ \psi_L + O(1)$.
- $h_{X,L} \geq O(1)$ outside the base locus of L.
- If L is ample, then for any fixed representative of $h_{X,L}$, any natural number d, and any real number ϵ, the set $\{a \in X(\mathbb{Q}^{\mathrm{alg}}) : [\mathbb{Q}(a) : \mathbb{Q}] \leq d \text{ and } h_{X,L}(a) < \epsilon\}$ is finite.
- $h_{X,L\otimes M} = h_{X,L} + h_{X,M} + O(1)$

Since it is more convenient to work with actual functions rather than equivalence classes of functions up to bounded functions, often specific choices are made. Ideally, there would be a canonical choice of such a representative. On projective varieties given with a polarized algebraic dynamical system there is such an associated canonical height. Here, a polarized algebraic dynamical system consists of a projective variety X, an ample line bundle L on X, and a self-map $f\colon X \to X$ for which $f^*L \approx L^{\otimes q}$ for some integer $q > 1$. Starting with any function h in the class of $h_{X,L}$ we may define the canonical height by

$$\widehat{h}_{X,L,f}(x) := \lim_{n\to\infty} \frac{1}{q^n} h(f^n(x)) \ .$$

This canonical height is the unique function $\widehat{h}$ in the class of $h_{X,L,f}$ satisfying the functional equation $\widehat{h} \circ f = q\widehat{h}$. It follows from this equation that for any algebraic point $a \in X(\mathbb{Q}^{\mathrm{alg}})$ we have $\widehat{h}_{X,L,f}(a) = 0$ if and only if a is f-preperiodic, that is, there are natural numbers $m < n$ with $f^m(a) = f^n(a)$.

On any abelian variety A there is an ample, symmetric line bundle L, where by "symmetric" we mean that if $[-1]\colon A \to A$ is the additive inverse map on A, then $L \approx [-1]^*L$. For such a choice of L we have $[2]^*L \approx L^{\otimes 4}$. The canonical height on A is then defined to be $\widehat{h}_{A,L,[2]}$. Specializing to this case of abelian varieties, we see that an algebraic point has canonical height zero if and only if it is torsion.

The canonical height is a quadratic form so that we may define an inner product on the real vector space $A(\mathbb{Q}^{\mathrm{alg}}) \otimes \mathbb{R}$ by the rule that $\langle P, Q\rangle := \frac{1}{2}(\widehat{h}(P+Q) - \widehat{h}(P) - \widehat{h}(Q))$.

The Faltings height gives fully well-defined heights. We refer the reader to the work of Bost, Gillet, and Soulé (1994) for the details about the theory of the Faltings height, comparisons between the Faltings height and related constructions, and most importantly, for the relevant definitions of metrized and adelic line bundles, of Chern classes of these, and of arithmetic intersection theory. The reader would also profit from the excellent survey of Chambert-Loir (2021). For a projective algebraic variety X over a number field K with an ample line bundle L on X and an adelic metric $|| \ ||$ on L, the height of an irreducible subvariety $Y \subseteq X$ of X is defined to be $h(Y) := \frac{\widehat{\deg}\widehat{c}_1(L|_Y, || \ ||)^{\dim Y+1}}{(\dim Y+1)\deg_L Y}$.

We will use a more down to earth definition of the height of a projective variety due to Philippon (1995) (which is a revision of the original definition of Philippon (1991) bringing this notion of height in line with the Faltings height, at least, up to a normalization factor). This definition is based on the parameterization of subvarieties of a given projective space by Chow forms which may themselves be regarded as points in some associated projective space. The height is then the height of the Chow form regarded as a point in a projective space with a certain adjustment to the archimedian contribution.

Let us recall first the construction of the Chow form. For an irreducible subvariety $X \subseteq \mathbb{P}^n$ of dimension d the space S of $n-(d+1)$-planes $V \subseteq \mathbb{P}^n$ which are incident to X is a hypersurface in the the Grassmannian of such planes. As such, S may be defined by a form f_X in variables $(x_{i,j})_{\{0 \leq i \leq \dim X, 0 \leq j \leq n\}}$, homogeneous in $(x_{i,0}, \ldots, x_{i,n})$ for each i. Assume now that X is defined over a number field K and that we have fixed a Chow form f_X. For each place v of K we define a local contribution $h_v(X)$ to the height $h(X)$ of X. One might regard $h_v(X)$ a local height, though, as is often the case when decomposing a height function into local constributions, it lacks some crucial features of a height function. For example, it depends on the choice of the Chow form f_X whereas $h(X)$ does not.

We define a normalizing factor by $d_v := \frac{[k_v:\mathbb{Q}_v]}{[k:\mathbb{Q}]}$ where k_v is the completion of k with respect to v and we are abusing notation by writing v for its restriction to $\mathbb{Q}$. The local height $h_v(X)$ of X at the place v is computed differently for ultrametric and archimedian places. In the ultrametic case, $h_v(X)$ is the normalizing factor d_v times the maximum of $\log |a_\ell|_v$ as a_ℓ runs through the coefficients of f_X. The case where v is archimedian corresponds to an embedding $k \hookrightarrow \mathbb{C}$. We define

$$h_v(X) := d_v \int_{(S^{N+1})^{\dim X+1}} \log |f_X|_v d\mu + (\deg f_X) \sum_{i=1}^{n} \frac{1}{2i}$$

where here S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$ and μ is the product of the the usual invariant measures on the sphere of total mass one. The height $h(X)$ of X is then the sum $\sum_v h_v(X)$.

2. Some relevant earlier approaches to the Mordell–Lang problem

2.1. Mumford's gap and Vojta's method

Mumford (1965a) opens with

> It is somewhat surprising that the systematic evaluation of the heights of rational points on a curve and its jacobian variety and particularly their relation to each other should yield any new information. Nonetheless this appears to be the case and the result is described in this article.

This evaluation is based on some simple observations about Euclidean spaces coupled with estimates on heights of differences of points on the curve. For any given angle θ with $0 < \theta < \frac{\pi}{2}$ there is a constant $c = c(\theta)$ so that the Euclidean space $\mathbb{R}^n$ may be covered by c^n sectors in which the angle between any two points is at most θ.

Consider an abelian variety A over a number field K, a finite rank subgroup $\Gamma \leq A(K^{\mathrm{alg}})$ of the K-rational points, and an algebraic subvariety $X \subseteq A$. Regard $\Gamma_{\mathbb{R}} := \Gamma \otimes \mathbb{R}$ as a Euclidean space using the inner product coming from the canonical height. Set $r := \dim_{\mathbb{R}} \Gamma_{\mathbb{R}}$.

Given an algebraic sub-variety $X \subseteq A$, we find some suitable $\epsilon > 0$ so that we may break the problem of describing $X(K) \cap \Gamma$ into the subproblems of describing the points on X of small height, meaning of height at most ϵ, and then also of the points large height, meaning of height greater than ϵ. We further break up the points of large height into those finitely many sectors on which the angles between the points are small meaning at most θ, again for some well chosen angle θ.

When working with K-rational points, the set of small points is finite. For the large points, Mumford (1965b) proves what is now known as *Mumford's gap* that the heights of points on X for which the angles between the points is small must grow exponentially.

Theorem 2.1. — *Let K be a number field and let X be a smooth projective curve of genus at least two embedded in its Jacobian. Then there are constants c_1, c_2, and c_3 with $0 < c_2 < 1$ and $1 < c_3$ where c_1 depends on X, but c_2 and c_3 do not, so that for any two algebraic points P and Q from $X(K^{alg})$ with $P \neq Q$ and $\widehat{h}(P) \leq \widehat{h}(Q)$ if $c_1 \leq \widehat{h}(P)$ and $\frac{\langle P,Q \rangle}{\widehat{h}(P)\widehat{h}(Q)} \geq c_2$, then $\widehat{h}(Q) \geq c_3 \widehat{h}(P)$.*

Interestingly, if we were to extend the theory of heights to function fields over finite fields, then Mumford's gap remains valid even though the naïve transposition of Mordell's conjecture is false. More importantly for purposes of studying the finiteness theorems uniformly, one should note that the points P and Q are merely algebraic.

Over number fields, Vojta (1991) proves a complementary inequality showing that the heights of points in the same sector must be comparable.

Theorem 2.2. — *With the hypotheses as in Theorem 2.1, there are positive constants κ_1, κ_2, and κ_3 with $0 < \kappa_2 < 1$ and $\kappa_3 > 1$ so that if P and Q are any two distinct algebraic points in $X(K^{alg})$ with $\widehat{h}(P) \geq \kappa_1$ and $\frac{\langle P,Q \rangle}{\widehat{h}(P)\widehat{h}(Q)} \geq \kappa_2$, then $\widehat{h}(Q) \leq \kappa_3 \widehat{h}(P)$.*

This inequality gives the finiteness of the set of rational points and combined with Mumford's gap may be used to give better bounds on the number of rational points which may be effectively computed from a presentation of the curve and the data of the height of some point in each of the sectors. Bombieri (1990) reworks Vojta's method replacing some of the sophisticated arguments based on arithmetic intersection theory with more explicit computations with polynomials.

When considering the Mordell–Lang problem of intersections $X \cap \Gamma$ where $X \subseteq A$ is a subvariety and $\Gamma \leq A(\mathbb{C})$ is a finite rank subgroup, it may happen that this intersection is infinite. For instance, this could happen if X were a translate of a positive dimensional algebraic subgroup of A, or even if X simply contains such a translate. We may restate the Mordell–Lang conjecture as a simple finiteness assertion by removing from X these potential counterexamples.

Definition 2.3. — *The Ueno locus* $\mathrm{Ueno}(X)$ *(sometimes also called the Kawamata locus) of X is the union of all translates of positive dimensional algebraic subgroups of B which are fully contained in X. We write X^0 for $X \setminus \mathrm{Ueno}(X)$.*

From the definition, $\mathrm{Ueno}(X)$ is merely a union of algebraic varieties. In fact, it is actually a closed subvariety of X.

Lemma 2.4. — *Let $X \subseteq A$ be an irreducible subvariety of the abelian variety A. Let $S = \{a \in A : a + X = X\}$ be the stabilizer of X in A. Then* $\mathrm{Ueno}(X)$ *is a closed subvariety of X which is a proper subvariety of X if and only if S is finite.*

The proof of Lemma 2.4 proceeds by induction on $\dim X$ by considering the family of varieties $X \cap (a + X)$ for which $X \neq (a + X)$ noting that under the hypothesis that S is finite the Ueno locus is contained in the union of the positive dimensional components of these intersections.

It follows from Lemma 2.4 that the Ueno locus varies algebraically in families. That is, if $\pi\colon A \to S$ is a family of abelian varieties and $X \subseteq A$ is a subvariety of A, then there is a subvariety $U \subseteq X$ of X so that for any $s \in S$ the fiber $U_s := (\pi|_U)^{-1}\{s\}$ is the Ueno locus of X_s.

As a standard dévissage, the Mordell–Lang problem for X, A, and Γ may be solved first by working with the irreducible components of X separately, and then after reducing to the case that X is irreducible, by passing to the image of X and Γ in the quotient of A by S so that we may assume that X is irreducible and has a trivial stabilizer. With this reduction, Mordell–Lang becomes the assertion that $X^0 \cap \Gamma$ is finite.

Faltings (1991) proves Mordell–Lang when Γ is finitely generated. Of particular relevance to the new uniform versions of Mordell–Lang is the work of Rémond (2000) giving an effective version of the Mordell–Lang conjecture. A key step in that proof is a higher dimensional version of Vojta's inequality.

Theorem 2.5. — *Let K be a number field, A an abelian variety with a fixed choice L of a symmetric ample line bundle coming from a projective embedding $A \hookrightarrow \mathbb{P}^n$, and $X \subseteq A$ an algebraic subvariety. There are positive constants c_1, c_2, and c_3 with $0 < c_2 < 1$ and $c_3 > 1$ so that whenever $P_0, \ldots, P_{\dim X}$ is a sequence of $\dim X + 1$ distinct algebraic points in $X(K^{alg})$ with*

- $\widehat{h}(P_0) \geq c_3$,
- $\widehat{h}(P_{i+1}) \geq c_1 \widehat{h}(P_i)$ *for* $0 \leq i < \dim X$, *and*
- $\frac{\langle P_i, P_{i+1} \rangle}{h(P_i) h(P_{i+1})} \geq c_2$ *for* $0 \leq i < \dim X$,

then at least one of the points P_i belongs to $\mathrm{Ueno}(X)$.

In Theorem 2.5, the constants c_1 and c_2 depend only geometric data, namely the dimension and degree of X and Rémond (2000) expresses them explicitly. Indeed, one may take $c_1 = \left(\max\{\deg(X), (6d)^{2d}\}\right)^{d^{3d^2}}$ where $d = \dim(X) + 1$ and $c_2 = 1 - \frac{1}{c_1}$. The constant c_3 is also expressed explicitly and it depends on arithmetic data, namely h_1, the height of the polynomials defining the group operations one A, $h(X)$, the height of X, and a constant c_{NT} bounding the difference between the canonical height $\widehat{h}$ and the naïve height coming from a fixed projective embedding of A. As an explicit formula we may take $c_3 = c_1 (n+1)^{(\dim X)^3} \max\{h(X), h_1, c_{\mathrm{NT}}, n \log(n+1)\}$.

2.2. Points of small height

The methods we have discussed so far work well to describe intersections $X \cap \Gamma$ for $X \subseteq A$ a subvariety of an abelian variety and $\Gamma \leq A$ a finitely generated subgroup of A since the kernel of the map from Γ to $\Gamma_{\mathbb{R}}$ is the finite torsion subgroup of Γ so that, in particular, there are only finitely many points of small height in Γ all told. When considering Γ which is merely finite rank, it is a nontrivial problem to describe the intersection of X with the set of small points in Γ.

The first instance of this problem is the Manin–Mumford conjecture which we formulate for higher dimensional varieties even though as originally posed by Manin and Mumford only curves were implicated.

Theorem 2.6. — *Let A be an abelian variety defined over an algebraically closed field K of characteristic zero. Let $\Gamma \leq A(K)$ be a rank zero subgroup. Let $X \subseteq A$ be an algebraic subvariety. Then $X \cap \Gamma$ is a finite union of cosets of subgroups of A.*

Hindry (1988) shows how to deduce the Mordell–Lang conjecture for finite rank groups formally from the case of finitely generated groups by using Theorem 2.6.

The original proof of Theorem 2.6 by Raynaud (1983) is p-adic in nature. Some earlier results in the context where A is replaced by a power of the multiplicative group were already known and many other proofs of the Manin–Mumford conjecture have been published over the years. The literature on the Manin–Mumford conjecture is so voluminous that we prefer to omit most of the references to its other proofs. However, there are two approaches that are relevant to the new the uniform theorems we are studying. First, the proof of Pila and Zannier (2008) introduces the o-minimal point counting method to diophantine geometry. This technique is implicit in the proofs of the Ax–Schanuel theorems we discuss in Section 3.3. Secondly, and more directly, the approach of Zhang (1998), which extends the case of curves considered by Ullmo (1998) and which is based on the equidistribution theorems of Szpiro, Ullmo, and Zhang (1997), is the one which is generalized and made more uniform in the newer work we are expositing.

The Bogomolov conjecture as proven by Zhang (1998) takes the following form.

Theorem 2.7. — *Let A be an abelian variety defined over a number field K and let $X \subseteq A_{K^{alg}}$ be an irreducible subvariety of A over the algebraic closure of K. We suppose that X is not a translate by a torsion point of an algebraic subgroup of A. Then there is a number $\epsilon > 0$ for which the set of points of small height on X, $\{a \in X(K^{alg}) : \widehat{h}(a) < \epsilon\}$, is not Zariski dense in X.*

Since the torsion points are those with canonical height zero, Theorem 2.6 may be deduced from Theorem 2.7 by Noetherian induction on X.

The proof of Theorem 2.7 passes through an equistribution result proven using Arakelov theory. To state the result we require some definitions. Let K be a number field and A an abelian variety defined over K. For an algebraic point $a \in A(K^{\mathrm{alg}})$ we write $O(a)$ for the set of Galois conjugates of a over K. We say that a sequence of points $a_n \in A(K^{\mathrm{alg}})$ (for $n \in \mathbb{N}$) is *strict* if for every proper subvariety $Y \subsetneq A_{K^{\mathrm{alg}}}$ of A over K^{alg} of the form $Y = \xi + B$ for some torsion point ξ and algebraic subgroup $B < A$ the set $\{n \in \mathbb{N} : a_n \in Y(K^{\mathrm{alg}})\}$ is finite. We say that the sequence is *small* if $\lim_{n\to\infty} \widehat{h}(a_n) = 0$.

Theorem 2.8. — *For every strict, small sequence $(a_n)_{n=0}^{\infty}$ of algebraic points in $A(K^{alg})$ the Galois orbits $O(a_n)$ are equidistributed with respect to the normalized Haar measure μ on $A(\mathbb{C})$. That is, for every continuous function $f\colon A(\mathbb{C}) \to \mathbb{R}$ we have*

$$\lim_{n\to\infty} \frac{1}{\#O(a_n)} \sum_{a' \in O(a_n)} f(a') = \int_{A(\mathbb{C})} f \, d\mu \ .$$

An alternative proof of Theorem 2.7 was given by David and Philippon (1998) using more classical methods of diophantine approximation. David and Philippon (2002) follows up with a quantitative version of the Bogomolov conjecture.

We define a measure of the degree to which points of small height avoid proper subvarieties of X as follows.

$$\widehat{\mu}^{\mathrm{ess}}(X) := \sup_{Y} \inf\{\widehat{h}(a) : a \in (X \smallsetminus Y)(K^{\mathrm{alg}})\}$$

Here, the supremum is taken over subvarieties $Y \subsetneq X$ of codimension one.

The Bogomolov conjecture asserts that when X is not a translate by a torsion point of an abelian subvariety of A, then $\widehat{\mu}^{\mathrm{ess}}(X)$ is positive. David and Philippon (2002) show that $\widehat{\mu}^{\mathrm{ess}}(X) \geq \frac{\widehat{h}(X)}{(\dim(X)+1)\deg(X)}$, which is strictly positive when X is not a translate of an algebraic subgroup of A by a torsion point.

3. The Betti map and Betti form

While many of the techniques used to prove the new uniform versions of the Mordell–Lang and Bogomolov conjectures mirror those appearing in earlier work, the analysis of Betti maps gives these proofs a fundamentally new character. Interestingly, the Betti maps themselves come from the classical approach of regarding a complex abelian variety as a complex torus. The novelty derives on the one hand from o-minimal geometry used to determine functional transcendence properties and on the other hand from a comparison between complex algebraic geometry and differential geometry to employ these maps to perform height computations.

Betti maps are explicitly introduced in Corvaja, Masser, and Zannier (2018) for the purpose of studying torsion sections of abelian schemes. The authors attribute the terminology to Bertrand and note that the construction is used implicitly in Voisin (2018). Indeed, Bertrand, Masser, Pillay, and Zannier (2016) make use of the term "Betti coordinates" for the first time. Without using the term, Masser and Zannier (2014) employ Betti corrdinates for the study of torsion points in families. These Betti maps underlie the construction of differential operators which have come to be known as Manin maps from Manin (1963). Mok (1991) constructs the Betti forms (though does not use this terminology) which serve as the differential geometric instantiations of the Betti coordinates and are used to establish the new height inequalities.

3.1. Definition and first properties of Betti maps

Consider a complex torus X. That is, X is a complex Lie group expressible as $X = V/\Lambda$ where $V = T_0 X$ is the tangent space of X at the identity and $\Lambda = \ker \exp_X$ is a lattice in V arising as the kernel of the Lie exponential map $\exp_X \colon V \to X$. Fix a basis $\omega_1, \ldots, \omega_{2g}$ of Λ as a $\mathbb{Z}$-module where $g = \dim_{\mathbb{C}} X$ is the dimension of X as a complex manifold. For this basis we may define a real analytic isomorphism

$$X \xrightarrow{\ b\ } (\mathbb{R}/\mathbb{Z})^{2g}$$

between X and the standard real torus $(\mathbb{R}/\mathbb{Z})^{2g}$ via

$$a \longmapsto (b_1(a) + \mathbb{Z}, \ldots, b_{2g}(a) + \mathbb{Z})$$

where

$$a = \exp_X(\sum_{i=1}^{2g} b_i(a)\omega_i) \ .$$

This map b_A gives a real analytic isomorphism between $A(\mathbb{C})$ and the real torus $(\mathbb{R}/\mathbb{Z})^{2g}$.

Suppose now that S is a connected complex manifold and $X \to S$ is a family of complex tori over S by which we mean that $\rho\colon X \to S$ is a map of complex manifolds and we have a complex analytic map $+\colon X \times_S X \to X$ over S giving each fiber the structure of a complex torus. Let $\varpi\colon \widetilde{S} \to S$ be the universal cover of S. Then the universal cover $\pi\colon \widetilde{X} \to X$ has form $\widetilde{X} = \widetilde{S} \times \mathbb{C}^g$ where g is the relative dimension of X over S. These maps fit into a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\pi"] \arrow[d, "\widetilde{\rho}"] & X \arrow[d, "\rho"] \\ \widetilde{S} \arrow[r, "\varpi"] & S \end{tikzcd}$$

where for any fixed $s \in S$ and lifting to $\widetilde{s} \in \varpi^{-1}\{s\}$, the restriction of π to $\{\widetilde{s}\} \times \mathbb{C}^g$ may be identified with $\exp_{X_s}\colon \mathbb{C}^g \to X_s$.

Fixing a base point $s_0 \in \widetilde{S}$, and a choice of a basis $\omega_1^0, \ldots, \omega_{2g}^0$ of $\ker(\exp_{X_{\varpi(s_0)}})$, because $\widetilde{S}$ is simply connected, there are complex analytic maps $\omega_j\colon \widetilde{S} \to \mathbb{C}^g$ for $1 \leq j \leq 2g$ so that $\omega_j(s_0) = \omega_j^0$ (again for $1 \leq j \leq 2g$) and for each $s \in \widetilde{S}$ the sequence $\omega_1(s), \ldots, \omega_{2g}(s)$ is a $\mathbb{Z}$-basis of $\ker \exp_{X_{\varpi(s)}}$. We define $\widetilde{b}\colon \widetilde{X} \to (\mathbb{R}/\mathbb{Z})^{2g}$ by the rule that

$$(s, \mathbf{v}) \longmapsto (\widetilde{b}_1(s, \mathbf{v}) + \mathbb{Z}, \ldots, \widetilde{b}_{2g}(s, \mathbf{v}) + \mathbb{Z})$$

just in case

$$\mathbf{v} \equiv \sum_{j=1}^{2g} \widetilde{b}_j(s, \mathbf{v})\omega_j(s) \mod \Lambda_{X_{\varpi(s)}} .$$

Even as a function on the universal cover $\widetilde{X}$, $\widetilde{b}$ depends on our choice of basis for $\ker \exp_{X_{\varpi(s_0)}}$ which gives an ambiguity by an action of $\mathrm{GL}_{2g}(\mathbb{Z})$. More seriously, even after fixing that basis, if we were to try to descend $\widetilde{b}$ to a function $b\colon X \to (\mathbb{R}/\mathbb{Z})^{2g}$ we would encounter monodromy. However, locally we may regard the Betti map as defined on X.

More precisely, for any simply connected $\Delta \subseteq S$, setting $X_\Delta := \rho^{-1}\Delta$, identifying X_Δ with one of the components of $\pi^{-1}X_\Delta \subseteq \widetilde{X}$ and then restricting $\widetilde{b}$ to that component, we have $b_\Delta\colon X_\Delta \to (\mathbb{R}/\mathbb{Z})^{2g}$. Fiberwise, b_Δ is a real analytic isomorphism of Lie groups. That is, for each $s \in \Delta$ we have

$$X_s \xrightarrow[b_{\Delta s}]{\sim} (\mathbb{R}/\mathbb{Z})^{2g} \quad .$$

Moreover, combining b_Δ with ρ we can trivialize X_Δ. That is, we have an isomorphism of real analytic spaces:

$$X_\Delta \xrightarrow[(b_\Delta, \rho)]{\sim} (\mathbb{R}/\mathbb{Z})^{2g} \times \Delta$$

While b_Δ is merely real analytic, its fibers are complex analytic. That is, for any $\mathbf{r} \in (\mathbb{R}/\mathbb{Z})^{2g}$ the fiber $b_\Delta^{-1}\{\mathbf{r}\} \subseteq X_\Delta$ is a complex analytic subvariety of X_Δ.

3.2. The Betti form

In this section when discussing moduli spaces for abelian varieties, to simplify the presentation we will drop the reference to a polarization type, implicitly restricting to the case of principally polarized abelian varieties. Fix a positive integer $g \geq 1$. Write the coordinates on $\mathbb{R}^{2g} \times \mathfrak{h}_g = \mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{h}_g$ as (x, y, Z). We define $\mathbb{R}^{2g} \times \mathfrak{h}_g \cong \mathbb{C}^g \times \mathfrak{h}_g$ by $(x, y, Z) \mapsto (x + Zy, Z)$. We define $\widehat{\omega}^{\mathrm{univ}}$ to be the form $2(dx)^\intercal \wedge dy$ regarded as a 2-form on $\mathbb{C}^g \times \mathfrak{h}_g$ via the above change of coordinates. A short computation shows that $\widehat{\omega}^{\mathrm{univ}}$ is a $(1,1)$-form.

For any arithmetic group $\Gamma \leq \mathrm{Sp}_{2g}(\mathbb{Z})$, $\widehat{\omega}^{\mathrm{univ}}$ descends to a $(1,1)$-form $\omega^{\mathrm{univ}} = \omega^{\mathrm{univ}}_\Gamma$ on $\mathfrak{A} = (\mathbb{Z}^{2g} \rtimes \Gamma)\backslash(\mathbb{C}^g \times \mathfrak{h}_g)$. For any principally polarized abelian scheme $A \to S$ with sufficient level structure, $A \to S$ fits into a pullback square

$$\begin{array}{ccc} A & \longrightarrow & \mathfrak{A} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{A} \end{array}$$

so that we may take ω to be the pullback of ω^{univ} along the map $A \to \mathfrak{A}$. This form ω is a Betti form for the abelian scheme $A \to S$.

The construction of ω from the Betti coordinates may be conceptually clear, but other presentations are better suited for computations. Indeed, if we write the coordinates on $\mathbb{C}^g \times \mathfrak{h}_g$ as (w, Z), then

$$\widehat{\omega}^{\mathrm{univ}} = \sqrt{-1}\partial\overline{\partial}(2(\operatorname{Im} w)^\intercal(\operatorname{Im} Z)^{-1}(\operatorname{Im} w)) \ .$$

Expanding this expression, we obtain another formula.

$$\widehat{\omega}^{\mathrm{univ}} = \sqrt{-1}(dZ(\operatorname{Im} Z)^{-1}\operatorname{Im}(w) - dw)^\intercal) \wedge (\operatorname{Im} Z)^{-1}(d\overline{Z}(\operatorname{Im} Z)^{-1}\operatorname{Im}(w) - d\overline{w}) \ .$$

The Betti form enjoys some favorable properties.

Proposition 3.1. — *Let $A \to S$ be a principally polarized abelian scheme with appropriate level structure and let ω be a Betti form on A. Then*

- *ω is a closed, semi-positive, $(1,1)$-form (where "semi-positive" means that the associated Hermitian form is positive semi-definite),*
- *for every integer N we have $[N]^*\omega = N^2\omega$ where $[N]\colon A \to A$ is the multiplication by N map, and*
- *if $U \subseteq S(\mathbb{C})$ is a Euclidean open subset, $X \subseteq A$ is an irreducible subvariety of dimension d for which the smooth locus of X meets A_U, then the restriction of $\omega^{\wedge d}$ to the smooth locus of X is nonzero if and only if there is a point x in the smooth locus of X_U at which the real-rank of the differential of the Betti map at x is $2d$.*

When the condition that the there is a smooth point on the variety X at which the Betti map has full rank holds we say that X is nondegenerate.

3.3. (Non)degenerate varieties and the Ax–Schanuel theorem

For a given family $X \to S$ of complex tori, the Betti map $b\colon X \to (\mathbb{R}/\mathbb{Z})^{2g}$ is only well-defined locally, and even then this is only true up to an automorphism of the real torus $(\mathbb{R}/\mathbb{Z})^{2g}$. However, for any point $x \in X$ the rank $\operatorname{rk} db_x$ of the differential of the Betti map is well-defined.

Definition 3.2. — *For $Y \subseteq X$ an irreducible complex analytic subvariety of X, we say that Y is* nondegenerate *if there is a point $x \in Y^{sm}$ in the smooth locus of Y for which $\operatorname{rk} db_x = \dim_{\mathbb{R}} Y = 2\dim_{\mathbb{C}}(Y)$. Otherwise, we say that Y is* degenerate.

An extreme way in which $Y \subseteq X$ might be degenerate would be for db to restrict to the zero map on TY, the tangent bundle of Y. Equivalently, taking $\widetilde{Y} \subseteq \widetilde{X}$ to be any component of the preimage of Y in the universal covering space $\widetilde{X}$ of X, there would be some fixed element $\mathbf{r} \in (\mathbb{R}/\mathbb{Z})^{2g}$ for which $\widetilde{Y}$ is contained in the graph of the map $\widetilde{S} \to \mathbb{C}^g$ given by $s \mapsto \sum_{j=1}^{2g} \mathbf{r}_j \omega_j(s)$ where $\omega_1, \ldots, \omega_{2g}$ is the sequence of analytic maps $\omega_j\colon \widetilde{S} \to \mathbb{C}^g$ described in Subsection 3.1. For example, for any positive integer N, this would happen if Y were taken to be a component of $X[N]$, the kernel of the fiberwise multiplication by N map on X.

Other varieties on which the rank of the Betti map is zero may be constructed by taking products. For example, start with complex torus A and any point $a \in A$. Let S be any connected complex manifold and set $X := A \times S$ with the structural morphism $\rho\colon X \to S$ being the projection to S. On the subvariety $Y := \{a\} \times S$ the Betti map has rank zero.

When we assume further that $X \to S$ comes from the analytification of an algebraic family of abelian varieties, then it follows from Manin's Theorem of the Kernel that all such examples can be constructed from these two. A full description of the degenerate varieties is made possible by an Ax–Schanuel theorem for mixed Shimura varieties.

Schanuel proposed the following transcendence inequality involving complex numbers and their exponentials.

Conjecture 3.3. — *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be complex numbers that are $\mathbb{Q}$-linearly independent. Then*

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)) \geq n \ .$$

Conjecture 3.3 encapsulates some known results on transcendental number theory including, for example, the Lindemann–Weierstrass theorem and the six-exponentials theorem. However, even for small values of n, the conjecture remains widely open. For example, the case of $n = 2$ with $\alpha_1 = 1$ and $\alpha_2 = \sqrt{-1}\pi$ would imply the algebraic independence of e and π.

Shortly after Conjecture 3.3 was proposed, Ax proved a functional version which we write in a form mirroring Conjecture 3.3.

Theorem 3.4. — *Let $\Delta \subseteq \mathbb{C}^m$ be an open, connected domain complex m-space for some natural number m. Write the variables on $\mathbb{C}^m$ as $z_1, \dots, z_m$. Let $\alpha_j \colon \Delta \to \mathbb{C}$ be holomorphic functions on Δ for $1 \leq j \leq n$. We presume that no nontrivial $\mathbb{Q}$-linear combination of the the α_js is constant. Then*

$$\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(\alpha_1, \dots, \alpha_n, \exp(\alpha_1), \dots, \exp(\alpha_n)) \geq n + \operatorname{rk}\left(\frac{\partial \alpha_j}{\partial z_i}\right)$$

where $\left(\frac{\partial \alpha_j}{\partial z_i}\right)$ is the Jacobian matrix of $\alpha = (\alpha_1, \dots, \alpha_n) \colon \Delta \to \mathbb{C}^n$.

Theorem 3.4 has been generalized for other special functions including abelian exponentials, modular functions, and period mappings associated to variations of Hodge structures. Theorem 3.4 itself played an important role in Zilber's program on the logic of the complex exponential function and set the stage for the Zilber–Pink conjecture. A version for Klein's j-function was proven in Pila (2011) using the o-minimal counting theorem of Pila and Wilkie (2006) and then used as a crucial step for the proof of the André-Oort conjecture for products of modular curves.

The strongest Ax–Schanuel theorems known to date are those for period mappings of variations of mixed Hodge structures Chiu (2021) and Gao and Klingler (2021) and the very general differential algebraic version of Blázquez Sanz, Casale, Freitag, and Nagloo (2021) from which the geometric forms may be deduced. For the applications to the Mordell–Lang and Bogomolov conjectures, we need only the Ax–Schanuel theorem for the universal abelian scheme over moduli spaces, itself an instance of the Ax–Schanuel theorem for mixed Shimura varieties of Kuga type due to Gao (2020). A crucial component of the proof in this case of mixed Shimura variety is the Ax–Schanuel theorem for pure Shimura varieties of Mok, Pila, and Tsimerman (2019).

Let us express Gao's Ax–Schanuel theorem in the language of bi-algebraic varieties, restricting to the case of the universal abelian variety. The covering space $\mathbb{C}^g \times \mathfrak{h}^g$ naturally sits as an open subset of the complex points of an algebraic variety, namely $\check{D} := \mathbb{G}_a^g \times S$, the product of the g^{th} Cartesian power of the additive group with the space of symmetric $g \times g$ matrices. We say that a subset of $\mathbb{C}^g \times \mathfrak{h}^g$ is algebraic if it is an intersection of the form $Y(\mathbb{C}) \cap (\mathbb{C}^g \times \mathfrak{h}_g)$ for some algebraic subvariety $Y \subseteq \check{D}$. We say that an irreducible subvariety $Z \subseteq \mathfrak{A}$ is *bi-algebraic* if there is some component of its preimage in $\mathbb{C}^g \times \mathfrak{h}_g$ which is algebraic. A general variety is bi-algebraic if all of its components are. The class of bi-algebraic varieties is closed under intersection, so that every subset $X \subseteq \mathfrak{A}$ has a bi-algebraic closure $\overline{X}^{\text{bi-alg}}$. Bi-algebraic varieties admit other characterizations. For example, the bi-algebraic subvarieties of $\mathcal{A}$ are what are called "varieties of Hodge type" in the literature. More generally, bi-algebraic varieties admit coverings by homogeneous spaces for algebraic subgroups of $\mathbb{G}_a^{2g} \rtimes \operatorname{Sp}_{2g}$. Morally, they should be components of group schemes over Hodge type varieties, but such a proposed characterization would be incorrect due to the existence of what are called "Ribet sections" as is observed by Bertrand, Masser, Pillay, and Zannier (2016).

Theorem 3.5. — *Let $\mathcal{Z}$ be an irreducible complex analytic subvariety of some open subset of the graph of the analytic covering map expressing $\mathfrak{A}$ as a quotient of $\mathbb{C}^g \times \mathfrak{h}_g$ by an arithmetic group. Let Z be the image of $\mathcal{Z}$ in $\mathfrak{A}$. Then*

$$\dim \overline{\mathcal{Z}}^{\textit{Zariski}} - \dim \mathcal{Z} \geq \dim(\overline{Z}^{\textit{bi-alg}}) \ .$$

Ax–Schanuel theorems, especially Gao's version, are instrumental in the characterization of degenerate varieties. Since sufficiently general Ax–Schanuel theorems were not yet available for Gao and Habegger (2019), they directly characterized degenerate subvarieties of abelian schemes over curves by proving an Ax–Schanuel-type theorem using the o-minimal Pila–Wilkie counting method. In some ways, this preliminary result is more satisfying than what is possible in general because the degenerate varieties have a much cleaner form.

Consider now a curve S defined over a field k of characteristic zero and $A \to S$ an abelian scheme over S. Let K be the algebraic closure of the function field $k(S)$ and let A_K be the base change to this algebraic closure of the generic fiber of $A \to S$. Recall from Section 1.1 the K/k-trace of A. We say that an irreducible subvariety $X \subseteq A$ is special if $X \to S$ is dominant and X_K is a finite union of varieties of the form $\rho(Y) + B + \xi$ where $Y \subseteq \mathrm{Tr}_{K/k}(A)$ is a subvariety of the trace defined over k, $B \leq A_K$ is an abelian subvariety (or is the trivial group), and ξ is some point of A_K. It is strongly special if ξ may be taken to be a torsion point of A_K. Gao and Habegger (2019) show that the degenerate subvarieties are precisely the strongly special subvarieties.

In the more general case of abelian schemes over higher dimensional bases, Theorem 3.5 can be used to show that nondegenerate varieties may be constructed from fiber powers. For purposes of computing bounds on intersections with finite rank groups or the sets of small height, it often suffices to work with these fiber powers. For studying the relations between different points on the variety X it can be useful to consider a certain image of these fiber powers coming from taking differences. Let us introduce some notation so that we may state the nondegeneracy theorem in both cases.

We are given an irreducible quasiprojective variety S defined over $\mathbb{Q}^{\mathrm{alg}}$, $\pi \colon A \to S$ an abelian scheme over S, and $X \subseteq A$ a subvariety. For a natural number m, we write $X^{[m]}$ for the m^{th} fiber power of X over S. We define $\mathcal{D}_m \colon A^{[m+1]} \to A^{[m]}$ by $(P_0, \dots, P_m) \mapsto (P_1 - P_0, \dots, P_m - P_0)$.

Proposition 3.6. — *With the notation as above, if the generic fiber $X_{\mathbb{Q}(S)}$ is a nonempty absolutely irreducible subvariety of $A_{\mathbb{Q}(S)}$ which generates $A_{\mathbb{Q}(S)}$ and which is not special , then there is a natural number m_0 so that for all $m \geq m_0$ the fiber power $X^{[m]}$ and the image $\mathcal{D}_m(X^{[m+1]})$ are nondegenerate subvarieties of $A^{[m]}$.*

4. Uniform height estimates

As we have seen, height inequalities for algebraic points on subvarieties of abelian varieties strong enough to deduce explicit, effective forms of the Mordell–Lang conjecture have been known for decades. The key to proving the new uniform versions of Mordell–Lang is to improve those earlier estimates by removing the dependence on the heights of the varieties and other related arithmetic parameters, or at the very least keeping them under sufficient control to permit the desired conclusions.

While Dimitrov, Gao, and Habegger (2021) ultimately conclude with a uniform bound on the number of large points in intersections of finite rank subgroups of abelian varieties with *curves*, at the heart of that paper a fundamental height inequality for general non degenerate subvarieties is proven. Let us set the context for the remainder of this section.

Let S be a quasiprojective variety over $\mathbb{Q}^{\mathrm{alg}}$ and let $\pi\colon A \to S$ be an abelian scheme. Fix an embedding $S \subseteq \mathbb{P}^m$ of S as locally closed subvariety of projective space and then also fix a compatible embedding $A \subseteq \mathbb{P}^n \times \mathbb{P}^m$. We assume that on the generic fiber $A_{\mathbb{Q}^{\mathrm{alg}}(S)}$ the embedding $A_{\mathbb{Q}^{\mathrm{alg}}(S)} \hookrightarrow \mathbb{P}^n$ comes from sections of a symmetric line bundle. We work with the height on A coming from this embedding which is a representative of the height $h_{\mathcal{O}(1,1)|A}$ given by the Weil height machine. Treating A as an algebraic dynamical system with the map $P \mapsto [2]P$ we deduce a canonical height $\widehat{h}$ on A. Some additional technical requirements are added in the original text, for example, that the polarization is a principal polarization and that $A \to S$ is given with suitable level structure so that the associated moduli problem is represented by a universal abelian variety over a fine moduli space. With this set-up the height estimate of Dimitrov, Gao, and Habegger (2021) reads as follows.

Theorem 4.1. — *With the hypotheses as given above, if $X \subseteq A$ is an irreducible, nondegenerate variety for which $X \to S$ is dominant, then there are two positive constants c_1 and c_2 and a Zariski dense, open set $U \subseteq X$ so that for all $P \in U(\mathbb{Q}^{alg})$ we have $\widehat{h}(P) \geq c_1 h(\pi(P)) - c_2$.*

Theorem 4.1 has the flavor of the uniform Bogomolov conjecture in that when $h(\pi(P)) > c_2/c_1$ the canonical height is bounded away from zero on U. One might hope to complete a proof of the uniform Bogomolov conjecture (with this additional hypothesis that $h(\pi(P))$ is large) by arguing by Noetherian induction looking next at the components of $X \smallsetminus U$ until one ends with the translates by torsion points of abelian varieties on which the canonical height will be zero. Such an approach does not quite work in that those components may be degenerate without coming from those height zero varieties. This issue is dealt with by considering fiber powers invoking Proposition 3.6. The more serious issue concerns the moduli points of small height.

The proof of Theorem 4.1 passes through a weaker statement about height bounds for our chosen naïve height.

Proposition 4.2. — *With the hypotheses as in Theorem 4.1, there is a constant c_1 so that for every natural number N which is a power of two there are a constant $c_2 = c_2(N)$ and a Zariski dense and open set $U_N \subseteq X$ so that for every $P \in U_N(\mathbb{Q}^{alg})$ we have $h([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N)$.*

To go from Proposition 4.2 to Theorem 4.1 requires a good way to compare the naïve height to the canonical height and then a trick which Masser calls "killing Zimmer constants" in Appendix C of the book by Zannier (2012).

The comparison between the canonical height and naïve height is given by a theorem of Silverman (1983): there is a constant $c > 0$ so that

$$|\widehat{h}(P) - h(P)| \leq c \max\{1, h(\pi(P))\} \leq c(1 + h(\pi(P)))$$

as P ranges through $A(\mathbb{Q}^{\mathrm{alg}})$.

As the argument based on Masser's trick is short, we reproduce it here. Fix N a power of two for which $N^2 \geq \frac{2c}{c_1}$. We will take $U = U_N$ and use new constants $c_1' := \frac{c_1}{2}$ and $c_2' := \frac{c_2(N)+c}{N^2}$ for the conclusion of Theorem 4.1 (in place of c_1 and c_2).

For $P \in U(\mathbb{Q}^{\mathrm{alg}})$ we compute.

$$\begin{aligned} N^2\widehat{h}(P) &= \widehat{h}([N]P) \\ &\geq N^2 h(P) - c(1 + h(\pi([N]P))) \\ &= N^2 h(P) - c(1 + h(\pi(P))) \\ &\geq c_1 N^2 h(\pi(P)) - c_2(N) - c(1 + h(\pi(P))) \end{aligned}$$

Thus,

$$\widehat{h}(P) \geq (c_1 - \frac{c}{N^2})h(\pi(P)) - \frac{c_2(N) + c_0}{N^2} \geq c_1' h(\pi(P)) - c_2'$$

Let us return to the argument for Proposition 4.2. The inequality is proven by relating it to a statement about the bigness of certain associated line bundles. For a natural number N we write X_N for the Zariski closure in $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^m$ of $\{(P, [N]P, \pi(P)) : P \in X(\mathbb{Q}^{\mathrm{alg}})\}$. For $P \in X(\mathbb{Q}^{\mathrm{alg}})$ we write $P' = (P, [N]P, \pi(P))$. Let $\mathcal{F} := \mathcal{O}(0, 1, 1)|_{X_N}$ and $\mathcal{M} := \mathcal{O}(0, 0, 1)|_{X_N}$. Then $h([N](P)) = h_{\mathcal{F}}(P')$ and $h(\pi(P)) = h_{\mathcal{M}}(P')$, or, perhaps we should say, the functions on the left of each of these equalities represent the heights given by the height machine. If we could find positive integers p and q for which $\mathcal{F}^p \otimes \mathcal{M}^{-qN^2}$ were big, then on a Zariski dense and open subset of X_N any representative of the height $h_{\mathcal{F}^{\otimes p} \otimes \mathcal{M}^{\otimes -qN^2}}$ would be bounded below. From the computation above, we see that $ph([N]P) - qN^2 h(\pi(P)) = h_{\mathcal{F}^{\otimes p} \otimes \mathcal{M}^{\otimes -qN^2}}(P')$ would be bounded below, say by c' Taking $c_1 := \frac{q}{p}$ and $c_2(N)$ to $\frac{c'}{p}$ we have the inequality of Proposition 4.2. Both of $\mathcal{F}$ and $\mathcal{M}$ are numerically effective so that a numerical criterion due to Siu may be used to check that $\mathcal{F}^{\otimes p} \otimes \mathcal{M}^{\otimes -qN^2}$ is big.

Siu's criterion takes the following form.

Theorem 4.3. — *Let Y be a projective variety of dimension d and let D and E be two numerically effective divisors on Y. If $(D^{\cdot d}) > d(D^{d-1} \cdot E)$, then $D - E$ is big.*

Applying Theorem 4.3 to the problem at hand, taking $d := \dim X = \dim X_N$ we need to check that $(\mathcal{F}^{\cdot d}) > dc_1(\mathcal{M}^{\otimes N^2} \cdot \mathcal{F}^{\cdot(d-1)})$, which is $dc_1 N^2(\mathcal{M} \cdot \mathcal{F}^{\cdot(d-1)})$. This is achieved by establishing a lower bound for $(\mathcal{F}^{\cdot d})$ and an upper bound for $(\mathcal{M} \cdot \mathcal{F}^{\cdot(d-1)})$.

We convert the problem into one estimating certain integrals. Let α be the pullback of the Fubini–Study form on $\mathbb{P}^{(n+1)(m+1)-1}$ to $\mathbb{P}^n \times \mathbb{P}^m$ under the Segre embedding and let $\rho\colon \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n \times \mathbb{P}^m$ be the projection onto those coordinates. Then $(\mathcal{F}^{\cdot d}) = \int_{X_N(\mathbb{C})} \rho^* \alpha^{\wedge d}$. This is the integral we need to bound from below. The main idea in this computation is to compare α to the Betti form ω (or, really, these forms times a suitable compactly supported function) making use of the positivity of the Betti form (due to X being nondegenerate) and its transformation rule $[N]^*\omega = N^2\omega$.

To find the lower bound, we restrict as follows. Fix a point $P_0 \in X$ at which the Betti map has full rank. Let $\Delta \subseteq S(\mathbb{C})$ be a relatively compact, contractible neighborhood of $\pi(P_0)$ in $S(\mathbb{C})$. Write A_Δ for $\pi^{-1}\Delta$. Fix a smooth bump function $\vartheta\colon S(\mathbb{C}) \to [0,1]$ with support contained in Δ and $\vartheta(\pi(P_0)) = 1$. Set $\theta := \vartheta \circ \pi$. Using the positivity of α and the relative compactness of Δ in S, we may find some constant $C > 0$ so that $C\theta\alpha_X - \theta\omega_X \geq 0$. Let $\kappa' := \int_{X(\mathbb{C})} (\theta\omega)^{\wedge d}$, which is strictly positive because ω (and, hence, $\theta\omega$) is semi-positive and $\omega|_X^{\wedge d} \neq 0$ at P_0. One shows that the inequality $(\mathcal{F}^{\cdot d}) \geq \left(\frac{\kappa'}{C^d}\right) N^{2d}$ holds for all N. (Remember that even though it is suppressed from the notation, $\mathcal{F}$ depends on N.)

In the other direction, it is shown that there is a constant c so that for all powers of two N one has $(\mathcal{M} \cdot \mathcal{F}^{\cdot(d-1)}) \leq cN^{2(d-1)}$. The argument for this inequality passes through algebraic intersection theory.

As an alternative to the proof we just outlined which is taken from the paper of Dimitrov, Gao, and Habegger (2021), Theorem 6.2.2 of Yuan and Zhang (2021) gives Theorem 4.1 for any polarized algebraic dynamical system.

5. Equidistribution and points of small height

Theorem 4.1 combined with the effective proof of the Mordell–Lang conjecture by Rémond, 2000 on its own can be used to give a proof of a uniform version of the Mordell–Lang conjecture in which we count only points of large height. To complete the argument for uniform Mordell–Lang we require bounds for points of small height as well. This is achieved through equidistribution theorems which can be seen as versions of the theorems of Szpiro, Ullmo, and Zhang (1997) in families.

Kühne (2021) proves an equidistribution theorem for small points on subvarieties of abelian schemes.

Theorem 5.1. — *Let S be an irreducible variety over a number field K, $A \to S$ an abelian scheme over S equipped with a symmetric ample line bundle, and $X \subseteq A$ an irreducible, nondegenerate subvariety. We suppose that there is a sequence of algebraic points $(x_n)_{n=0}^{\infty}$ for which $\lim_{n\to\infty} \widehat{h}(x_n) = 0$ and for every proper subvariety $Y \subsetneq X$ $\{n \in \mathbb{N} : x_n \in Y(\mathbb{Q}^{alg})\}$ is finite. (We call such a sequence a "small, generic sequence".) Then the canonical height of X is zero and there is a measure μ on $X(\mathbb{C})$ so that every small, generic sequence $(a_n)_{n=0}^{\infty}$ in $X(K^{alg})$ is equidistributed with respect to μ in the sense that for any compactly supported continuous function $f\colon X(\mathbb{C}) \to \mathbb{R}$ we have*

$$\lim_{n\to\infty} \frac{1}{[K(a_n) : K]} \sum_{a' \in O(a_n)} f(a') = \int_{X(\mathbb{C})} f \, d\mu$$

In the statement of Theorem 5.1 we have fixed a single complex embedding $K \hookrightarrow \mathbb{C}$, but the result is true for all of them. In fact, the theorem holds for nonarchimedian places as well using the formalism of Chambert-Loir (2006).

The proof of Theorem 5.1 passes through yet another way to realize the Betti form in terms of the equilibrium measure. As with the set up in Section 4 we have have embedded S as locally closed subvariety of some $\mathbb{P}^m$ and then A into $\mathbb{P}^n \times \mathbb{P}^m$ so that on the generic fiber this embedding comes from the sections of a very ample, symmetric line bundle. We then take $\iota\colon A \to \mathbb{P}^{(n+1)(m+1)-1}$ to be the composition with the Segre embedding. Let α be the Fubini–Study form on $\mathbb{P}^{(n+1)(m+1)-1}$ and for each natural number k we let $\gamma_k := \frac{(\iota\circ[2^k])^*\alpha}{4^k}$. It is shown that the Betti form is the limit of the γ_k in the sense that for each irreducible subvariety $Y \subseteq A$ and compactly supported continuous function $f\colon Y(\mathbb{C}) \to \mathbb{R}$ one has

$$\lim_{k\to\infty} \int_{Y(\mathbb{C})} f\gamma_k^{\wedge \dim Y} = \int_{Y(\mathbb{C})} f\beta^{\wedge \dim Y} \ .$$

The proof of Theorem 5.1 shares some features with the proof of Theorem 4.1. For example, a key lemma is based on a version of Siu's criterion for bigness for hermitian line bundles on arithmetic varieties due to Yuan (2021).

6. Deducing the uniform diophantine theorems

The uniform Bogomolov conjecture follows from Theorem 5.1.

Theorem 6.1. — *Let S be an irreducible variety over a number field K, $A \to S$ an abelian scheme over S equipped with a symmetric ample line bundle, and $X \subseteq A$ an irreducible subvariety. There are two positive constants c_1 and c_2 so that for every $s \in S(\mathbb{Q}^{alg})$*

$$\#\{P \in X_s^0(\mathbb{Q}^{alg}) : \widehat{h}(P) \leq c_1\} \leq c_2 \ .$$

When X is nondegenerate, Theorem 5.1 implies that there is some $\epsilon > 0$ and a proper subvariety $Y \subsetneq X$ so that $\{P \in X(\mathbb{Q}^{\mathrm{alg}}) : \widehat{h}(P) \leq \epsilon\} \subseteq Y(\mathbb{Q}^{\mathrm{alg}})$. We could then apply induction to Y to complete the argument. If X is degenerate because it is equal to its own Ueno locus, then we are also done. Otherwise, we can pass to the image under $\mathcal{D}_m$ of some fiber power using Proposition 3.6 to make $\mathcal{D}(X^{[m+1]})$ nondegenerate for some $m > 1$. We then note that the height bounds given by Theorem 5.1 imply similar height bounds on X up to possibly $m-1$ exceptions.

The proof of the uniform Mordell–Lang conjecture is deduced in stages. First, what has been called the New Gap Principle is demonstrated.

THEOREM 6.2. — *There are constants c_1 and c_2 depending only on the dimension of the abelian variety A polarized by the symmetric line bundle L and degree of the subvariety X, both defined over $\mathbb{Q}^{alg}$, so that the set $\{P \in X^0(\mathbb{Q}^{alg}) : \widehat{h}(P) \leq c_1 \max\{1, h(A)\}\}$ is contained in a proper subvariety of X of degree at most c_2.*

The proof of Theorem 6.2 proceeds by first using Theorem 4.1 together with Proposition 3.6 and induction to show that there are positive constants a_1, a_2, and a_3 depending just on the degrees and dimensions mentioned in Theorem 6.2 so that the set

$$\{P \in X^0(\mathbb{Q}^{\mathrm{alg}}) : \widehat{h}(P) \leq a_1 \max\{1, h(A) - a_2\}\}$$

is contained in a proper subvariety of X degree at most a_3. Likewise, Theorem 6.1 can be used to produce positive constants b_1 and b_3 so that the set

$$\{P \in X^0(\mathbb{Q}^{\mathrm{alg}}) : \widehat{h}(P) \leq b_1 \max\{1, h(A)\}\}$$

is contained in a proper subvariety of X degree at most b_3. A simple computation then shows that taking

$$c_1 := \min\{\frac{b_1}{\max\{1, \frac{2a_2}{a_1}\}}, \frac{a_1}{2}\}$$

and

$$c_2 := \max\{a_3, b_3\}$$

works.

The deduction of the uniform Mordell–Lang conjecture from the New Gap Principle uses the existing effective bounds of Theorem 2.5 of Rémond (2000). What prevents an immediate application is that Rémond's bounds depend on three height parameters: the modular height of A (what we have been calling $h(A)$), the constant c_{NT} bounding the difference between the canonical height and naïve Weil height, and the height h_1 of the polynomials defining the group operations.

We have already discussed how to bound c_{NT} using Silverman's theorem. A bound for h_1 linear in $h(A)$ is computed by observing that the coefficients of the group operations may be computed as rational functions of any Zariski dense set of points in $A \times A$. In particular, we may use the torsion points which have canonical height zero so that using Silverman's theorem again and the transformation rule for heights under rational functions, h_1 may be bounded in terms of $h(A)$.

With these computations completed, Theorem 2.5 now gives a bound of the form c^{r+1} for the number of points in X^0 on a finitely generated group Γ of rank r with height greater than $c_1 \max\{1, h(A)\}$ for appropriate constants c_1 and c depending only on the dimension of the abelian variety A and the degree of X relative to our chosen ample, symmetric line bundle on A. The New Gap Principle gives an absolute bound on the number of points on X^0 with height less than $c_2 \max\{1, h(A)\}$. If $c_2 \geq c_1$, we would be done, but that might be asking too much. Instead, a sphere packing argument is used to produce a bound on the small points of a similar quality to the bound for the points of large height so that together we obtain a uniform bound for the number of the points in total.

While this argument works only for finitely generated groups, since the bound is uniform and every finite rank group can be realized as direct limit of finitely generated groups of the same rank, the bounds for finitely generated groups imply the same bounds for finite rank groups.

The approach to the uniform Mordell–Lang conjecture we have outlined largely follows the methods of Dimitrov, Gao, and Habegger (2021), Kühne (2021), and Gao, Ge, and Kühne (2021). Yuan (2021) offers a different approach to the uniform Mordell–Lang conjecture for curves. As we have noted on other occasions, Yuan's method is based on the Yuan and Zhang (2021) theory of adelic line bundles and presents some advantages over the method we have exposited. For example, the method works in any characteristic giving a new proof of the theorem of Hrushovski (1996). In particular, this method is not dependent on Ax–Schanuel theorems to establish nondegeneracy.

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