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MOMENTS OF L-FUNCTIONS AND HOMOLOGICAL STABILITY [after Bergström–Diaconu–Petersen–Westerland and Miller–Patzt–Petersen–Randal-Willams]

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1. Introduction

Let q be a power of an odd prime number and $\mathbb{F}_q$ a finite field of cardinality q. For each integer $g \geq 1$, let $\mathcal{P}_{2g+1}$ denote the set of monic and squarefree polynomials $d \in \mathbb{F}_q[x]$ of degree $2g+1$. With each $d \in \mathcal{P}_{2g+1}$ is associated the L-function

$$L(s, \chi_d) = \sum_{\substack{m \in \mathbb{F}_q[t] \\ \text{monic}}} \frac{\chi_d(m)}{q^{s \deg m}},$$

where χ_d stands for the quadratic residue symbol. By the Weil conjectures for the hyperelliptic curve of affine equation $y^2 = d(x)$, this L-function turns out to be a polynomial of degree $2g$ in the variable q^{-s}, and satisfies the functional equation

$$L(s, \chi_d) = q^{g(1-2s)} L(1-s, \chi_d).$$

We will be interested in the distribution of the central values $L(\frac{1}{2}, \chi_d)$ as d varies. A conjecture of Andrade and Keating [2] predicts the asymptotic behaviour of the moments of this distribution as g tends to infinity: for every integer $r \geq 1$, there exists an explicit polynomial Q_r of degree $r(r+1)/2$ such that

$$\frac{1}{q^{2g+1}} \sum_{d \in \mathcal{P}_{2g+1}} L(\tfrac{1}{2}, \chi_d)^r - Q_r(2g+1)$$

tends to 0 as $g \to +\infty$. The goal of these notes is to survey on a recent breakthrough establishing this conjecture when q is very large with respect to r.

Theorem 1.1 (Bergström, Diaconu, Petersen, and Westerland [6, Theorem 1.1.14])
Let q be an odd prime power and let $r \geq 1$ be an integer. There exists an explicit polynomial Q_r of degree $r(r+1)/2$ and an explicit positive real number C independent of q and r such that the following estimate holds for all g:

$$\left| \frac{1}{q^{2g+1}} \sum_{d \in \mathcal{P}_{2g+1}} L(\tfrac{1}{2}, \chi_d)^r - Q_r(2g+1) \right| \leq C \cdot 4^{g(r+1)} q^{-(g+6)/12}. \tag{1}$$

In particular, for fixed $r \geq 1$, the conjecture holds for all fields $\mathbb{F}_q$ with $q > 2^{24(r+1)}$.

The conjecture by Andrade and Keating is an analogue over function fields in positive characteristic of a conjecture of Conrey, Farmer, Keating, Rubinstein, and Snaith [9] that predicts the moments of $L(\frac{1}{2}, \chi)$ as χ runs through the set of quadratic Dirichlet characters. One motivation for investigating the distribution of the central values of the L-functions associated with a family of automorphic representations is the long standing problem of estimating the moments of the Riemann zeta function on the critical line[(1)]. For example, one of the popular consequences of the Riemann hypothesis, the Lindelöf hypothesis, amounts to an estimate of the form $O(T^{1+\varepsilon})$ for all $\varepsilon > 0$. Based on the results of Hardy–Littlewood and Ingham for the first two moments, one expects for each integer $r \geq 1$ the asymptotic behaviour

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2r} dt \sim C_r T \log^{r^2} T$$

as T goes to infinity, for some explicit positive real number C_r. The value of this constant was conjectured by Keating and Snaith [19], based on the heuristics that the zeroes of the Riemann zeta function on the critical line are in some sense distributed as eigenvalues of random hermitian matrices. Pursuing this philosophy, Conrey, Farmer, Keating, Rubinstein, and Snaith [9] proposed a recipe to predict the leading term in the asymptotic behaviour of the moments of $L(\frac{1}{2}, \pi)$ for many families of automorphic representations π. In the case of quadratic Dirichlet characters, the same conjecture was obtained by Diaconu, Goldfeld, and Hoffstein [12] using multiple Dirichlet series. At the time of writing, this conjecture is only known for $r \leq 3$, thanks to the work of Jutila [18] and Soundararajan [26, Theorem 2]. Surprisingly, before the work of Bergström, Diaconu, Petersen, and Westerland [6] that we discuss here, the situation was the same over function fields, even though the Riemann hypothesis holds in this setting. The conjecture of Andrade and Keating was only established for $r \leq 3$, mainly by Florea [14, 15].

The proof of Theorem 1.1 relies on a new homological stability bound for the braid group, which is obtained in a companion paper by Miller, Patzt, Petersen, and Randal-Williams [22]. The application of homological stability results to questions in arithmetic statistics has witnessed spectacular developments in recent years; see, for example, the Bourbaki seminar of Randal-Williams [24]. Roughly speaking, the strategy consists in reinterpreting the moments as alternating sums of traces of Frobenius acting on the étale homology with twisted coefficients of the moduli space of hyperelliptic curves of genus g, identifying stable homology as the leading term, and bounding the error term by means of Deligne's estimates for the absolute values of Frobenius eigenvalues.

More precisely, let $X = \mathrm{Conf}_{2g+1}(\mathbb{A}^1)$ be the configuration space of $2g+1$ distinct unordered points on the affine line, considered as a scheme over $\mathbf{Z}$, and let $p \colon \mathcal{C} \to X$ be the family of affine hyperelliptic curves whose fiber over some given configuration is the double cover $y^2 = d(x)$ ramified at the points in question. Over the complex numbers, $X(\mathbf{C})$ is a classifying space for the Artin braid group β_{2g+1} on $2g+1$ strands,

(1)This corresponds to the continuous family of characters $|\cdot|^{it}$ indexed by real numbers t.

and the symplectic local system $\mathbb{V} = R^1 p_! \mathbf{Q}$ on X that assembles the first cohomology groups of the hyperelliptic curves can be identified with a representation $\beta_{2g+1} \to \mathrm{Sp}(V)$ classically known in topology as the rational reduced Burau representation.

Over the finite field $\mathbb{F}_q$, the points $X(\mathbb{F}_q)$ are in bijection with the set of polynomials $\mathcal{P}_{2g+1}$, and there is for each prime number ℓ that does not divide q a similarly defined ℓ-adic local system $\mathbb{V} = R^1 p_! \mathbf{Q}_\ell$ on $X \otimes \mathbb{F}_q$, along with an action of Frobenius. Letting $\omega_1, \ldots, \omega_{2g}$ denote the eigenvalues of Frobenius acting on the fiber of $\mathbb{V}$ at a geometric point above a polynomial $d \in \mathcal{P}_{2g+1}$, the central value is given by

$$L(\tfrac{1}{2}, \chi_d) = \prod_{j=1}^{2g} (1 - q^{-1/2} \omega_j).$$

By the expression of the coefficients of the characteristic polynomial as traces on exterior powers, it is also equal to the trace of Frobenius acting on the exterior algebra[(2)]

$$\Lambda \mathbb{V}(\tfrac{1}{2}) = \bigoplus_{i=0}^{2g} \Lambda^i \mathbb{V}(\tfrac{1}{2}).$$

Similarly, $L(\frac{1}{2}, \chi_d)^r$ is the trace of Frobenius acting on $(\Lambda \mathbb{V}(\frac{1}{2}))^{\otimes r}$, and hence the Grothendieck–Lefschetz trace formula gives the following expression for the moments:

$$\frac{1}{q^{2g+1}} \sum_{d \in \mathcal{P}_{2g+1}} L(\tfrac{1}{2}, \chi_d)^r = \sum_k (-1)^k \mathrm{Tr}\Big(\mathrm{Frob}_q \,|\, \mathrm{H}_k^{\text{ét}}(X \otimes \overline{\mathbb{F}}_q, (\Lambda \mathbb{V}(\tfrac{1}{2}))^{\otimes r})\Big).$$

The idea is now to decompose $(\Lambda \mathbb{V}(\frac{1}{2}))^{\otimes r}$ into irreducible representations $\mathbb{V}_\lambda(\frac{1}{2})$ of the symplectic group Sp_{2g}, indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_g \geq 0)$, and to study each contribution to the trace formula separately. An encouraging observation at this point is that the multiplicity with which a given $\mathbb{V}_\lambda$ occurs in $(\Lambda \mathbb{V})^{\otimes r}$ grows like a polynomial of degree $r(r+1)/2$ in g.

Artin's comparison isomorphism allows one to identify these étale homology groups over finite fields with group homology of the braid group over the complex numbers:

$$\mathrm{H}_k(\beta_{2g+1}, V_\lambda) \otimes \mathbf{Q}_\ell \cong \mathrm{H}_k^{\text{ét}}(X \otimes \mathbb{F}_q, \mathbb{V}_\lambda).$$

The key feature is then that the braid group satisfies homological stability with twisted coefficients, which means that in a given degree k the maps

$$\mathrm{H}_k(\beta_{2g+1}, V_\lambda) \longrightarrow \mathrm{H}_k(\beta_{2g+3}, V_\lambda),$$

induced from the inclusions $\beta_{2g+1} \subset \beta_{2g+3}$ and from considering V_λ as a representation of a bigger symplectic group by padding the partition λ with some zeroes, are isomorphisms for all large enough g. Such a statement was already known as a consequence of a general homological stability result by Randal-Williams and Wahl [25], with a bound on k that depends on the partition λ. However, it is crucial for the purposes of the proof to have a *uniform* stability bound at our disposal: this is the new contribution of Miller, Patzt,

(2) The half-integral Tate twist is an operation on ℓ-adic local systems over finite fields to the effect of multiplying by $q^{-1/2}$ the eigenvalues of Frobenius.

Petersen, and Randal-Williams [22], who show that for all partitions λ, the above map is an isomorphism in all degrees $k \leq 1 + g/6$.

The natural question is then to compute the *stable homology*

$$\mathrm{H}_k(\beta_\infty, V_\lambda) = \operatorname*{colim}_g \mathrm{H}_k(\beta_{2g+1}, V_\lambda),$$

which one may think of as a hyperelliptic analogue with twisted coefficients of the theorem of Madsen–Weiss (formerly known as the Mumford conjecture) identifying the stable homology of the mapping class group with an explicit polynomial algebra. The main results of Bergström, Diaconu, Petersen, and Westerland [6] are the computation of the generating series of dimensions of $\mathrm{H}_k(\beta_\infty, V_\lambda)$ and of the Galois action induced from Artin's isomorphism. This allows them to show that the expression

$$Q_r(2g+1) = \sum_{\lambda_1 \leq r} \sum_k (-1)^k \operatorname{Tr}(\mathrm{Frob}_q \,|\, \mathrm{H}_k^{\text{ét}}(\beta_\infty, \mathbb{V}_\lambda(\tfrac{|\lambda|}{2})) \dim V_{\lambda^\dagger(g)},$$

where $V_{\lambda^\dagger(g)}$ is a suitably defined representation of the symplectic group Sp_{2r}, defines a polynomial of degree $r(r+1)/2$ satisfying the estimate in Theorem 1.1. It turns out to match perfectly with the predictions of Andrade and Keating [2]. The bound on the right-hand side of (1) comes from Deligne's estimates on the absolute value of the eigenvalues of Frobenius, along with a rough estimate of the Betti numbers using the number of cells in a stratification of the configuration space.

The text is organised as follows. Section 2 gathers background material on the symmetric function formalism, quadratic L-functions, and moduli spaces of hyperelliptic curves both in the topological and the algebro-geometric setting. With these preliminaries out of the way, we give a precise statement of the main results concerning homological stability for the braid group with twisted coefficients in Section 3. Section 4 is devoted to the proof of Theorem 1.1 taking the results from Section 3 for granted. Finally, among the results of Section 3, we chose to present in Section 5 a sketch of some of the ideas leading to the computations of the stable homology.

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2. Preliminaries

2.1. Symmetric functions

Recall that a *partition* λ is a decreasing sequence of non-negative integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$$

that eventually reaches 0. We set $|\lambda| = \sum \lambda_i$ and denote by $\ell(\lambda)$ the *length* of the partition, *i.e.* the number of positive terms in the sequence. A useful way to represent a partition is through its Young diagram, which is the array of $\ell(\lambda)$ rows consisting of λ_i boxes each. The *conjugate partition* λ' is then obtained by flipping the Young diagram of λ along its diagonal, as illustrated in the figure below:

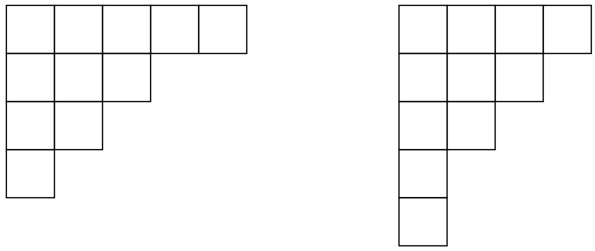

FIGURE 1. The partition $\lambda = (5,3,2,1)$ and its conjugate $\lambda' = (4,3,2,1,1)$

There is a standard one-to-one correspondence between partitions λ of an integer $|\lambda| = n$ and irreducible $\mathbf{Q}$-linear representations σ_λ, the so-called *Specht modules*, of the symmetric group $\mathfrak{S}_n$. The *Schur functor* S^λ is defined on $\mathbf{Q}$-vector spaces V as

$$S^\lambda(V) = V^{\otimes n} \otimes_{\mathbf{Q}[\mathfrak{S}_n]} \sigma_\lambda.$$

For example, the partition $\lambda = (n)$ corresponds to the trivial representation and the associated Schur functor is the symmetric power Sym^n, whereas $\lambda = (1, \dots, 1)$ corresponds to the sign representation and the associated Schur functor is the exterior power Λ^n. The space $S^\lambda(V)$ is non-zero if and only if $\ell(\lambda) \leq \dim V$. By *Schur–Weyl duality*, each $S^\lambda(V)$ is an irreducible representation of $\mathrm{GL}(V)$ and

$$V^{\otimes n} \cong \bigoplus_{|\lambda|=n} S^\lambda(V) \otimes \sigma_\lambda^\vee.$$

We will need a similar statement for the symplectic group $\mathrm{Sp}(V)$ on a vector space of dimension $2g$ equipped with a symplectic form $V \otimes V \to \mathbf{Q}$. In general, the restriction of $S^\lambda(V)$ to $\mathrm{Sp}(V)$ is not irreducible. For each partition λ, there is a unique subrepresentation $V_\lambda \subset S^\lambda(V)$ of $\mathrm{Sp}(V)$ satisfying

$$V^{\langle n \rangle} \cong \bigoplus_{|\lambda_n|} V_\lambda \otimes \sigma_\lambda^\vee \subset V^{\otimes n},$$

where $V^{\langle n \rangle}$ denotes the subspace of traceless tensors, *i.e.* the joint kernel of all the linear maps $V^{\otimes} \to V^{\otimes(n-2)}$ induced by the symplectic form. In other words,

$$V_\lambda = S^\lambda(V) \cap V^{\langle n \rangle}. \tag{2}$$

The space V_λ is non-zero if and only if $\ell(\lambda) \leq g$, and those V_λ form a complete set of pairwise non-isomorphic irreducible representations of $\mathrm{Sp}(V)$.

The *ring of symmetric functions* is defined as the limit

$$\Lambda = \lim_n \mathbf{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$$

with respect to the maps $\mathbf{Z}[x_1, \dots, x_{n+1}]^{\mathfrak{S}_{n+1}} \to \mathbf{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ that send to 0 the last coordinate x_{n+1}. It is isomorphic to the polynomial ring $\mathbf{Z}[h_1, h_2, \dots]$ in the variables

$$h_r = \sum_{i_1 \leq \cdots \leq i_r} \prod_{j=1}^{r} x_{i_j}. \tag{3}$$

The ring of symmetric functions has a grading $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ in which all x_i have degree 1, and Λ_n is isomorphic to the degree n part of $\mathbf{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$.

Recall also that the Frobenius characteristic map gives an isomorphism from the representation ring of $\mathfrak{S}_n$ to Λ_n. The image of σ_λ is the so-called *Schur polynomial*

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}}{\det\left(x_i^{n-j}\right)_{1\leq i,j\leq n}} \in \Lambda_n.$$

It is the character of the representation $S^\lambda(V)$ of $\mathrm{GL}(V)$. For example, the Schur polynomial corresponding to $\lambda = (n)$ is h_n.

In a similar vein, the character of the representation V_λ of $\mathrm{Sp}(V)$ is given by the *symplectic Schur function*, which is now a Laurent polynomial with integer coefficients:

$$s_{\langle\lambda\rangle}(x_1^\pm, \ldots, x_g^\pm) = \frac{\det\left(x_i^{\lambda_j+g-j} - x_i^{-(\lambda_j+g-j)}\right)_{1\leq i,j\leq g}}{\det\left(x_i^{g-j} - x_i^{-(g-j)}\right)_{1\leq i,j\leq g}}. \tag{4}$$

A *graded analytic functor* is an endofunctor F on the category of degreewise finite-dimensional graded vector spaces over $\mathbf{Q}$ of the form

$$A \longmapsto \bigoplus_{n\geq 0} V(n) \otimes_{\mathbf{Q}[\mathfrak{S}_n]} A^{\otimes n}, \tag{5}$$

where $(V(n))_{n\geq 0}$ is a sequence of $\mathbf{Q}$-linear representations of the symmetric group $\mathfrak{S}_n$ on bounded below degreewise finite-dimensional graded vector spaces. These representations $V(n)$ are called the *Taylor coefficients* of the analytic functor. Analytic functors form an abelian category $\mathrm{Ana}^{\mathrm{gr}}$, the Grothendieck group of which is isomorphic to

$$K_0(\mathrm{Ana}^{\mathrm{gr}}) \cong \prod_{n=0}^{\infty} \Lambda_n \otimes \mathbf{Z}(\!(z)\!).$$

We let $\widehat{\Lambda}^{\mathrm{gr}}$ denote the right-hand side, and define the *Taylor series* of an analytic functor F as in (5) as the element of $\widehat{\Lambda}^{\mathrm{gr}}$ given by[(3)]

$$[F] = \sum_n \sum_k [V(n)_k](-z)^k,$$

under the identification of Λ_n with the representation ring of $\mathfrak{S}_n$. Those analytic functors such that $V(0)$ lies in positive degrees have their Taylor series in

$$\widehat{\Lambda}_0^{\mathrm{gr}} = z\mathbf{Z}[\![z]\!] \oplus \prod_{n=1}^{\infty} \Lambda_n \otimes \mathbf{Z}(\!(z)\!) \subset \widehat{\Lambda}^{\mathrm{gr}}, \tag{6}$$

and satisfy the important property that $F \circ G$ is again an analytic functor. The Taylor series of such a composition is then given by $[F \circ G] = [F] \circ [G]$, where

$$\circ\colon \widehat{\Lambda}_0^{\mathrm{gr}} \times \widehat{\Lambda}_0^{\mathrm{gr}} \longrightarrow \widehat{\Lambda}_0^{\mathrm{gr}}$$

(3) The sign comes from the Koszul sign rule in the category of graded vector spaces.

is an operation called *plethysm*. In terms of this operation, the plethystic exponential and the plethystic logarithm are bijections inverse to each other

$$\mathbf{Exp}\colon \widehat{\Lambda}_0^{\mathrm{gr}} \longrightarrow 1+\widehat{\Lambda}_0^{\mathrm{gr}} \quad \text{and} \quad \mathbf{Log}\colon 1+\widehat{\Lambda}_0^{\mathrm{gr}} \longrightarrow \widehat{\Lambda}_0^{\mathrm{gr}},$$

given on an element $x \in \widehat{\Lambda}_0^{\mathrm{gr}}$ by the formulas

$$\mathbf{Exp}(x) = \exp\Big(\sum_{k\geq 1} \tfrac{p_k}{k}\Big) \circ x \quad \text{and} \quad \mathbf{Log}(1+x) = \sum_{k\geq 1} \tfrac{\mu(k)}{k} \log(1+p_k) \circ x,$$

where $p_k = x_1^k + x_2^k + \cdots \in \Lambda$ denotes the k^{th} power symmetric function and μ the Möbius function. They are a useful way to express the Taylor series of some analytic functors. For example, the symmetric algebra $A \mapsto \mathrm{Sym}(A)$ has Taylor series $\mathbf{Exp}(h_1) = \sum_{r\geq 0} h_r$.

2.2. Quadratic L-functions over function fields

Let $p \geq 3$ be a prime number and q a power of p. We fix an algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ and, for each $n \geq 1$, we denote by $\mathbb{F}_{q^n}$ the subfield of $\overline{\mathbb{F}}_p$ of cardinality q^n. Let C be a smooth projective geometrically connected curve over $\mathbb{F}_q$. By the Weil conjectures for curves over finite fields, the zeta function

$$Z_C(t) = \exp\Big(\sum_{n=1}^{\infty} |C(\mathbb{F}_q)| \frac{t^n}{n}\Big)$$

is a rational function of the form

$$Z_C(t) = \frac{P_C(t)}{(1-q)(1-qt)},$$

where P_C is a polynomial with integer coefficients of degree $2g$ and constant term equal to 1, the reciprocal roots of which are complex numbers $\omega_1, \ldots, \omega_{2g}$ of modulus $\sqrt{q}$ stable by the symmetry $\omega \mapsto q/\omega$. In fact, for every prime number ℓ different from p, the numerator of the zeta function is the characteristic polynomial of Frobenius

$$P_C(t) = \det(1 - t\mathrm{Frob}_q \mid \mathrm{H}^1_{\text{ét}}(C \otimes \overline{\mathbb{F}}_q, \mathbf{Q}_\ell)) = \prod_{i=1}^{2g} (1-\omega_i t)$$

acting on the first étale cohomology of C, and $\omega_1, \ldots, \omega_{2g}$ are the eigenvalues. We will think of the multiset of these eigenvalues as a conjugacy class Θ_C in the symplectic similitude group $\mathrm{GSp}_{2g}(\mathbf{C})$, or more conveniently, after unitarizing them, of

$$\overline{\Theta}_C = q^{-1/2}\Theta_C \tag{7}$$

as a conjugacy class in $\mathrm{USp}_{2g} = \mathrm{Sp}_{2g}(\mathbf{C}) \cap \mathrm{U}_{2g}$. Note that $\overline{\Theta}_C$ is stable under inversion.

When $d \in \mathbb{F}_q[x]$ is a monic squarefree polynomial of degree $2g+1$ and when we take as C the hyperelliptic curve C_d of genus g with affine equation $y^2 = d(x)$, the polynomial P can be expressed in terms of the quadratic L-function

$$L(s, \chi_d) = \sum_{\substack{m \in \mathbb{F}_q[t] \\ \text{monic}}} \frac{\chi_d(m)}{q^{s \deg m}} = \prod_{\substack{\pi \in \mathbb{F}_q[t] \text{ monic} \\ \text{irreducible}}} \frac{1}{1-\chi_d(\pi) q^{-s\deg \pi}},$$

where χ_d is the quadratic residue symbol $(\frac{d}{\cdot})$, defined as follows:

– if m is constant, then $\chi_d(m) = 1$;
– if m is irreducible, then $\chi_d(m)$ is equal to 0 if m divides d, to 1 if d is congruent to a square modulo m, and to -1 otherwise;
– if $m = \pi_1 \cdots \pi_k$ is a product of irreducible factors, then $\chi_d(m) = \chi_d(\pi_1) \cdots \chi_d(\pi_k)$.

Indeed, using the equality of the zeta function of C_d and that of the quadratic function field $\mathbb{F}_q(x)(\sqrt{d})$, the precise relation between the two functions is $L(s, \chi_d) = P_{C_d}(q^{-s})$. In particular, the central values we are interested in are given by the product

$$L(\tfrac{1}{2}, \chi_d) = \prod_{\omega \in \overline{\Theta}_d} (1 - \omega),$$

where we use the notation $\overline{\Theta}_d$ to abbreviate $\overline{\Theta}_{C_d}$.

2.3. Topological moduli spaces of hyperelliptic surfaces

We will consider hyperelliptic curves in a topological and algebro-geometric setting. Following tradition, we call them hyperelliptic surfaces in the topological setting.

Definition 2.1. — *A* hyperelliptic surface *is an oriented compact surface S, possibly with boundary, along with an involution ι such that $S/\langle\iota\rangle$ is a sphere with a finite number of punctures and holes and such that the quotient map $S \to S/\langle\iota\rangle$ is a branched cover. We do not allow ramification points on the boundary.*

Up to diffeomorphism, oriented compact surfaces are classified by two discrete invariants: the genus and the number of boundary components. From this, one can construct a topological moduli space as follows: let us fix a reference surface $S_{g,n}$ of genus g with n boundary components, and let $\mathrm{Diff}_\partial(S_{g,n})$ denote the group of diffeomorphisms $S_{g,n} \to S_{g,n}$ that restrict to the identity on some collar neighborhood of the boundary components [(4)]. Then we define the moduli space as the classifying space

$$M_g^n = \mathrm{BDiff}_\partial(S_{g,n}).$$

Except for a few small values of g and n, the connected components of $\mathrm{Diff}_\partial(S_{g,n})$ are contractible, and hence M_g^n is also the classifying space of the *mapping class group*

$$\mathrm{Mod}_g^n = \pi_0(\mathrm{Diff}_\partial(S_{g,n})).$$

When it comes to hyperelliptic surfaces, there are two kinds of boundary components to distinguish: those stable under the hyperelliptic involution (*Weierstrass components*) and those that are pairwise interchanged by it (*conjugate boundary pairs*); up to diffeomorphism, hyperelliptic surfaces are classified by their genus and their type of boundary components. For integers $n, m \geq 0$, let us fix a reference hyperelliptic surface $S_{g,n,m}$ of genus g with $n+2m$ boundary components, among which n are Weierstrass components and m are conjugate boundary pairs, and let

$$\mathrm{Hyp}_\partial(S_{g,n,m}) \subset \mathrm{Diff}_\partial(S_{g,n,m})$$

(4) In the case $n = 0$, the diffeomorphism is also required to preserve the orientation.

denote the subgroup of diffeomorphisms that commute with the hyperelliptic involution. As above, we define the moduli space of hyperelliptic surfaces as

$$H_g^{n,m} = \mathrm{BHyp}_\partial(S_{g,n,m}).$$

We will only need to consider the cases where (n, m) is $(1, 0)$ and $(0, 1)$, for which these spaces admit a more concrete description in terms of the configuration spaces $\mathrm{Conf}_r(\mathbb{D})$ of r distinct unordered points in the unit disk $\mathbb{D}$. Recall that this space is a $K(\pi, 1)$ with fundamental group the *Artin braid group* on r strands

$$\beta_r = \langle \sigma_1, \dots, \sigma_{r-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i - j| \geq 2\rangle.$$

Lemma 2.2. — *There are homotopy equivalences*

$$H_g^{0,1} \simeq \mathrm{Conf}_{2g+1}(\mathbb{D}) \quad \textit{and} \quad H_g^{1,0} \simeq \mathrm{Conf}_{2g+2}(\mathbb{D}).$$

In particular, $H_g^{0,1}$ and $H_g^{1,0}$ are classifying spaces for the braid groups β_{2g+1} and β_{2g+2}.

The space $H_g^{n,m}$ carries a local system V of rank $2g + n + 2m - 1$ with fiber

$$\mathrm{H}^1(S_{g,n,m}/\partial S_{g,n,m}, \mathbf{Z})$$

at the point $[S_{g,n,m}]$. For $(n, m) = (1, 0)$, it is the rank $2g$ local system with fiber the cohomology of the closed hyperelliptic curve S_g, and in the case $(n, m) = (0, 1)$ it is an extension of that one by the trivial rank-one local system. Through the identifications of Lemma 2.2, we can think of V as defining a representation

$$\beta_r \to \mathrm{GL}_{r-1}(\mathbf{Z}),$$

which is classically known as the reduced integral *Burau representation*.

For odd r, the Burau representation actually lands in the symplectic group $\mathrm{Sp}_{r-1}(\mathbf{Z})$, reflecting the fact that the local system V carries the symplectic pairing induced by the intersection product on the cohomology of $S_{g,1,0}$. For even r, it follows from the extension structure of V that the Burau representation takes values in a parabolic subgroup of $\mathrm{Sp}_r(\mathbf{Z})$, the stabilizer of a unimodular vector on the standard representation; this is what Gelfand and Zelevinsky [16] call the *odd symplectic group*.

We now discuss the multiplicative structure of the spaces $H_g^{n,m}$. In homotopy theory, the analogue of a monoid is called an E_1*-algebra*, and a commutative monoid is called an E_∞*-algebra*. In ordinary abstract algebra, a monoid may or may not be commutative, and there is nothing in between. But in homotopy theory there are intermediate notions of E_n*-algebras*, whose multiplication is roughly speaking commutative up to higher and higher orders of coherent homotopy. The disjoint union of configuration spaces $X = \coprod_{n\geq 0} \mathrm{Conf}_n(\mathbb{D})$ carries the structure of an E_2*-algebra*, with a multiplication map given by putting two configurations of points next to each other. It follows that

$$\coprod_{g\geq 0} H_g^{1,0} \sqcup H_g^{0,1}$$

is also equipped with the structure of an E_2-algebra. Multiplication by a fixed element of $H_0^{1,0}$ gives rise to a sequence of maps

$$\cdots \longrightarrow H_g^{1,0} \longrightarrow H_g^{0,1} \longrightarrow H_{g+1}^{1,0} \longrightarrow H_{g+1}^{0,1} \longrightarrow \cdots \tag{8}$$

which can also be interpreted in more concrete terms as gluing a pair of pants along the waist in the case of $H_g^{1,0} \to H_g^{0,1}$ and along the two legs in the case of $H_g^{0,1} \to H_{g+1}^{1,0}$. On fundamental groups, they induce the natural inclusions of braid groups.

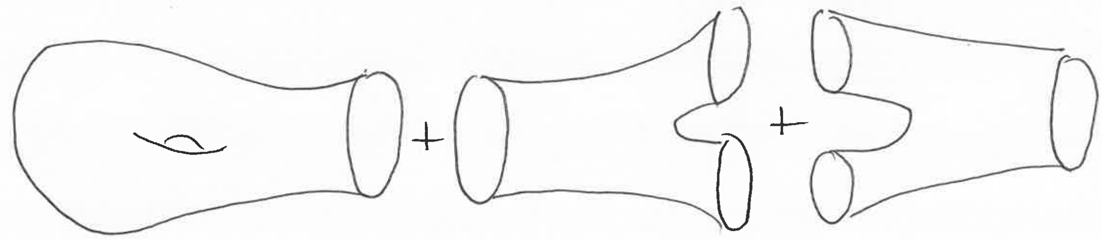

Figure 2. Moduli interpretation of the maps $H_g^{1,0} \to H_g^{0,1}$ and $H_g^{0,1} \to H_{g+1}^{1,0}$

We denote the homotopy colimits with respect to the above transition maps by

$$H_\infty^{1,0} = \operatorname{hocolim} H_g^{1,0} \quad \text{and} \quad H_\infty^{0,1} = \operatorname{hocolim} H_g^{0,1}.$$

2.4. Algebro-geometric moduli spaces of hyperelliptic curves

We now turn to the algebraic setting:

Definition 2.3. — *A* hyperelliptic curve *is a smooth projective geometrically connected curve C over some field along with an involution $\iota\colon C \to C$ such that the quotient $C/\langle\iota\rangle$ is isomorphic to $\mathbb{P}^1$. The fixed points of the hyperelliptic involution are called the* Weierstrass points *of C.*

For integers $n, m \geq 0$, let $\mathcal{H}_g^{n,m}$ be the moduli stack parameterizing hyperelliptic curves of genus g with $n+m$ distinct ordered marked points and a non-zero tangent vector at each of them, with the requirement that n points are Weierstrass points and m points are non-Weierstrass points, none of which is conjugate to another under the involution. It is a smooth stack over $\mathbf{Z}[\frac{1}{2}]$, equipped with the universal family $p\colon \mathcal{C} \to \mathcal{H}_g^{n,m}$ of hyperelliptic curves of genus g with $n + 2m$ punctures. For each prime number ℓ, this family gives rise to an ℓ-adic local system $\mathbb{V} = R^1 p_! \mathbf{Q}_\ell$ on $\mathcal{H}_g^{n,m}$. Besides, the complex points $\mathcal{H}_g^{n,m}(\mathbf{C})$ have the homotopy type of the topological moduli space $H_g^{n,m}$ from Section 2.3, and $\mathbb{V}$ induces the local system $V \otimes \mathbf{Q}_\ell$ on $H_g^{n,m}$. By Artin's comparison theorem, for every partition λ, there is an isomorphism

$$\mathrm{H}_k^{\mathrm{sing}}(H_g^{n,m}, S^\lambda(V)) \otimes \mathbf{Q}_\ell \cong \mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{n,m} \otimes \overline{\mathbf{Q}}, S^\lambda(\mathbb{V})).$$

Moreover, Abramovich, Corti and Vistoli [1] construct a smooth proper stack $\overline{\mathcal{H}_g^{n,m}}$ over $\mathbf{Z}[\frac{1}{2}]$ containing $\mathcal{H}_g^{n,m}$ as the complement of a normal crossing divisor. For every odd prime power q different from ℓ, the local system $\mathbb{V}$ has tame ramification at infinity, and hence Artin's comparison theorem also provides us with an isomorphism

$$\mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{n,m} \otimes \overline{\mathbf{Q}}, S^\lambda(\mathbb{V})) \cong \mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{n,m} \otimes \overline{\mathbb{F}}_q, S^\lambda(\mathbb{V})).$$

Recall that $H_g^{n,m}$ is a $K(\pi,1)$-space, and hence the homology of a local system can be computed as group homology. Putting everything together, we find an isomorphism

$$\mathrm{H}_k^{\text{sing}}(\beta_{2g+1}, S^\lambda(V)) \otimes \mathbf{Q}_\ell \cong \mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{1,0} \otimes \overline{\mathbb{F}}_q, S^\lambda(\mathbb{V})), \tag{9}$$

and similarly for the symplectic coefficients V_λ or for the group homology of the braid group β_{2g+2} on an even number of strands.

3. The stable cohomology of the braid group with twisted coefficients

We are now able to state the main new results concerning homological stability of the braid group with twisted coefficients. Let us start with the general setting. Let

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

be a sequence of group homomorphisms, and let

$$M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots$$

be a sequence of group representations, where M_i is a representation of G_i and the map $M_i \to M_{i+1}$ is G_i-equivariant with respect to the action of G_i on M_{i+1} via the morphism $G_i \to G_{i+1}$. For each degree k, we get induced maps in group homology

$$\mathrm{H}_k(G_0, M_0) \longrightarrow \mathrm{H}_k(G_1, M_1) \longrightarrow \mathrm{H}_k(G_2, M_2) \longrightarrow \cdots$$

and there are two basic questions:

a) For a given k, are these maps isomorphisms for all big enough n? The degrees for which this holds are then called the *stable range*.
b) If so, can one compute the *stable homology*

$$\mathrm{H}_k(G_\infty, M_\infty) = \operatorname*{colim}_n \mathrm{H}_k(G_n, M_n)?$$

We specialize this general setting to the case where G_n is the Artin braid group β_n and the sequence of group homomorphisms $\beta_n \to \beta_{n+1}$ are the natural inclusions obtained by adding one strand to the braid. In fact, these maps fit into a diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \beta_{2g+1} & \longrightarrow & \beta_{2g+2} & \longrightarrow & \beta_{2g+3} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{Mod}_g^1 & \longrightarrow & \mathrm{Mod}_g^2 & \longrightarrow & \mathrm{Mod}_{g+1}^1 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{Sp}_{2g}(\mathbf{Z}) & \longrightarrow & \mathrm{Sp}_{2g+1}(\mathbf{Z}) & \longrightarrow & \mathrm{Sp}_{2g+2}(\mathbf{Z}) & \longrightarrow & \cdots \end{array} \tag{10}$$

where Mod_g^n stands for the mapping class group and $\mathrm{Sp}_{2g+1}(\mathbf{Z})$ for the odd symplectic group from Section 2.3. The horizontal maps $\beta_{2g+1} \to \beta_{2g+2} \to \beta_{2g+3}$ coincide with those induced on fundamental groups by the sequence (8), and the second row is also obtained by gluing pairs of pants. The vertical maps arise from naturally defined maps on moduli spaces. Topologically, with a configuration of r points one associates the double cover ramified at these points: for $r = 2g+1$, this is an oriented compact surface of genus g with one boundary component; for $r = 2g+2$, it is still of genus g but has two boundary components exchanged by the hyperelliptic involution. The map $\mathrm{Mod}_g^1 \to \mathrm{Sp}_{2g}(\mathbf{Z})$ sends a mapping class of S_g to the symplectic transformation that it induces on the cohomology $\mathrm{H}^1(S_g, \mathbf{Z})$, and the composition

$$\mathrm{Mod}_g^2 \longrightarrow \mathrm{Mod}_{g+1}^1 \longrightarrow \mathrm{Sp}_{2g+2}(\mathbf{Z})$$

takes values in the odd symplectic group $\mathrm{Sp}_{2g+1}(\mathbf{Z})$. The composition of the vertical maps is nothing but the Burau representation, so we also obtain a sequence

$$\cdots \longrightarrow V(2g+1) \longrightarrow V(2g+2) \longrightarrow V(2g+3) \longrightarrow \cdots$$

of representations $V(n)$ of β_n as in the general setting of homological stability. In more algebro-geometric terms, the “odd” vertical maps are induced on fundamental groups by

$$\mathcal{H}_g^{1,0} \longrightarrow \mathcal{M}_g^1 \longrightarrow \mathcal{A}_g,$$

where $\mathcal{M}_g^1 \to \mathcal{A}_g$ is the Torelli map from the moduli stack of smooth genus g curves with a marked point and a non-zero tangent vector at it to the moduli stack $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. There is also a more complicated interpretation of the “even” vertical maps along the same lines (see [6, 1.3.10]).

All the groups appearing in diagram (10) satisfy homological stability for the trivial representation: for braid groups, this is due to Arnold [3]; for mapping class groups, to Harer [17]; and for usual symplectic groups, to Charney [8]. Since the original results, there have been many improvements on the stability range that we do not attempt to survey. The rational stable homology of the braid group is not very interesting:

$$\mathrm{H}_k(\beta_\infty, \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{if } k \text{ is 0 or 1,} \\ 0 & \text{otherwise.} \end{cases}$$

With twisted coefficients, a general homological stability result implies:

Theorem 3.1 (Randal-Williams and Wahl, [25]). — *Let* λ *be a partition. The map*

$$\mathrm{H}_k(\beta_n, S^\lambda(V)) \longrightarrow \mathrm{H}_k(\beta_{n+1}, S^\lambda(V))$$

is an isomorphism in all degrees satisfying $2k \leq n - 2 - |\lambda|$.

Notice that the stability bound in Theorem 3.1 depends on the partition λ. It cannot be otherwise, even in degree zero, since the multiplicity of the trivial representation in the restriction of $S^\lambda(V)$ from $\mathrm{GL}(V)$ to $\mathrm{Sp}(V)$ depends on λ. For the application to the moments of L-functions, it is however crucial to have a uniform bound that only depends on n, and this is why we need to switch to the symplectic coefficients[(5)] V_λ.

Definition 3.2. — *A* uniform stability bound *is an increasing function* $\theta\colon \mathbb{R} \to \mathbb{R}$ *such that, for all* n, *the map*

$$\mathrm{H}_k(\beta_n, V_\lambda(n)) \longrightarrow \mathrm{H}_k(\beta_{n+1}, V_\lambda(n)) \tag{11}$$

is an isomorphism for all partitions λ *with* $\ell(\lambda) \leq n/2$ *and all* $k \leq \theta(n)$.

We will shortly see that a non-trivial uniform stability bound exists, but let us first state the main topological result of Bergström, Diaconu, Petersen, and Westerland [6].

Theorem 3.3 (Bergström, Diaconu, Petersen, and Westerland [6, Theorem 4.4.12])
The equality

$$\sum_i \sum_\lambda \dim \mathrm{H}_i(\beta_\infty, V_\lambda)(-z)^i s_{\lambda'} = \mathbf{Exp}\Big(z^{-1}\mathbf{Log}(z + \sum_{j\geq 0} h_{2j} z^j) - 1 - h_2\Big) \tag{12}$$

holds in the completion Λ_0^{gr} *from* (6), *where* **Exp** *and* **Log** *stand for the plethystic exponential and logarithm, and* h_{2j} *are the symmetric functions* (3).

An explicit inspection of the degrees appearing in the right-hand side of this expression then yields the vanishing of the stable homology in certain low degrees.

Corollary 3.4 ([6, Theorem 7.0.2] and [6, Theorem 7.0.12])
For each partition λ, *the stable homology* $\mathrm{H}_k(\beta_\infty, V_\lambda)$ *vanishes in*

– *all degrees* $k < |\lambda|/4$;
– *all degrees* $k < \ell(\lambda)/2$.

Another consequence concerns the growth of the stable Betti numbers.

Corollary 3.5 ([6, Theorem 7.0.15]). — *For each partition* λ, *the power series*

$$\sum_k \dim \mathrm{H}_k(\beta_\infty, V_\lambda)(-z)^k$$

is a rational function of z *with all poles on the unit circle. If* $\lambda_1 \leq |\lambda|/2$, *then the Betti numbers grow as a polynomial of degree* $|\lambda|/2 - 1$ *in* k.

(5) In the case of the odd symplectic groups, the representations V_λ are again defined by the formula (2), even if the restriction of the symplectic form is not a perfect pairing. They are indecomposable representations of a non-semisimple group.

For the application to the asymptotic behaviour of the moments of L-functions, one also needs to bound the eigenvalues of Frobenius acting on stable homology. As often happens, the answer is as simple as possible.

THEOREM 3.6 (Bergström, Diaconu, Petersen, and Westerland [6, Theorem 9.2.2]) *The Galois representation on the stable homology groups*

$$\mathrm{H}_k^{ét}(\mathcal{H}_\infty^{0,1} \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda(\tfrac{|\lambda|}{2}))$$

is pure Tate of weight $-2k$, i.e. Frobenius acts as multiplication by q^{-k}.

The main difficulty here lies in proving that the Frobenius action on the different spaces $\mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{0,1} \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda)$ is compatible, via Artin's comparison isomorphism, with the transcendentally defined stabilization maps. Contrary to the vertical maps, the horizontal maps in diagram (10), for example the inclusions $\beta_{2g+1} \to \beta_{2g+2}$, are not induced by morphisms of algebraic varieties $\mathrm{Conf}_{2g+1}(\mathbb{A}^1) \to \mathrm{Conf}_{2g+2}(\mathbb{A}^1)$ giving rise to Galois-equivariant maps on étale homology. To remedy this, the authors resort to logarithmic algebraic geometry. Using the Abramovich–Corti–Vistoli compactification $\overline{\mathcal{H}_g^{n,m}}$, they replace $H_g^{n,m}$ with an oriented real blow-up and show that the stabilisation maps can be realised as the inclusions of certain boundary strata[(6)].

We finally turn to uniform homological stability.

THEOREM 3.7 (Miller, Patzt, Petersen, and Randal-Willams, [22, Theorem 1.4]) *Consider the function $\theta(n) = (n + 11)/12$. The map (11) is an isomorphism for all $k \leq \theta(n)$, provided that the representations $V_\lambda(n)$ and $V_\lambda(n + 1)$ are both non-zero.*

Combined with the computation of stable homology from Theorem 3.3, this implies the existence of a non-trivial uniform stability bound, namely:

COROLLARY 3.8. — *The function $\theta(n) = (n + 11)/12$ is a uniform stability bound for the twisted homology of the braid group.*

4. Proof of Theorem 1.1

4.1. Relating the moments to the representation theory of symplectic groups

The first step in the proof of Theorem 1.1 consists in expressing the r^{th} moment of the distribution of the central values in more representation-theoretic terms.

DEFINITION 4.1. — *For a partition λ, define*

$$\mathrm{tr}_\lambda(g) = \frac{1}{q^{2g+1}} \sum_{d \in \mathcal{P}_{2g+1}} s_{\langle\lambda\rangle}(\overline{\Theta}_d), \tag{13}$$

where $s_{\langle\lambda\rangle}(\overline{\Theta}_d)$ denotes the symplectic Schur function $s_{\langle\lambda\rangle}$ from (4) evaluated at the eigenvalues in the unitarized conjugacy class $\overline{\Theta}_d$ from (7).

(6) This is one of the advantages of working with the spaces $\mathcal{H}_g^{1,0}$ instead of $\mathrm{Conf}_{2g+1}(\mathbb{A}^1)$.

Given $r \geq 1$ and $g \geq 1$, we let $\lambda \subseteq (r^g)$ denote partitions whose Young diagrams fit inside a $g \times r$ rectangle, *i.e.* satisfying $\lambda_1 \leq r$ and $\ell(\lambda) \leq g$. From the conjugate partition λ' as in Section 2.1, we build a new partition

$$\lambda^\dagger(g) = (g - \lambda'_r \geq \cdots \geq g - \lambda'_1 \geq 0),$$

adding some extra zero parts λ'_j if needed, and we let $V_{\lambda^\dagger(g)}$ denote the associated representation of the symplectic group Sp_{2r}.

Lemma 4.2. — *The following identity holds:*

$$\frac{1}{q^{2g+1}} \sum_{d \in \mathcal{P}_{2g+1}} L(\tfrac{1}{2}, \chi_d)^r = \sum_{\lambda \subseteq (r^g)} \mathrm{tr}_\lambda(g) \dim V_{\lambda^\dagger(g)}.$$

This follows from a formal identity originally due to Jimbo and Miwa

$$\prod_{i=1}^{g} \prod_{j=1}^{r} (1 + t_j x_i)(1 + t_j x_i^{-1}) = (t_1 \cdots t_r)^g \sum_{\lambda \subseteq (r^g)} s_{\langle \lambda \rangle}(x_1^\pm, \ldots, x_g^\pm) s_{\langle \lambda^\dagger(g) \rangle}(t_1^\pm, \ldots, t_r^\pm) \tag{14}$$

by evaluation at $\{x_1^\pm, \ldots, x_g^\pm\} = \overline{\Theta}_d$ and $t_1 = \cdots = t_r = 1$, on noting that $\mathrm{tr}_\lambda(g)$ vanishes for all partitions λ such that $|\lambda|$ is odd.

4.2. Applying the Grothendieck–Lefschetz trace formula

The next step consists in reinterpreting the quantity $\mathrm{tr}_\lambda(g)$ in (13) as an alternating sum of traces of Frobenius acting on the étale homology of the moduli space $\mathcal{H}_g^{1,0}$. For this, we use the Grothendieck–Lefschetz trace formula for an ℓ-adic local system M on an algebraic stack $\mathcal{X} = [X/G]$ obtained as the global quotient of a scheme X by a connected algebraic group G over $\mathbb{F}_q$ (see [4, Corollary 6.4.10]). Once we express it in terms of étale homology rather than cohomology using the Poincaré isomorphism

$$\mathrm{H}^k_{\text{ét},c}(\mathcal{X}, M) \xrightarrow{\sim} \mathrm{H}^{\text{ét}}_{2\dim \mathcal{X} - k}(X, M(\dim \mathcal{X}))$$

given by cap product with the fundamental class in Borel–Moore homology, we get

$$q^{\dim \mathcal{X}} \sum_k (-1)^k \mathrm{Tr}(\mathrm{Frob}_q \,|\, \mathrm{H}_k^{\text{ét}}(\mathcal{X} \otimes \overline{\mathbb{F}}_q, M)) = \sum_{x \in X(\mathbb{F}_q)/G(\mathbb{F}_q)} \mathrm{Tr}(\mathrm{Frob}_x \,|\, M_{\overline{x}}).$$

We apply this formula to the $2g$-dimensional stack $\mathcal{X} = \mathcal{H}_g^{1,0} = [\mathrm{Conf}_{2g+1}(\mathbb{A}^1)/\mathbb{G}_a]$ and the ℓ-adic symplectic local system $M = \mathbb{V}_\lambda(\frac{|\lambda|}{2})$. The $\mathbb{F}_q$ points of $\mathrm{Conf}_{2g+1}(\mathbb{A}^1)$ are in bijection with $\mathcal{P}_{2g+1}$. Because of the half-integral Tate twist, the eigenvalues of Frobenius acting on the stack of M at each point represented by $d \in \mathcal{P}_{2g+1}$ are the elements of $\overline{\Theta}_d$, and hence the trace of Frob_d acting on $\mathbb{V}_\lambda(\frac{|\lambda|}{2}) \subset S^\lambda(\mathbb{V}(\frac{1}{2}))$ is precisely the symplectic Schur function $s_{\langle \lambda \rangle}(\overline{\Theta}_d)$. Summarizing, we get the following:

Lemma 4.3. — *The quantity* $\mathrm{tr}_\lambda(g)$ *is given by*

$$\mathrm{tr}_\lambda(g) = \sum_k (-1)^k \mathrm{Tr}(\mathrm{Frob}_q \,|\, \mathrm{H}_k^{\acute{e}t}(\mathcal{H}_g^{1,0} \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda(\tfrac{|\lambda|}{2}))). \tag{15}$$

In view of this formula, to study the asymptotic behaviour of $\operatorname{tr}_\lambda(g)$ as g goes to infinity, it is natural to consider the traces of Frobenius on stable homology

$$T_\lambda = \sum_k (-1)^k \operatorname{Tr}(\mathrm{Frob}_q \,|\, \mathrm{H}_k^{\text{ét}}(\mathcal{H}_\infty^{1,0} \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda(\tfrac{|\lambda|}{2}))). \tag{16}$$

Notice that this is now an infinite sum, so convergence needs to be proved. But we know from Theorem 3.6, after taking the twist into account, that the stable homology groups are pure of weight $-2k$, and hence T_λ is bounded by

$$\sum_k \dim \mathrm{H}_k^{\text{ét}}(\mathcal{H}_\infty^{1,0} \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda) q^{-k},$$

which converges thanks to Corollary 3.5.

Definition 4.4. — *The polynomial in Theorem 1.1 is defined as*

$$Q_r(2g+1) = \sum_{\lambda_1 \leq r} T_\lambda \dim V_{\lambda^\dagger(g)}.$$

The sum runs now over the larger set of partitions[7] *where $\ell(\lambda) \leq g$ is not required.*

That Q_r is indeed a polynomial of degree $r(r+1)/2$ follows from the explicit expression

$$g \mapsto \dim V_{\lambda^\dagger(g)} = \frac{\prod_{1 \leq i < j \leq n} (\mu_i^2 - \mu_j^2) \prod_{i=1}^r \mu_i}{(2r-1)!!},$$

where $\mu_i = g - \lambda'_{r+1-i} + r - i + 1$. In view of this definition, the difference

$$\frac{1}{q^{2g+1}} \sum_{d \in \mathcal{P}_{2g+1}} L(\tfrac{1}{2}, \chi_d)^r - Q_r(2g+1)$$

that needs to be bounded is equal to

$$\sum_{\lambda \subseteq (r^g)} \sum_k (-1)^k \operatorname{Tr}\Big(\mathrm{Frob}_q \,|\, \mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{1,0} \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda(\tfrac{|\lambda|}{2}))\Big) \dim V_{\lambda^\dagger(g)}$$
$$- \sum_{\lambda_1 \leq r} \sum_k (-1)^k \operatorname{Tr}\Big(\mathrm{Frob}_q \,|\, \mathrm{H}_k^{\text{ét}}(\mathcal{H}_\infty^{1,0} \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda(\tfrac{|\lambda|}{2}))\Big) \dim V_{\lambda^\dagger(g)}.$$

Let θ be a uniform stability bound in the sense of Definition 3.2. Then all terms indexed by $k \leq \theta(2g+1)$ cancel and it remains to bound two kinds of terms:

a) those with $\lambda \subseteq (r^g)$ and $k > \theta(2g+1)$;
b) those with $\lambda_1 \leq r$ but $\ell(\lambda) > g$ or $k > \theta(2g+1)$.

[7] When $\ell(\lambda) > g$, the formula for $\lambda^\dagger(g)$ still makes sense, but the resulting weight is not dominant. One needs to change the definition of $V_{\lambda^\dagger(g)}$ by considering another weight α as in [6, 11.3.10]

4.3. Bounding the unstable homology

We start with a). Since $\mathcal{H}_g^{1,0}$ is smooth and $\mathbb{V}_\lambda(\frac{|\lambda|}{2})$ is a local system of weight 0, the étale homology group $\mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{1,0}\otimes\overline{\mathbb{F}}_q,\mathbb{V}_\lambda(\frac{|\lambda|}{2}))$ has weights in $[-2k,-k]$ by Deligne's main theorem of Weil II [11], and hence we can bound the absolute values of the Frobenius eigenvalues by $q^{-k/2}$. This results in the estimate

$$\Big|\sum_{\lambda\subseteq(r^g)}\sum_{k>\theta(2g+1)}(-1)^k\,\mathrm{Tr}\Big(\mathrm{Frob}_q\,|\,\mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{1,0}\otimes\overline{\mathbb{F}}_q,\mathbb{V}_\lambda(\tfrac{|\lambda|}{2}))\Big)\dim V_{\lambda^\dagger(g)}\Big|$$
$$\leq\sum_{\lambda\subseteq(r^g)}\sum_{k>\theta(2g+1)}q^{-k/2}\dim\mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{1,0}\otimes\overline{\mathbb{F}}_q,\mathbb{V}_\lambda)\cdot\dim V_{\lambda^\dagger(g)}.$$

By the comparison isomorphism (9), it suffices to estimate the dimension of the group homology of the braid group by the formula

$$\dim\mathrm{H}_k(\beta_n,V)=\dim\mathrm{H}_c^{2n-k}(\mathrm{Conf}_n(\mathbf{C}),V)\leq\binom{n-1}{k}\dim V,$$

which relies on the existence of a cellular chain complex, associated with the *Fuks's stratification* of $\mathrm{Conf}_n(\mathbf{C})$ (see, for example, the exposition in [13, 4.1]), with $\binom{n-1}{k}$ cells of dimension $2n-k$ that computes its cohomology with compact support in degree $2n-k$. We can thus bound the contribution from unstable homology by

$$\sum_{k>\theta(2g+1)}q^{-k/2}\binom{2g}{k}\sum_{\lambda\subseteq(r^g)}\dim V_\lambda\cdot\dim V_{\lambda^\dagger(g)},$$

and use the identity

$$\sum_{\lambda\subseteq(r^g)}\dim V_\lambda\dim V_{\lambda^\dagger(g)}=4^{gr},$$

obtained by specializing to 1 all variables in the formula (14), to conclude

$$\Big|\sum_{\lambda\subseteq(r^g)}\sum_{k>\theta(2g+1)}(-1)^k\,\mathrm{Tr}\Big(\mathrm{Frob}_q\,|\,\mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{1,0}\otimes\overline{\mathbb{F}}_q,\mathbb{V}_\lambda(\tfrac{|\lambda|}{2}))\Big)\dim V_{\lambda^\dagger(g)}\Big|\leq C\cdot4^{g(r+1)}q^{-\theta(2g+1)/2},$$

where we can for instance take the constant to be $C=(3+\sqrt{3})/2$.

4.4. Bounding the remaining terms

We now deal with b) by splitting the sum in two according to the size of $|\lambda|$. In the case $|\lambda|\leq 2g$, we only need to consider degrees $k>\theta(2g+1)$. Indeed, since the stable homology vanishes for $k<\ell(\lambda)/2$ by Corollary 3.4, for those partitions with $\ell(\lambda)>g$ we get vanishing in the range $k<g/2$, and $g/2$ is at least $\theta(2g+1)$. By Theorem 3.6, the stable homology of degree k is pure of weight $-2k$, whence a bound

$$\Big|\sum_{\substack{|\lambda|\leq 2g\\ \lambda_1\leq r}}\sum_{k>\theta(2g+1)}(-1)^k\,\mathrm{Tr}\Big(\mathrm{Frob}_q\,|\,\mathrm{H}_k^{\text{ét}}(\mathcal{H}_g^{1,0}\otimes\overline{\mathbb{F}}_q,\mathbb{V}_\lambda(\tfrac{|\lambda|}{2}))\Big)\dim V_{\lambda^\dagger(g)}\Big|$$
$$\leq\sum_{\substack{|\lambda|\leq 2g\\ \lambda_1\leq r}}\sum_{k>\theta(2g+1)}q^{-k}\dim\mathrm{H}_k^{\text{ét}}(\mathcal{H}_\infty^{1,0}\otimes\overline{\mathbb{F}}_q,\mathbb{V}_\lambda)\dim V_{\lambda^\dagger(g)}.$$

For every λ with $\lambda_1 \leq r$, the dimension of $V_{\lambda^\dagger(g)}$ is bounded by $4^{gr} \dim S^{\lambda'}(\mathbf{C}^{2r})$, and hence it suffices to bound

$$4^{gr} \sum_{w=0}^{2g} \sum_{|\lambda|=w} \sum_{k>\theta(2g+1)} q^{-k} \dim \mathrm{H}_k^{\text{ét}}(\mathcal{H}_\infty^{1,0} \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda) \dim S^{\lambda'}(\mathbf{C}^{2r}).$$

This is done by identifying, for fixed k and w, the sum over λ with the coefficient of $z^k t^w$ in the generating series in two variables

$$\mathbf{Exp}\Big(z^{-1}\mathbf{Log}(z + \sum_{k\geq 0} h_{2j}(t) z^j) - 1 - h_2(t)\Big)$$

obtained by specializing to $x_1 = \cdots = x_w = t$ and $x_i = 0$ for $i > w$ the infinitely many variables in the right-hand side of (12). The case $|\lambda| > 2g$ is dealt with similarly, relying instead on the other vanishing from Corollary 3.4.

Theorem 1.1 follows from these estimates after plugging in the uniform stability bound $\theta(2g+1) = g/6 + 1$ from Corollary 3.8.

5. Some ideas in the computation of stable homology

The proof of Theorem 3.3 follows a strategy implemented by Randal-Williams [23] to compute the stable homology of the mapping class group with twisted coefficients, a result that had been previously obtained by Looijenga [20]. Very roughly, it consists in defining a moduli space $H_g^{n,m}(X)$ of maps from marked hyperelliptic surfaces to a background space X, and choose X to be an Eilenberg–MacLane space KA associated with a graded $\mathbf{Q}$-vector space A. The goal is to determine the stable homology

$$A \longmapsto \mathrm{H}_*(H_\infty^{0,1}(KA)) \cong \mathrm{H}_*(H_\infty^{1,0}(KA))$$

as a graded *analytic functor*, in the sense of Section 2.1, in two different ways:

a) From the Serre spectral sequence of the fibration $H_g^{0,1}(X) \to H_g^{0,1}$, one gets that the Taylor coefficients are given by the groups of interest, namely

$$\mathrm{H}_*(H_g^{0,1}, S^\lambda(V)).$$

b) From the group completion theorem, we know that $\mathrm{H}_*(H_\infty^{0,1}(KA))$ is the stable homology of a double loop space for all A. A scanning argument provides us with an explicit model $\Omega^2 W(A)$, from which the homology can be computed using techniques from Koszul duality and rational homotopy theory.

5.1. Hyperelliptic surfaces in a background space

Let X be a pointed topological space with an action of $\mathbf{Z}/2\mathbf{Z}$. Consider the space

$$\mathrm{map}_*^{\mathbf{Z}/2\mathbf{Z}}(S_{g,n,m}/\partial S_{g,n,m}, X)$$

of continuous maps $S_{g,n,m} \to X$ that send the boundary components of $S_{g,n,m}$ to the basepoint of X and are $\mathbf{Z}/2\mathbf{Z}$-equivariant for the action on $S_{g,n,m}$ by the hyperelliptic involution. We endow it with the compact-open topology. The group $\mathrm{Hyp}_\partial(S_{g,n,m})$ acts on the space of such maps by precomposition, and we define the topological moduli space $H_g^{n,m}(X)$ as the homotopy quotient

$$H_g^{n,m}(X) = \mathrm{map}_*^{\mathbf{Z}/2\mathbf{Z}}(S_{g,n,m}/\partial S_{g,n,m}, X) /\!\!/ \mathrm{Hyp}_\partial(S_{g,n,m}).$$

As before, it will suffice for our purposes to treat the cases where (n,m) is $(1,0)$ and $(0,1)$. By construction, the space $H_g^{n,m}(X)$ fits into a fiber sequence

$$\mathrm{map}_*^{\mathbf{Z}/2\mathbf{Z}}(S_{g,n,m}/\partial S_{g,n,m}, X) \longrightarrow H_g^{n,m}(X) \longrightarrow H_g^{n,m}, \tag{17}$$

and the beginning of the proof consists in letting X be an Eilenberg–MacLane space, and computing the homology of $H_g^{n,m}(X)$ using the Serre spectral sequence.

Recall that the *generalized Eilenberg–MacLane space* KA associated to a graded $\mathbf{Q}$-vector space $A = \bigoplus_{r\geq 0} A_r$ concentrated in non-negative degrees is the geometric realization of the simplicial $\mathbf{Q}$-vector space obtained from A via the Dold–Kan correspondence; up to homotopy, it is the product $\prod_{r\geq 0} K(A_r, r)$ of the usual Eilenberg–MacLane spaces, and it satisfies

$$\pi_*(KA)\otimes\mathbf{Q} = A \quad \text{and} \quad \mathrm{H}_*(KA,\mathbf{Q}) = \mathrm{Sym}(A).$$

One advantage of working with these spaces is that there is a quasi-isomorphism

$$\mathrm{map}_*(S, KA) \simeq K(\widetilde{\mathrm{H}}^*(S, A)) \tag{18}$$

for any pointed space S, where the reduced cohomology is placed in non-positive degrees.

Proposition 5.1. — *Let A be a graded $\mathbf{Q}$-vector space in degrees ≥ 2, and consider the action of $\mathbf{Z}/2\mathbf{Z}$ on KA induced by $x\mapsto -x$ on A. For each g, the Serre spectral sequence associated with the fibration (17) induces an isomorphism*

$$\mathrm{H}_*(H_g^{0,1}(KA),\mathbf{Q}) \cong \bigoplus_\lambda S^\lambda(A)\otimes \mathrm{H}_*(H_g^{0,1}, S^\lambda(V[-1])), \tag{19}$$

where V denotes the local system with fiber $\mathrm{H}^1(S_{g,0,1}/\partial S_{g,0,1},\mathbf{Q})$.

Proof. — Specializing to $S = S_{g,0,1}/\partial S_{g,0,1}$ the isomorphism (18) and taking invariants under the action of $\mathbf{Z}/2\mathbf{Z}$, we get

$$\mathrm{map}_*^{\mathbf{Z}/2\mathbf{Z}}(S_{g,0,1}/\partial S_{g,0,1}, KA) = K(A\otimes V[-1]),$$

and hence the homology of the fibers of (17) is the symmetric algebra

$$\mathrm{Sym}(A\otimes V[-1]) = \bigoplus_\lambda S^\lambda(A)\otimes S^\lambda(V[-1]).$$

The Serre spectral sequence

$$E^2_{p,q} = \mathrm{H}_p(H_g^{0,1}, \mathrm{H}_q(\mathrm{map}_*^{\mathbf{Z}/2\mathbf{Z}}(S_{g,0,1}/\partial S_{g,0,1}, KA),\mathbf{Q})) \Longrightarrow \mathrm{H}_{p+q}(H_g^{0,1}(KA),\mathbf{Q})$$

degenerates in the second page, since all its differentials are equivariant for the action of $\mathrm{GL}(A)$, which acts with different weights on the homology of the fibers. □

5.2. Analytic functors and the group completion theorem

We now restate Proposition 5.1 as saying that the assignment $A \mapsto \mathrm{H}_*(H^{0,1}_\infty(KA), \mathbf{Q})$ is a graded analytic functor. Indeed, plugging the definition of the Schur functor $S^\lambda(A)$ in the right-hand side of (19), the functor is given by

$$A \longmapsto \bigoplus_n \Big(\bigoplus_{|\lambda|=n} \mathrm{H}_*(H^{0,1}_\infty, S^\lambda(V[-1])) \otimes \sigma_\lambda \Big) \otimes_{\mathbf{Q}[\mathfrak{S}_n]} A^{\otimes n}. \tag{20}$$

Therefore, its Taylor series is equal to

$$\sum_\lambda \sum_k \dim \mathrm{H}_k(H^{0,1}_\infty, S^\lambda(V[-1]))(-z)^k s_\lambda = \sum_\lambda \sum_k \dim \mathrm{H}_k(H^{0,1}_\infty, S^\lambda(V))(-z)^{k-|\lambda|} s_{\lambda'},$$

where the second equality uses the identification $S^\lambda(V[-1]) = S^{\lambda'}(V)[-|\lambda|]$.

Let us briefly recall the setting of the group completion theorem. For each pointed space X, the loop space ΩX is a topological E_1-*algebra*, which essentially means that ΩX is equipped with a continuous multiplication map that is associative up to homotopy. The functor $X \mapsto \Omega X$ has a left adjoint B, given by the classifying space of an E_1-algebra M. The *group completion* is the functor $M \mapsto \Omega BM$, which has the effect of completing the monoid $\pi_0(M)$ to a group, and the goal of the group completion theorem is to describe the cohomology of ΩBM in terms of that of M. More precisely, assume that $\pi_0(M)$ is the monoid $\mathbf{Z}_{\geq 0}$, so that the connected components M_r of M are indexed by integers, and let $M_r \to M_{r+1}$ denote the maps given by multiplication by a generator of $\pi_0(M)$. Then the group completion theorem (for instance, in the version of McDuff and Segal [21]) gives an isomorphism

$$\mathrm{H}_*(\Omega BM, \mathbf{Z}) \cong \mathrm{H}_*(\mathbf{Z} \times M_\infty, \mathbf{Z}) \cong \mathbf{Z} \times \operatorname*{colim}_r \mathrm{H}(M_r, \mathbf{Z}).$$

By restricting to the base component, this implies that $\mathrm{H}_*(\Omega_0 BM, \mathbf{Z})$ is given by the stable homology of M. Therefore, applied to the E_2-algebra $M = \coprod_{g\geq 0} H^{0,1}_g(KA)$, the theorem implies that the rational stable homology of $H^{0,1}_g(KA)$ is isomorphic to

$$\mathrm{H}_*(H^{1,0}_\infty(KA), \mathbf{Q}) \cong \mathrm{H}_*(\Omega_0 BM, \mathbf{Q}).$$

Besides, there is a *recognition principle* that says that the group completion ΩBM of an E_2-algebra is a double loop space. The next step in the proof is to compute an explicit twofold delooping from which the analytic functor can be computed directly.

5.3. An explicit delooping via scanning

Let A be a graded $\mathbf{Q}$-vector space in degrees ≥ 2. Recall that KA is endowed with the involution ι induced from multiplication by -1 on A. The origin O of KA is a fixed point of ι, and hence the inclusion $O \subset KA$ induces a map $K(\mathbf{Z}/2, 1) \to KA /\!\!/ \langle\iota\rangle$ on homotopy quotients. Besides, the homotopy fiber W of the map $S^2 \to K(\mathbf{Z}/2, 2)$ corresponding to the non-trivial class in $\mathrm{H}^2(S^2, \mathbf{Z}/2)$ fits in a fiber sequence $K(\mathbf{Z}/2, 1) \to W \to S^2$.

DEFINITION 5.2. — *The space $W(A)$ is the homotopy pushout*

$$\begin{array}{ccc} K(\mathbf{Z}/2,1) & \longrightarrow & KA /\!\!/ \langle i\rangle \\ \downarrow & & \downarrow \\ W & \longrightarrow & W(A) \end{array}$$

with respect to the maps described above.

The space $W(A)$ is simply connected and the natural map $W(A) \to S^2 \vee KA/\langle \iota \rangle$ is a homotopy equivalence after localizing away from the prime 2, so for the purposes of computing the rational homology we will be able to work with this simpler space.

THEOREM 5.3 (Bergström, Diaconu, Petersen, and Westerland [6, Theorem 5.0.3]) *Let A be a graded $\mathbf{Q}$-vector space in degrees ≥ 2. Then the group completion of the E_2-algebra $\amalg_{g\geq 0} H_g^{0,1}(KA)$ is the double loop space $\Omega^2 W(A)$.*

To prove this theorem, the authors first observe that the E_2-algebra $\amalg_{g\geq 0} H_g^{0,1}(KA)$ is weakly equivalent to a Hurwitz stack $\mathrm{Hur}_{[0,1]^2}$ parameterizing triples consisting of

- a finite subset $z \subset (0,1)^2$,
- a branched double cover $E \to M$ ramified precisely at z,
- a $\mathbf{Z}/2\mathbf{Z}$-equivariant map $E \to KA$.

The idea is then to adapt to the setting of Hurwitz spaces the *scanning argument* of Segal and McDuff that describes the group completion of usual configuration spaces[8].

Given a locally compact Hausdorff space X and a subspace $A \subset X$, let $\mathrm{Fin}_{X,A}$ denote the set of all finite subsets of X disjoint from A, with basepoint the empty configuration. We endow $\mathrm{Fin}_{X,A}$ with a topology where a basis of open neighborhoods is indexed by compact subsets $K \subset X$ and finite families $\mathfrak{U}$ of disjoint open subsets $U \subset K$, by considering those configurations $z \in \mathrm{Fin}_{X,A}$ that only meet K at the open subsets in $\mathfrak{U}$, and exactly once each of them. For a manifold X, we abbreviate $\mathrm{Fin}_{X,\partial X}$ by Fin_X.

Example 5.4. — The configuration space $\mathrm{Fin}_{\mathbf{R}^n}$ has the homotopy type of the sphere S^n. This can be seen by replacing $\mathbf{R}^n$ with the open unit ball B^n and observing that Fin_{B^n} deformation retracts to the subspace of configurations with at most one point, which is nothing but the one-point compactification of B^n by the empty configuration.

For a compact manifold M, there is an equivalence

$$\mathrm{Fin}_{M\times[0,1]} \cong \coprod_{k\geq 0} \mathrm{Conf}_k(M^\circ \times (0,1))$$

---

[8] From a variant of this technique, Bianchi [7] gets an alternative proof of the Mumford conjecture, which has the new feature of not needing the description of the group completion of $\amalg_{g\geq 0}\mathrm{BMod}_g^1$ as an infinite loop space (Madsen–Weiss theorem) but only as a double loop space; see also [10].

that endows $\mathrm{Fin}_{M\times[0,1]}$ with the structure of an E_1-algebra. The goal is to describe its group completion. In a first approximation, the scanning argument of Segal and McDuff says that the group completion is given by the map

$$\mathrm{Fin}_{M\times[0,1]} \longrightarrow \Omega\mathrm{Fin}_{M\times\mathbb{R}}$$

that sends a configuration z to the loop γ given at time t by the part of the configuration that lies on a small horizontal strip[(9)] $M\times(t-\varepsilon, t+\varepsilon)$. Iterating this construction in the case $M=[0,1]$, one finds that the group completion of $\mathrm{Fin}_{[0,1]^2}$ is $\Omega^2 S^2$.

Bergström, Diaconu, Petersen, and Westerland proceed in a similar way to show in [6, Theorem 5.3.10] that, for any interval L, the scanning map $\mathrm{Hur}_{L\times[0,1]}\to\Omega\mathrm{Hur}_{L\times\mathbf{R}}$ is a group completion. In the case of interest where $L=[0,1]$, it remains to understand the Hurwitz space $\mathrm{Hur}_{[0,1]^2}$. There is still a deformation retract to the subspace of double covers branched at one point at most, and the subspace $W(A)$ from Definition 5.2 is the result of describing this locus as a cover of two open subsets.

5.4. The final computation

It only remains to compute the rational homology of $\Omega^2 W(A)$.

Theorem 5.5. — *The analytic functor $A\mapsto \mathrm{H}_*(\Omega^2_0 W(A),\mathbf{Q})$ has Taylor series*

$$\mathbf{Exp}\Big(z^{-1}\mathbf{Log}(z+\sum_{j\geq 0} h_{2j}z^{-j})-1\Big). \tag{21}$$

Sketch of proof. — From the fact that $W(A)$ has the rational homotopy type of the wedge sum $S^2\vee KA/\langle\iota\rangle$, it is straightforward to see that $A\mapsto\mathrm{H}_*(W(A),\mathbf{Q})$, considered as an analytic functor of quadratic coalgebras, has Hilbert–Poincaré series

$$F(t,z)=1+tz^2+\sum_{r\geq 1} t^r h_{2r}.$$

Indeed, $\mathrm{H}_*(S^2,\mathbf{Q})$ contributes with the term $1+tz^2$, and $A\mapsto\mathrm{H}_*(KA,\mathbf{Q})$ is the symmetric algebra $\mathrm{Sym}(A)$, whose Taylor series is given by $\mathbf{Exp}(th_1)=\sum_{r\geq 0}t^r h_r$; the effect of taking the quotient $KA/\langle\iota\rangle$ is to restrict the sum to even indices. Now, the space $W(A)$ is formal in the sense of rational homotopy theory and its homology is a Koszul coalgebra. By a result of Berglund [5, Theorem 4], this implies that the Taylor series of $A\mapsto\mathrm{H}_*(\Omega W(A),\mathbf{Q})$ is given by

$$\frac{1}{F(\frac{t}{z},z)}=\frac{1}{1+tz+\sum_{r\geq 1}t^r z^{-r}h_{2r}}.$$

(9) In the correct definition, one replaces both the configuration space and the loop space by monoids where multiplication is strictly associative. For this, we fix some $\varepsilon>0$, and we let C consist of pairs of a $t\in[0,\infty)$ and a configuration $z\in\mathrm{Fin}_{M\times[0,t]}$ that does not meet the fibers above $[0,\varepsilon)$ and $(t-\varepsilon,t]$. The *scanning map* $C\to\Omega'\mathrm{Fin}_{M\times(-\varepsilon,\varepsilon)}$ is defined by sending (t,z) to the Moore loop γ of length t where $\gamma(t_0)$ is the part of the configuration on the horizontal strip $(t_0-\varepsilon,t_0+\varepsilon)$.

Forgetting the length grading, which amounts to setting $t = 1$ in the Hilbert–Poincaré series, we get that the analytic functor $A \mapsto \mathrm{H}_*(\Omega W(A), \mathbf{Q})$ has Taylor series

$$G(z) = \frac{1}{z + \sum_{r\geq 0} z^{-r} h_{2r}}.$$

Besides, the space $W(A)$ is simply connected, and hence the homology of the loop space $\Omega W(A)$ is the symmetric algebra on $\pi_*(\Omega W(A)) \otimes \mathbf{Q}$. Since taking the symmetric algebra of an analytic functor corresponds to applying the plethystic exponential on Taylor series, $A \mapsto \pi_*(\Omega W(A)) \otimes \mathbf{Q}$ has Taylor series

$$-\mathbf{Log}(z + \sum_{r\geq 0} z^{-r} h_{2r}).$$

Using $\pi_k(\Omega^2 W(A)) = \pi_{k+1}(\Omega W(A))$ and the fact that shifting the degree down by one amounts to dividing by $-z$, the analytic functor $A \mapsto \pi_*(\Omega^2 W(A)) \otimes \mathbf{Q}$ has Taylor series

$$z^{-1}\mathbf{Log}(z + \sum_{r\geq 0} z^{-r} h_{2r}) = 1 + \text{higher order terms}.$$

The constant term 1 corresponds to $\pi_0(\Omega^2 W(A)) \cong \mathbf{Z}$, so that restricting to the base component has the effect of subtracting it. Using again that $A \mapsto \mathrm{H}_*(\Omega^2_0 W(A))$ is the symmetric algebra on $\pi_*(\Omega^2_0 W(A)) \otimes \mathbf{Q}$, we finally find the Taylor series (21). □

There are two sources of discrepancy between the expression (21) that we just obtained and that from Theorem 3.3. The first one comes from the shift by $-|\lambda|$ in the Taylor series of the analytic functor (20), which has the effect of multiplying h_{2r} by z^{2r}. The second one comes from the difference between the homology of $S^\lambda(V)$ and V_λ, and using the relation between symplectic Schur functions and usual Schur polynomials amounts to multiplying the whole generating series by $\mathbf{Exp}(-h_2)$.

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