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MOTIVIC HOMOTOPY THEORY AND STABLE HOMOTOPY GROUPS
[after Morel–Voevodsky, Isaksen–Wang–Xu,...]

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Contents

Introduction

The idea of applying topological methods to algebraic geometry dates back at least to Lefschetz, who in 1924 envisioned extending the nascent techniques of Analysis Situs to the study of algebraic varieties. Through the use of Lefschetz pencils, he introduced a topological viewpoint on phenomena such as degenerations and the behavior of varieties near their boundaries — ideas that already foreshadowed the modern concept of rational equivalence, which fundamentally encodes the deformation of cycles along the projective line $\mathbb{P}^1$, and suggested a deep connection between geometry and homotopy theory. This foundational perspective was later profoundly reshaped by Grothendieck, who elevated the use of sheaves and topos theory as central tools for structuring and understanding cohomological invariants, culminating in his visionary and revolutionary, yet unfinished, theory of motives.

Beilinson revived this vision by formulating the concept of motivic cohomology, conceived as a universal cohomology theory for algebraic varieties, akin to singular cohomology in topology. His conjectures, especially those concerning the relation with algebraic K-theory and cycles, became a major driving force in the development of the theory. On the topological side, the advent of stable homotopy theory, initiated by Adams and his successors, introduced a new language centered around generalized cohomology and orientation theory; Brown representability, characteristic classes, and formal group laws became essential tools, shaping the modern understanding of stable phenomena. These culminated in the development of chromatic homotopy theory, a guiding philosophy that now serves as a central framework for understanding the layered structure of stable homotopy.

Motivic homotopy theory lies at the crossroads of these ideas and structures. Introduced by Voevodsky and later developed systematically with Morel, it seeks to import the methods of algebraic topology into the realm of algebraic geometry, first by defining homotopies using the affine line $\mathbb{A}^1$, and second by building their homotopy theory within the topos of sheaves over smooth schemes, building on the foundational work of Illusie, Joyal, and Jardine. Motivic homotopy theory draws inspiration from several sources: from the theory of motives, with its Tate twists and philosophy of weights; from motivic cohomology, grounded in the theory of algebraic cycles and Chow groups; and from stable homotopy theory, especially through the examples of cobordism and Morava K-theories. Over time, it has developed into a rich and coherent framework that unifies these diverse perspectives.

At the heart of this unifying approach lies the theory of motivic homotopy sheaves, whose internal structure is governed by unramified cohomology. While the latter was originally developed by Gersten, Bloch–Ogus, and others, it reappears in motivic homotopy theory as an intrinsic phenomenon, encoded in the properties of motivic homotopy sheaves with respect Gersten resolutions. These ideas are central to Voevodsky's approach to motivic complexes and play a foundational role in Morel's generalization to the full $\mathbb{A}^1$-homotopy framework over a field. Through this perspective, motivic homotopy sheaves reflect the local-to-global nature of algebraic phenomena and serve as a key organizing tool in both the unstable and stable settings. Remarkably, they are also accessible to explicit computation, a feature we will illustrate throughout these notes.

The impact of motivic homotopy theory has been both broad and deep. It played a central role in Voevodsky's proof of the Milnor and Bloch–Kato conjectures, which establish powerful bridges between motivic cohomology and étale cohomology, Galois cohomology and and Milnor K-theory. It also led to the development of the theory of motivic complexes and the six functors formalism, and to Ayoub's theory of the motivic Galois group. Morel's foundational work introduced new quadratic invariants — such as the Chow–Witt groups — which opened new perspectives on quadratic enumerative geometry lead by Levine and Wickelgren, giving for example the emerging notion of quadratic L-functions. The decomposition of stable motivic homotopy theory into $\pm$-parts has revealed previously unseen structures in algebraic geometry, particularly over real fields, where motivic realization functors extend the classical links with complex geometry into genuinely new territory connected to real algebraic geometry.

Motivic obstruction theory has also provided fresh approaches to the classification of algebraic vector bundles, through classifying spaces and characteristic classes valued in Chow–Witt groups, as developed notably by Asok and Fasel (building on an idea of Morel) in their work on Murthy's splitting conjecture. More broadly, motivic homotopy invariants often mirror those in classical homotopy theory. Over time, increasingly rich connections between motivic and classical stable homotopy theory have been uncovered, thanks in part to the structural insights made possible by the Milnor and Bloch–Kato conjectures. These links have profoundly transformed the computation of classical stable stems, notably through the motivic approach advanced in its latest stage by Isaksen, Wang, and Xu (2023), which will form the central focus of the final part of these notes. Let us finally mention that the motivic approach, via synthetic homotopy theory, also led to a proposed resolution of the last remaining case of the Kervaire invariant one problem, in recent work by Lin, Wang, and Xu (2025).

The present text is structured to support a gradual conceptual progression and to highlight the central ideas of motivic homotopy theory. The first part lays out guiding principles, beginning with complex and real realization functors, naïve $\mathbb{A}^1$-homotopies, and the choice of topology, gradually introducing model categories and ∞-categories only as needed. The second part is devoted to unstable motivic homotopy theory, using the language of sheaves and localizations more extensively, and motivating the definition of motivic homotopy sheaves. The third part presents stable homotopy theory via $\mathbb{P}^1$-stabilization, and studies motivic stable stems through the lens of Morel's degree, slice filtration and higher homotopy tools. The final part is devoted to recent developments, in particular the computations of motivic and classical stable stems based on the motivic Adams spectral sequence and the deformation of homotopy theories via the motivic class τ, drawing from the work of Isaksen, Wang, and Xu (2023).

This text is written to be accessible at multiple levels. The first two parts are aimed at readers with a basic background in algebraic topology and algebraic geometry. The latter sections require more familiarity with stable homotopy theory, and in particular the last part fully adopts the language of ∞-categories. A brief review of this formalism is included at the end of the first section. We have tried throughout the text to maintain a pedagogical style, giving all necessary definitions and providing references to the literature.

We also refer the reader to several excellent surveys on motivic homotopy theory, each offering a distinct point of view; see e.g. Antieau and Elmanto (2017), Wickelgren and Williams (2020), Asok and Østvær (2021)

Conventions

Throughout these notes, k denotes a fixed base field.

We work in the language of schemes: all schemes are assumed to be of finite type over k. By a (algebraic) variety over k, we mean a quasi-projective scheme, that is, a scheme locally defined by polynomial equations in an affine or projective space. Readers unfamiliar with the language of schemes may safely replace k-schemes with algebraic varieties over k; the general theory remains unaffected.

Unless stated otherwise, the term "monoidal" means "symmetric monoidal". Units of monoidal categories are typically denoted by $\mathbb{1}$.

A "space" always refers to a simplicial set. A "presheaf" (resp. "sheaf") means, without explicitly stated otherwise, a presheaf (resp. Nisnevich sheaf) over the category Sm_k of smooth schemes over k. A "k-space" is a presheaf of simplicial sets on Sm_k.

In the last section, all formal group laws are assumed to be commutative.

Concerning foundational choices: the unstable part of the theory is formulated, as far as possible, using a combination of model categories and ∞-categories, in order to help the reader navigate the existing literature. The stable part, however, relies more systematically on the ∞-categorical formalism.

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1. A few guiding principles

1.1. Complex and real homotopy types of algebraic varieties

The main motivation behind motivic homotopy theory is to extend the invariants of classical homotopy theory to algebraic varieties. As envisioned by Lefschetz one century ago, the main guide to do so is to use the topological space underlying complex algebraic varieties. In fact, one of the appealing features of motivic homotopy is that it also naturally incorporate the homotopy of real algebraic varieties.

Let us consider a real or complex embedding $\sigma\colon k \to E = \mathbb{R}, \mathbb{C}$ of our base field k, and X be an algebraic k-variety. When σ is a complex embedding, we let $X^\sigma(\mathbb{C}) = \mathrm{Hom}_k(\mathrm{Spec}(\mathbb{C}), X)$ be the set $\mathbb{C}$-points of X, where $\mathrm{Spec}\,\mathbb{C}$ is viewed as a k-scheme via σ, endowed with its natural analytic topology — coming from its canonical structure of complex analytic variety: see Grothendieck, 2003, XII, Th. 1.1. Similarly, when σ is a real embedding, we let $X^\sigma(\mathbb{R}) = \mathrm{Hom}_k(\mathrm{Spec}(\mathbb{R}), X)$ endowed with canonical euclidean topology (similarly coming from its structure of real analytic variety). In any case, one can define the *σ-Betti homotopy type* of the k-scheme X, that is the isomorphism class of the topological space underlying $X^\sigma(E)$ in the homotopy category $\mathrm{H}^{\mathrm{top}}$. It is instructive to determine this purely topological invariant of k-schemes in a few cases, left as exercises to the reader.

k-schemes	real case	complex case
$\mathbb{A}^n$	$*$	$*$
$\mathbb{P}^1$	S^1	S^2
$\mathbb{G}_m$	S^0	S^1
$\mathbb{A}^n - \{0\}$	S^{n-1}	S^{2n-1}
$x^2 = y^3$ (cuspidal curve)	$*$	$*$
$y^2 = x^2(x+1)$ (nodal curve)	S^1	S^1
$Q_{2n-1} := V_{\mathbb{A}^{2n}}(\sum_{i=1}^n x_i y_i = 1)$	S^{n-1}	S^{2n-1}
$Q_{2n} := V_{\mathbb{A}^{2n+1}}(\sum_{i=1}^n x_i y_i = z(1+z))$	S^n	S^{2n}

σ-Betti homotopy type

Therefore, with the aim to define a motivic homotopy type which admits both a real and complex realization, this table tells us two things:

1. the affine line should be contractible;
2. there are several algebraic models whose motivic homotopy type should look like a sphere.

1.2. Naive $\mathbb{A}^1$-homotopies.

As suggested by the considerations of the previous subsection, Voevodsky's fundamental idea emerges: to use the affine line $\mathbb{A}^1_k$ to parameterize motivic homotopies. This echoes the notion of rational equivalence on algebraic cycles, already considered by Lefschetz, except one uses the projective line in the case of cycles. One introduces the following definition:[(1)]

Definition 1.1. — *One defines the* naive $\mathbb{A}^1$-homotopy equivalence relation *on morphisms of k-schemes as the transitive relation generated by the following symmetric and reflexive relation between two morphisms $f, g\colon Y \to X$ of algebraic k-varieties:*

$$\exists H\colon \mathbb{A}^1 \times Y \to X \mid H \circ s_0 = f, H \circ s_1 = g$$

where s_0 and s_1 are respectively the zero and unit sections of $\mathbb{A}^1_k$.

A k-scheme X is naively $\mathbb{A}^1$-contractible *if it admits a rational point x and the identity map Id_X is naively $\mathbb{A}^1$-equivalent to the composition $X \to \mathrm{Spec}(k) \xrightarrow{x} X$.*

In particular, both the algebraic affine k-variety $\mathbb{A}^n_k$ and the cuspidal curve $V: x^2 = y^3$ are naively $\mathbb{A}^1$-contractible.

This equivalence relation is compatible with composition. Therefore one can define the *naive motivic homotopy category* as the category whose objects are smooth k-schemes and morphisms are given by naive $\mathbb{A}^1$-homotopy equivalence classes simply denoted by $[X, Y]^N$. As in topology, we will see that this definition is not strong enough to define a suitable motivic homotopy theory (see in particular 2.36). But remarkably, it is already possible to make computations with this definition. Let us first introduce the pointed version.

Definition 1.2. — *A base point of a k-scheme is a rational point $x \in X(k)$. One defines* naive pointed $\mathbb{A}^1$-homotopy equivalence relation *between two pointed morphisms $f, g\colon (Y, y) \to (X, x)$ as above, but requiring that the homotopies $H(t, -)$ are all pointed maps.*

We let $[(Y, y), (X, x)]^N_\bullet$ be the naive $\mathbb{A}^1$-homotopy classes of pointed maps modulo this equivalence relation.

1.2.1. *Witt monoid.* — One of the striking aspects of motivic homotopy theory, as discovered by Morel, is its surprising connection to the theory of quadratic forms — more accurately, inner products, to accommodate characteristic 2. Let us recall some basic definitions of this rich and fascinating subject — the reference book Milnor and Husemoller (1973) will be well-suited to our point of view.

The isomorphism classes of finite-dimensional non-degenerate symmetric bilinear forms over k form a monoid under orthogonal sum which we denote by $(\mathrm{MW}(k), +)$ and refer to as the *Witt monoid*. Note moreover that the tensor product induces a semi-ring structure on $\mathrm{MW}(k)$, denoted simply by $(\mathrm{MW}(k), +, \times)$. Let $Q(k) := k^\times/(k^\times)^2$ be the *quadratic classes* of units of k, equipped with its group structure. Then one obtains a canonical morphism of (multiplicative) monoids: $(Q(k), \times) \to (\mathrm{MW}(k), \times)$ which to a

[(1)] In Morel and Voevodsky (1999, p. 88-89), it was called a *strict* $\mathbb{A}^1$-homotopy equivalence. The following terminology seems to be the more commonly used now.

unit u associates the symmetric bilinear form $u.xy$, whose isomorphism class is denoted by $\langle u\rangle$. Standard notations in this context are:

- $\langle a_1,\dots,a_n\rangle = \langle a_1\rangle + \cdots + \langle a_1\rangle$, for the obvious diagonal symmetric bilinear form;
- $h = \langle 1,-1\rangle$, the *hyperbolic form*.

One then defines two associated rings:

- the *Witt ring* $\mathrm{W}(k)$ of k which is the quotient of $(\mathrm{MW}(k),+,\times)$ with respect to the classical Witt equivalence relation;
- the *Grothendieck–Witt ring* $\mathrm{GW}(k)$ of k which is the group completion of the additive monoid $(\mathrm{MW}(k),+)$ equipped with the induced ring structure.

Note that one deduces from these definitions that $\mathrm{W}(k) = \mathrm{GW}(k)/(h)$.

One has two universal invariants associated with an element of $\mathrm{MW}(k)$, say given by the isomorphism class of (V,φ): the *rank* $\dim_k(V)$, and the *discriminant* $\mathrm{disc}(V,\varphi)\in Q(k)$.

Note also that in characteristic not 2, the monoid $(\mathrm{MW}(k),+)$ is cancellative, leading to a monomorphism: $\mathrm{MW}(k)\to\mathrm{GW}(k)$. In characteristic 2, we let $\mathrm{MW}^s(k)$ be the universal cancellative monoid associated with $\mathrm{MW}(k)$, so that $\mathrm{MW}^s(k)\to\mathrm{GW}(k)$ is the universal monomorphism.

1.2.2. *Cazanave's theorem.* — Let us now consider our motivic sphere $\mathbb{P}^1_k$, pointed by ∞ according to Definition 1.2. A pointed endomorphism of $\mathbb{P}^1_k$ is represented by a rational function $f=\frac{A}{B}$ where $A,B\in k[t]$ are coprime monic polynomials such that $n=\deg(A)>\deg(B)$. In this situation, one classically associates to (A,B) the Bezout matrix $\mathrm{Bez}(A,B)$, which is a symmetric bilinear form whose determinant is given by $(-1)^{n(n+1)/2}.\mathrm{res}(A,B)$. We then define a canonical application:

$$\mathrm{End}_\bullet(\mathbb{P}^1_k)\to \mathrm{MW}^s(k)\times_{Q(k)} k^\times f\mapsto \Big(\mathrm{Bez}(A,B),(-1)^{n(n+1)/2}.\mathrm{res}(A,B)\Big) \tag{1.2.a}$$

The following theorem was obtained in Cazanave (2012, Cor. 3.10, Th. 4.6).

Theorem 1.3. — *Consider the above notation. Then the above map induces a bijection of pointed sets:*

$$[\mathbb{P}^1_k,\mathbb{P}^1_k]^N_\bullet \to \mathrm{MW}^s(k)\times_{Q(k)} k^\times$$

and in fact an isomorphism of semi-ring, for suitably defined addition on the left hand-side, multiplication being composition.

This result will allow us to measure the difference between naive $\mathbb{A}^1$-homotopies and weak $\mathbb{A}^1$-homotopies; see Example 2.36.

1.2.3. *Presentations.* — The additive group of the Witt ring $(\mathrm{W}(k),+)$ has a well-known presentation, due to Witt, as the abelian group generated by $\langle\overline{u}\rangle$ for a quadratic class $\overline{u}\in Q(k)$ modulo the relations:

$$h=0,\quad \langle\overline{u},\overline{v}\rangle=\langle\overline{u}+\overline{v},(\overline{u}+\overline{v})\overline{uv}\rangle,\overline{u},\overline{v}\in Q(k),u+v\neq 0 \tag{1.3.a}$$

One also gets the presentation of $(\mathrm{GW}(k),+)$ as the abelian group genereated by $\langle\overline{u}\rangle$ for a quadratic class $\overline{u}\in Q(k)$ modulo the single relations

$$\langle\overline{u},\overline{v}\rangle=\langle\overline{u}+\overline{v},(\overline{u}+\overline{v})\overline{uv}\rangle,\overline{u},\overline{v}\in Q(k),u+v\neq 0 \tag{1.3.b}$$

One deduces from the above theorem the following very explicit computation of naive $\mathbb{A}^1$-homotopy classes.

Corollary 1.4. — *The pointed set $[\mathbb{P}^1_k,\mathbb{P}^1_k]^N_\bullet$ has a structure of abelian groups which is generated by the symbols (u) for the class of a unit $\overline{u}\in k^\times/(\pm 1)$, corresponding to the endomorphism $(x:y)\mapsto(ux:y)$,*[2] *modulo the unique relation:*

$$(\overline{u},\overline{v})=(\overline{u}+\overline{v},(\overline{u}+\overline{v})\overline{uv}),u+v\neq 0$$

where $(\overline{u},\overline{v})=(\overline{u})+(\overline{v})$.

1.3. The choice of topology

1.3.1. — The category of algebraic varieties is too coarse to allow for homotopy theoretic constructions such as path spaces, mapping cylinders, etc. As stated in the introduction the solution is to embed k-schemes into an appropriate *topos*, that is using sheaves for an appropriate Grothendieck topology.

For motivic homotopy theory, the Zariski topology is not strong enough. Indeed, smooth k-varieties look like $\mathbb{A}^n_k$ only étale locally. On the other hand, the étale topology is too strong: algebraic K-theory, as well as motivic cohomology, do not satisfy étale descent.[3] The *Nisnevich topology* is a topology intermediate between the Zariski and étale topologies that captures some good features of both.[4] whose covers are given by the étale surjective families $(p_i:X_i\to X)_{i\in I}$ such that for any $x\in X$, there exists $i\in I$, and $x_i\in X_i$ such that $p_i(x_i)=x$ and the induced extension of residue fields $\kappa(x_i)/\kappa(x)$ is trivial.

Here are the main advantages of the Nisnevich topology:

1. the cohomology of a point (spectrum of a field) is trivial;
2. algebraic K-theory does satisfy Nisnevich descent (Thomason and Trobaugh, 1990);

(2) with the convention that $(0:1)=\infty$

(3) For motivic cohomology, the failure of étale descent is quantified by the Beilinson-Lichtenbaum conjecture, now a theorem of Voevodsky. See Riou (2014, Conj. 1.16), and also Section 4.2.

(4) It was introduced by Nisnevich in Nisnevich (1989).

3. a closed immersion $Z \subset X$ of smooth k-schemes looks Nisnevich-locally like the 0-section of an affine space.[(5)]

In particular, the Grothendieck site chosen for motivic homotopy theory is the category of smooth k-schemes Sm_k, equipped with the Nisnevich topology.[(6)] Below we recall for the comfort of the reader a few important aspects of this particular Grothendieck topology (see Morel and Voevodsky, 1999, §3.1).

1.3.2. *Excision property.* — A (Nisnevich/elementary) *distinguished square* is a cartesian square of smooth k-schemes

$$\begin{array}{ccc} W & \to & V \\ \downarrow & \Delta & \downarrow p \\ U & \underset{j}{\to} & X \end{array}$$

such that p is étale, $Z = X - U$ is seen as a reduced closed subscheme of X, and the induced morphism $T = p^{-1}(Z) \to Z$ is an isomorphism.

In fact, the family of morphisms of the form (p, j) attached to a distinguished square as above generates the Nisnevich topology. In this situation, one deduces by taking quotient a morphism of pointed sheaves,[(7)]

$$X/U = X/(X - Z) \to V/(V - T) = V/W.$$

which can be seen to be an isomorphism. This is the so-called *excision property.*

Example 1.5. — 1. The obvious inclusion induces an isomorphism $\mathbb{A}^1/\mathbb{G}_m \to \mathbb{P}^1/\mathbb{A}^1$ of pointed sheaves.

2. Let $Z \to X$ be a closed immersion of smooth k-schemes. Assume there exists an étale morphism $p\colon X \to \mathbb{A}^n_k$ such that $Z = p^{-1}(\mathbb{A}^c_k)$. Then there exist an open subscheme $\Omega \subset (X \times_{\mathbb{A}^n_k} \mathbb{A}^c_Z)$ containing Z as a closed subscheme and such that the obvious projection map induces isomorphism of pointed sheaves:

$$X/(X - Z) \xleftarrow{\sim} \Omega/(\Omega - Z) \xrightarrow{\sim} \mathbb{A}^c_Z/(\mathbb{A}^c_Z - Z).$$

This property gives another concrete interpretation of point 3. in Section 1.3.1.

1.3.3. *Points.* — A point of the Nisnevich site Sm_k is given by a pair (X, x) where X is a smooth k-scheme and $x \in X$ an element of the underlying set. The *fiber* of a presheaf/sheaf F on Sm_k at the point (X, x) is defined as

$$F_{X,x} = \varinjlim_{V/X} F(V)$$

(5) i.e., for all $x \in Z$, one gets a (non-canonical) isomorphism $X^h_{(x)} \simeq (\mathbb{A}^c_{\kappa(x)})^h_{(0)}$, where $X^h_{(x)}$ is the Nisnevich localization of X at x, in other words, the spectrum of the henselization of the local ring of X at x.

(6) The restriction to *smooth* k-schemes is primarily justified by point (3).

(7) Indeed, this type of quotients always gives canonically pointed objects by the map $* = U/U \to X/U$ of sheaves of sets.

where V ranges over the Nisnevich neighborhoods of x in X.[(8)] This defines a *fiber functor* $F \mapsto F_{X,x}$, i.e., commutes with colimits and finite limits. The family of these fiber functors is *conservative* on the category of sheaves of sets on Sm_k: it preserves and detects isomorphisms (and monomorphisms).

Let E/k be a finitely generated separable extension of k. Note that E is the filtering colimit of its sub-k-algebra $A \subset E$ with A/k smooth (of finite type). Given any (pre)sheaf F over Sm_k, we put:

$$F(E) := \varinjlim_{A \subset E} F(\mathrm{Spec}(A)).$$

Obviously, given any smooth connected k-scheme X with generic point η, and an isomorphism $E \simeq \kappa(X) = \mathcal{O}^h_{X,\eta}$, one gets a canonical isomorphism $F(E) \simeq F_{X,\eta}$, so that $F \mapsto F(E)$ is a fiber functor of the Nisnevich site on Sm_k.

1.4. Tools from higher homotopy theory

This section can be avoided on a first reading. Motivic homotopy relies on two theoretical tools in order to study localized categories — an operation introduced in Gabriel and Zisman (1967) — along a set of morphisms: model categories and ∞-categories. Nowadays, model categories are often viewed as presentations of ∞-categories. The ∞-categorical perspective is therefore more synthetic and elegant. However, since most of the current literature in motivic homotopy theory still uses model categories, we have tried to provide the reader with sufficient tools to navigate between the two points of view.

1.4.1. *Model categories.* — Model categories were introduced in Quillen (1967), as a generalization of the homological algebra of Cartan and Eilenberg, and as a synthesis between the homotopy theories of topological spaces and simplicial sets (after Daniel Kan). They became the central tool for all generalizations of homotopy theory in other context than topology. This is particularly the case of motivic homotopy, which was developed using a particular model category on simplicial Nisnevich sheaves.

Model categories are designed to study categories obtained by formally inverting a collection $\mathscr{W}$ of morphisms, generically called *weak equivalences*, in a category whose objects are considered as *models*. The main idea is to build two kinds of resolutions, called *fibrant* (analogue of injective) and *cofibrant* (analogue of projective), out of the so-called small object argument[(9)] The central result is that morphisms in the localized category from a cofibrant object to a fibrant object can be computed by classes of morphisms in the original category up to an explicit equivalence relation. To achieve this, one requires the existence of two collections $\mathcal{F}ib$ and $\mathcal{C}of$ of morphisms called *fibrations* and *cofibrations* respectively. They are bound to satisfy a clear and simple axiomatic

(8) i.e., the étale X-scheme with a point $v \in V$ over x such that the residual extension $\kappa(v)/\kappa(x)$ is trivial.

(9) which actually originally appeared in Cartan and Eilenberg (1956) to prove the existence of injective resolutions.

that draw inspiration from both the properties of injective and projective objects in homological algebra, and from the *homotopy lifting property* that characterizes fibrations in topology.[10] An important tool of model categories is that, under appropriate assumptions, they can be localized, by adding more weak equivalences. We will review this procedure below.

1.4.2. *∞-categories.* — After developing his theory, Quillen had the major intuition that model categories concealed a deeper structure:

> [p. 0.4 of Quillen, 1967] Presumably there is a higher order structure ([8], [17]) on the homotopy category which forms the part of the homotopy theory of a model category, but we have not been able to find an inclusive general definition of this structure with the propery that this structure is preserved when there are adjoint functors which establish an equivalence of homotopy theories.

The said *higher order structure* took many years to be uncovered. It is the theory of **∞-categories**, due to many mathematicians and which was presented at the Bourbaki seminar by Cisinski (2016). As in *op. cit.*, we take Lurie (2009) as a reference — therefore, we use Joyal's theory of *quasicategories* as a model for ∞-categories. Here are the important points to keep in mind:

– Any category admits an associated ∞-category via the nerve functor (see §2 in Cisinski, 2016).
– Many (if not all) classical constructions and concepts from category theory extends to ∞-categories: adjoint functors, equivalences, Kan extensions, limits/colimits, pro/ind-objects, (co)fibred category, etc.
– Every ∞-category $\mathscr{C}$ admits a mapping space functor denoted by $\mathrm{Map}_{\mathscr{C}}(X, Y)$ for objects X and Y. In fact, one can view any ∞-category as a simplicial category; see §12, and Theorem 12.4, in Cisinski (2016).

1.4.3. *Localizations.* — In both model categories and ∞-categories, the fundamental operation is that of localization, the procedure of adding invertible morphisms.

Under a suitable assumption, there is a context, seemingly first formulated by Bousfield, in which this localization procedure is more structured and better suited for computations. This corresponds to the property of being *combinatorial* for model categories, and *presentable* for ∞-categories.

Let us describe this particular type of localization for ∞-categories, called *left localization* in Cisinski (2016). Let $\mathscr{C}$ be a presentable ∞-category and W be a set of morphisms. One introduces the following definitions:

– an object X of $\mathscr{C}$ is *W-local* if for any $f_0 \in W$, the map $\mathrm{Map}_{\mathscr{C}}(f_0, X)$ is a weak equivalence;

[10] In fact, according to the usual abuse of terminology, we call model category what Quillen defined as a *closed model category* in Quillen (1967, I.5 Definition 1).

– a morphism f on $\mathscr{C}$ is a *W-local weak equivalences* (or simply *weak W-equivalence*) if for any W-local object X, the map $\mathrm{Map}_{\mathscr{C}}(f, X)$ is a weak equivalence.

Letting $\overline{W}$ be the set of weak W-equivalences, there exists a functor $\pi\colon \mathscr{C} \to \mathscr{C}[\overline{W}^{-1}]$ of ∞-categories[(11)] which admits a right adjoint ν which is *fully faithful* and with essential image the W-local objects. We put $L_W = \nu \circ \pi$ and call it the *W-localization functor*.

Example 1.6. — The main example for us comes from *∞-topoi* which are left localizations of the presentable ∞-category of presheaves on some Grothendieck site (see Cisinski, 2016): we will work with the **∞-topos of Nisnevich sheaves on the site Sm_k of smooth k-schemes**.

1.4.4. *Monoidal structures.* — Among the many advantages of ∞-categories, it is possible to transport all the definitions and constructions from the theory of monoidal categories. As ∞-categories must always encode all the higher structures in a precise way, the theory has its roots in the point of view of operads. We refer the reader to the excellent account of Groth (2020, §3, 4) for the definitions of symmetric monoidal ∞-category, and commutative monoid objects in them.

2. Unstable motivic homotopy

2.1. Homotopy theory of Nisnevich sheaves

2.1.1. *k-Spaces.* — Our basic "homotopical objects" will be the simplicial presheaves on the category Sm_k:

$$\mathcal{X}\colon (\mathrm{Sm}_k)^{op} \to \Delta^{\mathrm{op}}\mathrm{Set},$$

simply called *k-spaces*. The corresponding category, with morphisms the natural transformations, is denoted by $\mathrm{PSh}(k, \Delta^{\mathrm{op}}\mathrm{Set})$. Note that it contains as a full subcategory the category of Nisnevich simplicial sheaves $\mathrm{Sh}(k, \Delta^{\mathrm{op}}\mathrm{Set})$. Both categories will serve as *models* for motivic homotopy types.

Example 2.1. —
1. Let X be an arbitrary k-schemes. Then the (pre)sheaf it represents $X(-) = \mathrm{Hom}(-, X)$ can be seen as a discrete simplicial sheaf, and therefore as a (discrete) k-space. When X is a smooth k-scheme, we will abusively denote it by X.[(12)]
2. Given an arbitrary simplicial set $E_\bullet$, one can consider the constant simplicial presheaf $U/S \mapsto E_\bullet$ and view it as a k-space. We will abusively denote it by $E_\bullet$.

[(11)] satisfying the usual universal property, and in particular unique in the ∞-categorical sense;

[(12)] This is harmless because of the Yoneda lemma. Beware however that it is different for singular schemes: as an example, if X is a non reduced k-scheme, with reduction X_{red}, the nil-immersion $\nu\colon X_{red} \to X$ induces an isomorphism of k-spaces $\nu_*\colon X_{red}(-) \to X(-)$.

2.1.2. *Pointed k-spaces.* — According to the notation of the previous example, the point $*$ seen as a k-space coincide with $\mathrm{Spec}(k)$. It is the final object of the category of k-spaces. As in topology, a base point of $\mathcal{X}$ is a map $x\colon * \to \mathcal{X}$, and we say that $(\mathcal{X}, x)$ is a pointed space. Given a k-space $\mathcal{X}$, we also denote by $\mathcal{X}_+$ the pointed k-space with a free base point added. As in topology, one defines a natural symmetric monoidal structure on pointed k-spaces whose product is the *smash product*:

$$\mathcal{X} \wedge \mathcal{Y} = \mathcal{X} \times \mathcal{Y}/[(* \times \mathcal{Y}) \cup (\mathcal{X} \times *)].$$

Similarly, the coproduct in pointed k-spaces is denoted by $\vee$, and called the *wedge product*.

Example 2.2. — (See Joyal and Tierney, 1993, §4, Morel and Voevodsky, 1999, §4.1, p. 123, 128) Let G be a smooth group scheme over k. We define the pointed k-space BG as the presheaf:

$$U \mapsto B\big(G(U)\big)$$

where $G(U)$ is seen as a group and $B(-)$ denotes the usual simplicial classifying space construction. The base point of BG is given by the identity element $e \in G(k) = BG_0(k)$. Note that BG is clearly a simplicial sheaf.

In fact, this construction of classifying spaces in a topos appears for the first time in Joyal and Tierney (1993, §4), and makes sense for any sheaf of groups — actually any sheaf of groupoids in *loc. cit.* This pioneering work of Joyal and Tierney is part of the long story for the quest for ∞-stacks.

2.1.3. *Weak equivalences.* — As explained in the introduction, one can do homotopy theory using simplicial (pre)sheaves. This was originally introduced by Illusie (1971). The reader can find a thorough account in Jardine (2015). Here is the particular case of this theory that is relevant to us.

Definition 2.3. — *A morphism of k-spaces $\mathcal{Y} \to \mathcal{X}$ is a (Nisnevich-)local weak equivalence*[13] *if for any Nisnevich point (X, x) on Sm_k, the induced morphisms $\mathcal{Y}_{X,x} \to \mathcal{X}_{X,x}$ on the fiber at (X, x) is a weak equivalence of simplicial sets.*

Remark 2.4. — A *global* weak equivalence is a morphism of k-spaces $\mathcal{Y} \to \mathcal{X}$ such that for all smooth k-schemes U, the induced morphism $\mathcal{Y}(U) \to \mathcal{X}(U)$ on global sections over U is a weak equivalence. This terminology is due to Jardine.

[13] Illusie says *quasi-isomorphism* in *op. cit.* This terminology has been replaced by the present one in the second period of study of simplicial sheaves, after Heller, Joyal and Jardine.

2.1.4. *Homotopy sheaves.* — Let $\mathcal{X}$ be a k-space (resp. $(\mathcal{X}, x)$ be a pointed k-space). One defines its 0-th (resp. n-th) homotopy sheaf as the (Nisnevich) sheaf associated with the presheaf of pointed sets (resp. groups if $n = 1$, abelian groups if $n > 1$)

$$U \mapsto \pi_0\big(\mathcal{X}(U)\big), \text{ resp. } \pi_n\big(\mathcal{X}(U), x_U\big).$$

We say that a map of pointed k-spaces $f\colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$ is a local weak equivalence if it is so after forgetting the base points. Then it induces an isomorphisms on homotopy sheaves for any $n \geq 0$

$$f_*\colon \underline{\pi}_n(\mathcal{Y}, y) \to \underline{\pi}_n(\mathcal{X}, x). \tag{2.4.a}$$

Remark 2.5. — One should be careful, however, that the latter condition is not enough to guarantee that f is a Nisnevich weak equivalence in the sense of Definition 2.3. In fact, one must consider all possible choices of base points; see Illusie (1971, §2.2.1).

Here is a particular case where this issue does not arise. We will say that a k-space $\mathcal{X}$ is *locally connected* if $\underline{\pi}_0(\mathcal{X})$ is the constant sheaf. Then a pointed morphism $f\colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$ between locally connected k-spaces is a weak equivalence if and only if the maps (2.4.a) are isomorphisms for all $n > 0$.

Example 2.6. — Let G be a smooth group scheme over k. Then, as expected, one deduces from its construction that BG is locally connected. Moreover, one gets:

$$\pi_n(BG, e) = \begin{cases} G & n = 1, \\ * & n \neq 1. \end{cases}$$

In other words, the k-space BG is a "$K(G, 1)$".

2.1.5. *Associated homotopy category.* — As in Illusie (1971, Def. 2.3.5), one introduces[(14)]:

Definition 2.7. — *We define the* Nisnevich homotopy category $\mathrm{H}^{\mathrm{Nis}}(k)$ *(resp.* Nisnevich pointed homotopy category $\mathrm{H}^{\mathrm{Nis}}_*(k)$*) as the localization of the category of k-spaces (resp. pointed k-spaces) with respect to the local weak equivalences.*

Morphisms between two k-spaces $\mathcal{X}$ and $\mathcal{Y}$ in $\mathrm{H}^{\mathrm{Nis}}(k)$ are simply called weak homotopy classes and denoted by

$$[\mathcal{X}, \mathcal{Y}]^{\mathrm{Nis}} := \mathrm{Hom}_{\mathrm{H}^{\mathrm{Nis}}_*(k)}(\mathcal{X}, \mathcal{Y}).$$

We similarly denote by $[(\mathcal{X}, x), (\mathcal{Y}, y)]^{\mathrm{Nis}}_\bullet$ in the pointed case.

Example 2.8. — 1. Let X and Y be two smooth k-schemes. Then the canonical map $\mathrm{Hom}_k(X, Y) \to [X, Y]^{\mathrm{Nis}}$ is a bijection, reflecting the fact that X and Y are discrete k-spaces. In particular, local weak equivalences are insensitive to the geometry of the underlying smooth k-schemes. For $n > 0$, we further get:

$$[S^n \wedge X_+, Y_+]^{\mathrm{Nis}} = *.$$

(14) see also *op. cit.* Th. 2.3.6 for an equivalent construction using Gabriel-Zisman calculus of fractions

2. Let G be a smooth group scheme over k. Then for any smooth k-scheme X and any integer $n \geq 0$, one gets using the notation from 2.2:

$$[S^n \wedge X_+, BG]^{\mathrm{Nis}} = \begin{cases} H^1_{\mathrm{Nis}}(X, G) & n = 0, \\ G(X) & n = 1, \\ 0 & n > 1. \end{cases}$$

On the first line, the right-hand side denotes the set of Nisnevich-local G-torsors on X. The second point can be reformulated by saying that $\Omega BG = G$, where Ω is the loop space functor extended to k-spaces. We refer the reader to Morel and Voevodsky (1999, p. 120) for more details.

Remark 2.9. — In the homotopy category $\mathrm{H}^{\mathrm{Nis}}(k)$, one can always identify a simplicial presheaf $\mathcal{X}$ with its associated Nisnevich sheaf $a(\mathcal{X})$. Indeed, the canonical map

$$\mathcal{X} \to a(\mathcal{X})$$

is readily seen to be a local weak equivalence. This justifies our choice to refer to a k-space as a simplicial presheaf, although the reader is free to work simplicial (Nisnevich) sheaves instead — a choice that was in fact adopted in Morel and Voevodsky (1999).

2.1.6. *Computational tools.* — As mentioned in Section 1.4, one can enhance the homotopy category $\mathrm{H}^{\mathrm{Nis}}(k)$ with several model category structures,[15] and a canonical ∞-categorical structure simply denoted here by $\mathscr{H}^{\mathrm{Nis}}(k)$, encompassing in the general notion of ∞-topos; see 2.1.7 below. One of the advantages of these structures is to allow the definition of a (Nisnevich-local) mapping space between k-spaces (resp. pointed k-spaces) $\mathcal{X}$ and $\mathcal{Y}$:

$$\mathrm{Map}^{\mathrm{Nis}}(\mathcal{X}, \mathcal{Y}) \text{ resp. } \mathrm{Map}^{\mathrm{Nis}}_{\bullet}(\mathcal{X}, \mathcal{Y}).$$

It has the important property that:

$$\pi_0 \mathrm{Map}^{\mathrm{Nis}}(\mathcal{X}, \mathcal{Y}) = [\mathcal{X}, \mathcal{Y}]^{\mathrm{Nis}} \text{ resp. } \pi_i \mathrm{Map}^{\mathrm{Nis}}_{\bullet}(\mathcal{X}, \mathcal{Y}) = [S^i \wedge \mathcal{X}, \mathcal{Y}]^{\mathrm{Nis}}_{\bullet}.$$

Remark 2.10. — 1. From the model category perspective, the mapping space is obtained by deriving an appropriate basic mapping space functor.[16] For example,

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[15] One can retain two such model structures:

- The *Joyal model structure*, equivalently *injective Nisnevich-local model structure* used by Morel and Voevodsky: the base category is that of k-spaces that are Nisnevich sheaves, cofibrations are monomorphisms, weak equivalence are local weak equivalences, and fibrations are the maps with the RLP with respect to the acyclic cofibrations.
- The *Blander model structure*, equivalently *Nisnevich-localized projective model structure* first considered in Blander, 2001 in this context: the base category is that of k-spaces, cofibrations are the maps with the LLP with respect to epimorphisms that are term-wise weak-equivalences, fibrations are the maps with the LLP with respect to the acyclic cofibrations.

The advantage of the second model structure is that all representable k-spaces are cofibrant and a k-space is fibrant if and only if it is term-wise a Kan complex and is Nisnevich excisive in the sense of Definition 2.11.

[16] One says that the model category is *simplicial*, see Quillen, 1967, II.2, Definition 2.

with Morel and Voevodsky's notation, one puts for any k-spaces $\mathcal{X}$ and $\mathcal{Y}$

$$\mathrm{Map}^{\mathrm{Nis}}(\mathcal{X}, \mathcal{Y}) = S(\mathcal{X}_c, \mathcal{Y}_f)$$

where $\mathcal{X}_c$ (resp. $\mathcal{Y}_f$) is a cofibrant (resp. fibrant) resolution of the sheaf associated with $\mathcal{X}$ (resp. $\mathcal{Y}$) for the Nisnevich local injective model structure that they use.

2. From the ∞-categorical perspective, the mapping space comes readily out of the ∞-categorical structure (as explained in Section 1.4.2). Moreover, we can view the ∞-category $\mathscr{H}^{\mathrm{Nis}}(k)$ as a simplicial category (as any ∞-category, see Cisinski, 2016, §12, Th. 12.4).

Definition 2.11. — *Let $\mathcal{X}$ be a k-space. After Morel and Voevodsky, we say that $\mathcal{X}$ is* Nisnevich excisive[(17)] *if $\mathcal{X}(\varnothing)$ is contractible*[(18)] *and for any distinguished square Δ as in Section 1.3.2, the resulting square of simplicial sets*

$$\begin{array}{ccc} \mathcal{X}(X) & \xrightarrow{p^*} & \mathcal{X}(V) \\ {\scriptstyle j^*}\downarrow & & \downarrow \\ \mathcal{X}(U) & \to & \mathcal{X}(W) \end{array}$$

is homotopy cartesian (i.e., cartesian in the ∞-category $\mathscr{H}$).

This is a kind of "homotopical excision property".[(19)] It was first considered by Brown and Gersten, but for the Zariski topology (i.e., $V \to X$ is an open immersion). In practice, it is mostly used by applying the following proposition.

Proposition 2.12. — *Let $\mathcal{X}$ be a k-space which is Nisnevich excisive. Then for any smooth k-scheme X, the canonical map*

$$\mathcal{X}(X) \to \mathrm{Map}^{\mathrm{Nis}}(X, \mathcal{X})$$

is a weak equivalence. In particular,

$$[X, \mathcal{X}]^{\mathrm{Nis}} = \pi_0\big(\mathcal{X}(X)\big)$$

and when $\mathcal{X}$ is pointed,

$$[S^i \wedge X_+, \mathcal{X}]^{\mathrm{Nis}}_{\bullet} = \pi_i\big(\mathcal{X}(X)\big).$$

Proof. — This can be derived from the Blander model structure on k-spaces mentioned in footnote 15 (use Blander, 2001, Theorem 1.6, Lemmas 1.8 and 4.1). For an ∞-categorical proof, we refer the reader to Lurie (2011, Th. 2.9). □

(17) Morel and Voevodsky originally used the term "B.G.-property", which appears to have been replaced in later literature by the terminology adopted here;
(18) This will be automatic if $\mathcal{X}$ is a Nisnevich sheaf.
(19) For example, one can interpret it by saying that the induced map on the homotopy fibers of the vertical map with respect to any choice of base point in $\mathcal{X}(W)$ is a weak equivalence. Indeed, these homotopy fibers are respectively the derived sections of $\mathcal{X}$ at X with support in $(X - U)$ and at V with support in $(V - W)$.

Example 2.13. — 1. Let X be an eventually singular k-scheme. The associated k-space $X(-)$ is obviously Nisnevich excisive, as it is a Nisnevich sheaf.

2. Given a k-space $\mathcal{X}$, the Godement resolution provides a weak equivalence $\mathcal{X} \to G(\mathcal{X})$, where the target is a Nisnevich excisive k-space (see Jardine, 2015, Prop. 3.3 or Morel and Voevodsky, 1999, §1, 1.66). We call G the *Godement simplicial resolution functor*.

2.1.7. *The ∞-categorical description.* — We can restate the previous constructions in light of the tools from higher category theory. The category $\mathrm{H}^{\mathrm{Nis}}(k)$ is the homotopy category associated with the ∞-topos $\mathscr{H}^{\mathrm{Nis}}(k)$ of Nisnevich sheaves over Sm_k.

Let us recall the construction of the latter. We start from the ∞-category of presheaves on Sm_k, $\mathscr{P}\mathscr{S}\mathrm{h}(\mathrm{Sm}_k) = \mathscr{F}\mathrm{un}((\mathrm{Sm}_k)^{\mathrm{op}}, \mathscr{S})$ (see Cisinski, 2016, §14). The objects of this category are genuinely k-spaces in the sense introduced earlier; the only difference from the ordinary category $\mathrm{PSh}(k)$ is the presence of higher morphisms.[20] The ∞-category $\mathscr{H}^{\mathrm{Nis}}(k)$ is obtained by left localization with respect to local weak equivalences as defined above. In particular, we have a notion of local objects — these are simply called (∞-)sheaves — and a localization endofunctor L_{Nis} of $\mathscr{P}\mathscr{S}\mathrm{h}(k)$. In fact, a k-space $\mathcal{X}$ is local if and only if it is Nisnevich excisive. An explicit model of the localization functor L_{Nis} is given by the Godement simplicial functor described above.

Note finally that we can formulate the excision property in $\mathscr{H}^{\mathrm{Nis}}(k)$ by saying that any distinguished square Δ induces a homotopy cartesian[21] square in $\mathscr{H}^{\mathrm{Nis}}(k)$. This amounts to say that the canonical map

$$V//W \to X//U$$

is a local weak equivalence, i.e., an isomorphism in the ∞-category $\mathscr{H}^{\mathrm{Nis}}(k)$. Here we have denoted by $X//U$ the homotopy cofiber[22] of the map $U \to X$.

2.2. The $\mathbb{A}^1$-localization

Having prepared the ground in the preceding sections, we can now apply Voevodsky's main idea of using the affine line as an interval for doing homotopy with algebraic varieties over k.

Definition 2.14. — *A k-space $\mathcal{X}$ will be called* $\mathbb{A}^1$-local *if for any smooth k-scheme U, the following map of spaces induced by the obvious projection is a weak equivalence:*

$$p^*\colon\ \mathrm{Map}^{\mathrm{Nis}}(U, \mathcal{X}) \to \mathrm{Map}^{\mathrm{Nis}}(\mathbb{A}^1_U, \mathcal{X}).$$

(20) Recall that there is an ∞-functor $\mathrm{N}\,\mathrm{PSh}(k) \to \mathscr{P}\mathscr{S}\mathrm{h}(k)$, allowing us to view (ordinary) morphisms of k-spaces as 1-morphisms in $\mathscr{P}\mathscr{S}\mathrm{h}(k)$.

(21) This really means cartesian in the ∞-categorical sense. We use this expression to avoid possible confusions with the same notion in the underlying category of simplicial sheaves

(22) This really means the quotient/cokernel in the ∞-categorical sense. The same remark as in the previous footnote is in order: we use this expression to avoid possible confusions.

We say that a morphism of k-spaces $f\colon \mathcal{Y} \to \mathcal{X}$ is a weak $\mathbb{A}^1$-equivalence *if for any $\mathbb{A}^1$-local k-space $\mathcal{Z}$, the following map of spaces is a weak equivalence:*

$$f^*\colon \operatorname{Map}^{\mathrm{Nis}}(\mathcal{X}, \mathcal{Z}) \to \operatorname{Map}^{\mathrm{Nis}}(\mathcal{Y}, \mathcal{Z}).$$

We define the $\mathbb{A}^1$-homotopy category $\mathrm{H}(k)$ over k as the localization of the (Nisnevich) homotopy category $\mathrm{H}^{\mathrm{Nis}}(k)$ of k-spaces with respect to weak $\mathbb{A}^1$-equivalences. One also denotes by

$$[\mathcal{X}, \mathcal{Y}]^{\mathbb{A}^1}$$

the morphisms in $\mathrm{H}(k)$, called weak $\mathbb{A}^1$-homotopy classes. One defines similarly the corresponding pointed category, denoted by $\mathrm{H}_\bullet(k)$.

This definition follows the classical pattern of left Bousfield localization.[23] In particular, weak $\mathbb{A}^1$-equivalences are exactly the morphisms of k-spaces that become isomorphisms in the motivic homotopy category $\mathrm{H}(k)$. It is clear that naive $\mathbb{A}^1$-equivalences (Definition 1.1) are a special case. We will see in Example 2.36 that this inclusion is strict.

Example 2.15. — $\mathbb{A}^1$**-Local objects.** One considers the notation of Example 2.8.

1. Let X be a smooth k-scheme. One says that X is $\mathbb{A}^1$-*rigid* if the associated k-space is $\mathbb{A}^1$-invariant: for any smooth k-scheme Y, the application

$$\operatorname{Hom}(Y, X) \to \operatorname{Hom}(\mathbb{A}^1_Y, X)$$

is a bijection. Canonical examples are: $\mathbb{G}_m$, abelian varieties, smooth proper curves.

If the k-scheme X is $\mathbb{A}^1$-rigid, one deduces that the k-space X is $\mathbb{A}^1$-local — use Example 2.8(1). One further deduces that for any smooth k-scheme Y, the following canonical application is a bijection:

$$\operatorname{Hom}(Y, X) \to [Y, X]^{\mathbb{A}^1}.$$

In particular, the $\mathbb{A}^1$-rigid smooth k-schemes are *discrete* from the point of view of motivic homotopy: $\mathbb{A}^1$-weak equivalences between them are exactly the isomorphisms of k-schemes.

2. A more interesting example of an $\mathbb{A}^1$-local k-space is provided by the classifying k-space $B\mathbb{G}_m$ associated with the multiplicative group. This follows from the above definition and Example 2.8(2), and yields the computation

$$[S^n \wedge X_+, B\mathbb{G}_m] = \begin{cases} \operatorname{Pic}(X) & n = 0 \\ \mathcal{O}(X)^\times & n = 1 \\ 0 & n > 1. \end{cases}$$

[23] This is presented in abstract terms in 1.4.3.

The attentive reader will recognize familiar motivic cohomology groups. In fact, $B\mathbb{G}_m$ is the first example of a *motivic Eilenberg-MacLane space*: $B\mathbb{G}_m = K(\mathbb{Z}(1), 1)$.

2.2.1. *Computational tools.* — As explained in Section 1.4, the motivic homotopy category admits the $\mathbb{A}^1$-localized versions of the adequate model structures on k-spaces.[24] It also admits a canonical ∞-categorical structure denoted by $\mathscr{H}(k)$: one considers the localization of the Nisnevich ∞-topos with respect to weak $\mathbb{A}^1$-equivalences. Using any of these enhancements, one deduces the definition of an $\mathbb{A}^1$-local mapping space simply denoted by $\mathrm{Map}^{\mathbb{A}^1}(\mathcal{X}, \mathcal{Y})$. One gets:

$$[\mathcal{X}, \mathcal{Y}]^{\mathbb{A}^1} = \pi_0 \mathrm{Map}^{\mathbb{A}^1}(\mathcal{X}, \mathcal{Y}).$$

Moreover, the following principles hold:

- A morphism between $\mathbb{A}^1$-local k-spaces is a weak $\mathbb{A}^1$-equivalence if and only if it is a local weak equivalence (in the sense of Definition 2.3).
- For any k-space $\mathcal{X}$, there exists an $\mathbb{A}^1$-local k-space $L_{\mathbb{A}^1}\mathcal{X}$ and a weak $\mathbb{A}^1$-equivalence $\mathcal{X} \to L_{\mathbb{A}^1}\mathcal{X}$ (which can even be chosen functorially in the model category case).

Except for the last statement, these principles follow from the general procedure of localization in higher categories (see 1.4.3).

Remark 2.16. — In fact, Morel and Voevodsky provide a more concrete construction of the $\mathbb{A}^1$-localization functor using the functor $\mathrm{Sing}_*^{\mathbb{A}^1}$ of "Suslin's singular chains" (see Morel and Voevodsky, 1999, Lem. 3.20, with the addition that for Nisnevich sheaves on Sm_k, one may take the countable cardinal).

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[24] Let us be explicit and extend footnote 15 page 15. One defines two model structures whose homotopy category is $\mathrm{H}(k)$:

- The $\mathbb{A}^1$*-localized Joyal model structure*: the base category is that of k-spaces that are Nisnevich sheaves, cofibrations are monomorphisms, weak equivalences are weak $\mathbb{A}^1$-equivalences, and fibrations are the maps with the RLP with respect to the $\mathbb{A}^1$-acyclic cofibrations.
- The $\mathbb{A}^1$*-localized Blander model structure*: the base category is that of k-spaces, cofibrations are the maps with the LLP with respect to epimorphisms that are term-wise Nisnevich-local weak-equivalences, fibrations are the maps with the RLP with respect to the $\mathbb{A}^1$-acyclic cofibrations.

The first model structure is the one used in Morel and Voevodsky (1999), and in most of the works on motivic homotopy categories. The second model structure has the advantage that representable k-spaces are cofibrant objects, fibrant k-spaces are k-spaces (simplicial presheaves) $\mathcal{X}$ which are Nisnevich excisive, termwise Kan complexes, and such that $\mathcal{X}(X) \to \mathcal{X}(\mathbb{A}^1_X)$ are weak equivalences (of Kan complexes) for each smooth k-scheme X. Also, the two pairs of adjoint functors, (f^*, f_*) and $(p_\sharp, p^*)$ for p smooth, that appear on S-spaces when one works with a scheme instead of a field are Quillen adjunctions for the second model structure, but not for the first one.

2.2.2. *Realizations.* — We consider the notation of Section 1.1.

We first consider a complex embedding $\sigma\colon k \to \mathbb{C}$. Then the canonical functor $X \mapsto X^\sigma(\mathbb{C})$ with values in topological spaces, admits a canonical extension to k-spaces. It can be shown following either Dugger and Isaksen (2004, §5.1), or Panin, Pimenov, and Röndigs (2009, §A.4) that it sends weak $\mathbb{A}^1$-equivalences to weak equivalences of topological spaces, therefore inducing an ∞-functor called the *σ-realization functor*, or complex realization when σ is clear:

$$\rho_\sigma\colon \mathscr{H}(k) \to \mathscr{H}$$

For $\sigma = \mathrm{Id}_\mathbb{C}$, we simply write $\rho_\mathbb{C}$.

Next we consider a real embedding $\sigma\colon k \to \mathbb{R}$. Given a smooth k-scheme, instead of looking at the real points $X^\sigma(\mathbb{R})$, it is more accurate to look at the complex points $X^\sigma(\mathbb{C})$ with the canonical action ν_X^σ of $\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The canonical functor $X \mapsto (X^\sigma(\mathbb{C}), \nu_X^\sigma)$ from smooth k-schemes to $\mathbb{Z}/2$-equivariant topological spaces also admits a canonical extension to k-spaces, and one shows this extension maps weak $\mathbb{A}^1$-equivalences to weak equivalences of $\mathbb{Z}/2$-equivariant topological spaces (see Dugger and Isaksen, 2004, Th. 5.5). One deduces the *equivariant σ-realization functor*:

$$\rho_\sigma^{\mathbb{Z}/2}\colon \mathscr{H}(k) \to \mathscr{H}_{\mathbb{Z}/2}$$

Taking $\mathbb{Z}/2$-homotopy fixed points then induces the σ-realization functor:

$$\rho_\sigma\colon \mathscr{H}(k) \to \mathscr{H}\,.$$

In particular, for a smooth k-scheme X, one gets: $\rho_\sigma(X) = X(\mathbb{C})^{h\mathbb{Z}/2} = X(\mathbb{R})$. When σ is clear, one simply uses the terminology "equivariant real" and "real realization".

Example 2.17. — As expected, a smooth k-scheme X is said to be *$\mathbb{A}^1$-contractible* if its structural morphism p is a weak $\mathbb{A}^1$-equivalence.

The study of this particular kind of varieties has been a rich question, started by Asok and Doran (2007) in their pioneering work. There are many interesting families of $\mathbb{A}^1$-contractible smooth algebraic varieties. We refer the interested reader to Asok and Østvær (2021). Let us mention a few examples:

1. *Open varieties in quadrics*: Consider the affine quadric

$$Q_{2n} : \left\{\sum_i x_i y_i = z(1+z)\right\} \subset \mathbb{A}^{2n+1}$$

and the closed subvariety

$$E_n : \{x_1 = \cdots = x_n = 0, z = -1\} \subset Q_{2n}.$$

Then it was shown by Asok, Doran, and Fasel (2017, §3.1) that for all integers $n \geq 1$, $X_n = Q_{2n} - E_n$ is a quasi-affine smooth k-variety which is $\mathbb{A}^1$-contractible, but not a unipotent quotient of an affine space for $n \geq 3$.[25]

[25] This works not only over an arbitrary field k, but even over an arbitrary base.

2. *Koras–Russell 3-fold of the first kind*: they are smooth affine hypersurfaces defined by an equation of the form:

$$K_{m,r,s} : \{x^m z = y^r + t^s + x\} \subset \mathbb{A}^4_{\mathbb{C}}$$

where $m, r, s \geq 2$, and r and s are coprime. These complex algebraic varieties are known to be topologically contractible, non isomorphic to $\mathbb{A}^3_{\mathbb{C}}$. Dubouloz and Fasel (2018) proved they are $\mathbb{A}^1$-contractible, even working over any characteristic 0 field k.

Given the realization functors defined in 2.2.2, it is known that an $\mathbb{A}^1$-contractible smooth k-variety is topologically contractible for all complex and real embeddings of the base field k. One may naturally ask whether the converse holds. This question was resolved in Choudhury and Roy (2024, Th. 1.1).

Theorem 2.18. — *Let X be a smooth affine algebraic surface over a field k of characteristic 0. Then X is $\mathbb{A}^1$-contractible if and only if X is isomorphic to $\mathbb{A}^2_k$.*

It follows that there are smooth affine complex algebraic surfaces which are topologically contractible but not $\mathbb{A}^1$-contractible: one can consider either a tom Dieck-Petri surface or the Ramanujam surface (see *op. cit.* after Th. 1.2).

Remark 2.19. — Note that this implies that, even when restricted to the subcategory of compact objects, the complex realization functor with source $\mathrm{H}(\mathbb{C})$ is not conservative. This is in contrast with Beilinson's well-known conservativity conjecture for constructible rational mixed motives.

Example 2.20 (Motivic spheres and Thom spaces). — We will always assume that $\mathbb{P}^1_k$ (resp. $\mathbb{G}_m$, $\mathbb{A}^n - \{0\}$) is pointed by ∞ (resp. 1, $(0, \dots, 0, 1)$). Given a vector bundle V over a smooth k-scheme, one defines the *Thom (k-)space of V* as the quotient[(26)]

$$\mathrm{Th}(V) \coloneqq V/V^\times$$

where $V^\times \to V$ is the open immersion of the complement of the zero section.

Then one obtains the following computations in the pointed motivic homotopy category $\mathrm{H}_\bullet(k)$:

- $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$
- $\mathbb{A}^n - \{0\} \simeq S^{n-1} \wedge \mathbb{G}_m^n$
- $\mathrm{Th}(\mathbb{A}^n_k) \simeq (\mathbb{P}^1)^{\wedge,n}$
- $Q_{2n-1} \simeq S^{n-1} \wedge \mathbb{G}_m^{\wedge,n}$
- $Q_{2n} \simeq S^n \wedge \mathbb{G}_m^{\wedge,n}$

(26) the base point is given by the canonical map $* = V^\times/V^\times \to \mathrm{Th}(V)$

The first three isomorphisms are good exercises.[27] The last two isomorphisms are due to Asok, Doran, and Fasel (2017, Th. 2/2.2.5). The reader can now check, from the table in Section 1.1, that all these computations are compatible with the real and complex realizations as defined in 2.2.2.

Example 2.21. — Let us finish with a more advanced example. Assume our base field k is algebraically closed field of characteristic 0. Given an integer $n > 0$ and a separable polynomial $P(z)$ of degree $d > 0$, the associated *Danielewski surface* is defined as the affine hypersurface

$$D_{n,P} : x^n y = P(z) \subset \mathbb{A}^3_k.$$

It was shown by Danielewski and Fieseler (see Dubouloz, 2005) that $D_{n,P}$ is a Zariski-local torsor under a line bundle over the affine line with a d-fold origin. This implies that the associated k-space has the motivic homotopy type of a wedge[28] of spheres:

$$D_{n,P} \simeq (\mathbb{P}^1_k)^{\vee,d}.$$

2.3. $\mathbb{A}^1$-homotopy sheaves

As in topology, one can encode weak $\mathbb{A}^1$-equivalences via the appropriate notion of homotopy groups — or rather homotopy sheaves, as in Section 2.1.4.

Definition 2.22. — *Let $\mathcal{X}$ be a k-space. One defines its* sheaf of $\mathbb{A}^1$-connected components $\underline{\pi}_0^{\mathbb{A}^1}(\mathcal{X})$ *as the (Nisnevich) sheaf of sets on* Sm_k *associated with the presheaf:*

$$V \mapsto [V, \mathcal{X}]^{\mathbb{A}^1}.$$

Assume that $\mathcal{X}$ is pointed. Then one defines for any integer $n > 0$ the n-th $\mathbb{A}^1$-homotopy sheaf $\underline{\pi}_n^{\mathbb{A}^1}(\mathcal{X})$ *associated with $\mathcal{X}$ as the sheaf associated with the presheaf:*

$$V \mapsto [S^n \wedge V_+, \mathcal{X}]_\bullet^{\mathbb{A}^1}.$$

The properties of the $\mathbb{A}^1$-localization functor imply the following important formula, for any $n \geq 0$:

$$\underline{\pi}_n^{\mathbb{A}^1}(\mathcal{X}) = \underline{\pi}_n(L_{\mathbb{A}^1}\mathcal{X}) \tag{2.22.a}$$

where the right-hand side sheaf was defined in Section 2.1.4.

Example 2.23. — *Loop spaces.–* We recalled the definition of the smash product on pointed k-spaces; in Section 2.1.2. This operation corresponds to a closed monoidal; structure on the homotopy category $\mathrm{H}_\bullet(k)$.

In particular, one gets an internal pointed hom functor, and one can define several loop k-spaces, associated with our different motivic spheres:

- simplicial: $\Omega_{S^1} = \underline{\mathrm{Hom}}_\bullet(S^1, -)$;
- $\mathbb{G}_m$-loops: $\Omega_{\mathbb{G}_m} = \underline{\mathrm{Hom}}_\bullet(\mathbb{G}_m, -)$;
- $\mathbb{P}^1$-loops: $\Omega_{\mathbb{P}^1} = \underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1_k, -)$.

(27) Hint: use the excision property and the $\mathbb{A}^1$-contractibility of $\mathbb{A}^n$.
(28) see Section 2.1.2;

We can iterate as usual these constructions. It follows formally that:

$$\underline{\pi}_n^{\mathbb{A}^1}(\mathcal{X}) \simeq \underline{\pi}_0^{\mathbb{A}^1}(\Omega^n_{S^1}\mathcal{X}).$$

Then it can be shown as in topology that $\Omega_{S^1}\mathcal{X}$ has an h-group structure: this gives another proof for the fact that $\underline{\pi}_1^{\mathbb{A}^1}(\mathcal{X})$ is a sheaf of groups. As is classical in topology, one also deduces that for $n > 1$, the sheaf $\underline{\pi}_n^{\mathbb{A}^1}(\mathcal{X})$ has two compatible group structures hence is abelian.

Let us also finally mention that one deduces from Example 2.20 the identity $\Omega_{\mathbb{P}^1} = \Omega_{S^1}\Omega_{\mathbb{G}_m}$. In particular, any $\mathbb{P}^1$-loop space has an h-group structure.

Remark 2.24. — As is typical in homotopy theory, it is crucial that the closed monoidal structure on $\mathrm{H}_\bullet(k)$ corresponding to the smash product exists at the model or ∞-categorical level (see also Section 1.4.4). First, this is the only way to get such a structure on the homotopy category. Second, it will be used to define the stable motivic homotopy category in Section 3. For the explicit construction in the ∞-categorical case, we refer the reader to Robalo (2015, §2.4.2).

Definition 2.25. — *Let $n \geq 0$ be an integer and $\mathcal{X}$ be a k-space, pointed if $n > 0$.*

One says that $\mathcal{X}$ is n-$\mathbb{A}^1$-connected *if for any $0 \leq i \leq n$, we have $\underline{\pi}_i^{\mathbb{A}^1}(\mathcal{X}) = *$. We say* $\mathbb{A}^1$-connected *for 0-$\mathbb{A}^1$-connected and* simply $\mathbb{A}^1$-connected *for 1-$\mathbb{A}^1$-connected.*

Example 2.26. — One easily deduces from Example 2.15 the following computations.

1. Let X be a smooth k-scheme. Then the following conditions are equivalent:
   (i) X is $\mathbb{A}^1$-rigid.
   (ii) The canonical morphism of sheaves $X \to \underline{\pi}_0(X)$ is an isomorphism.
   In this case, one also deduces that for any base point $x \in X(k)$ and any $n > 0$, $\underline{\pi}_n^{\mathbb{A}^1}(X, x) = *$.
2. $\underline{\pi}_n^{\mathbb{A}^1}(B\mathbb{G}_m) = \mathbb{G}_m$ if $n = 1$, and $*$ otherwise. This correctly reflects Example 2.15(2).
3. Following Morel, one says that a sheaf of groups $\mathcal{G}$ is *strongly* $\mathbb{A}^1$*-invariant* if for any smooth k-scheme X, both maps

$$\mathcal{G}(X) \to \mathcal{G}(\mathbb{A}^1_X) \text{ and } H^1(X, \mathcal{G}) \to H^1(\mathbb{A}^1_X, \mathcal{G})$$

   are isomorphisms. On the right-hand side, we considered pointed sets of isomorphisms classes of Nisnevich-local $\mathcal{G}$-torsors on X.

   One can check that the following properties are equivalent:
   (a) $\mathcal{G}$ is strongly $\mathbb{A}^1$-invariant.
   (b) The k-space $B\mathcal{G}$ (see Example 2.2) is $\mathbb{A}^1$-local.
   As in the case of $\mathbb{G}_m$, when these properties hold, one gets: $\underline{\pi}_n^{\mathbb{A}^1}(B\mathcal{G}) = \mathcal{G}$ if $n = 1$, and $*$ otherwise.

2.3.1. *Unramified sheaves.* — To state the next theorem we introduce the following terminology, based on classical considerations but due to Morel. Let F be a Zariski sheaf of sets on Sm_k. Given a smooth k-scheme X and a point $x \in X$, we let

$$F(\mathcal{O}_{X,x}) = \varinjlim_{x \in U \subset X} F(U) \tag{2.26.a}$$

be the Zariski fiber of F over X at the point x. When x is a generic point with residue field E, we let $F(E) = F(\mathcal{O}_{X,x})$.

Let us assume that for any dense open immersion $j\colon U \to X$ of smooth k-schemes, the induced morphism $F(X) \to F(U)$ is injective. Then when X is connected with function field E, the set $F(U)$, and similarly $F(\mathcal{O}_{X,x})$ for any $x \in X$, can be identified with a subset of $F(E)$. Using this identification, we say that F is *unramified* when in addition to the preceding condition we have for any connected smooth k-scheme X

$$F(X) = \bigcap_{x \in X^{(1)}} F(\mathcal{O}_{X,x}),$$

where the intersection runs over the set $X^{(1)}$ of points of codimension 1 in X.

Examples of unramified abelian sheaves come from Bloch–Ogus theory (Bloch and Ogus, 1974, Th. 4.2), and also from Voevodsky's theory of homotopy invariant Zariski sheaves with transfers (see Mazza, Voevodsky, and Weibel, 2006, Th. 24.11). The following fundamental structure theorem is due to Morel, inspired by the latter example.

Theorem 2.27. — *Let $\mathcal{X}$ be a pointed k-space.*

1. *The sheaf of groups $\underline{\pi}_1^{\mathbb{A}^1}(\mathcal{X})$ over Sm_k is unramified and strongly $\mathbb{A}^1$-invariant.*
2. *Assume the field k is perfect. Then for any $n > 1$, the sheaf of abelian groups $\underline{\pi}_n^{\mathbb{A}^1}(\mathcal{X})$ is unramified and* strictly $\mathbb{A}^1$-invariant*: it has $\mathbb{A}^1$-invariant cohomology.*

This theorem is due to Morel (2012): point (1) is Theorem 6.1, and point (2) would follow from Theorem 5.46. We warn the reader that the proof of the latter contains an as yet unproved claim.(29) However, two alternative proofs, heavily based on *loc. cit.* but which avoid the problem, are given by Bachmann (2024, Th. 1.6, Cor. 1.8) and by Ayoub (2022).

(29)This problem was raised by Niels Feld. Here is a detailed summary where, unless explicitly stated, references concern Morel (2012):

1. The existence of well-defined transfers on strongly $\mathbb{A}^1$-invariant sheaves F is only proven on those of the form $F = M_{-2}$ (Theorem 4.27, see Feld, 2021a, Th. 5/6.1.5 for a more precise statement), but claimed to exist on sheaves of the form $F = M_{-1}$ in Remark 4.31;
2. Corollary 5.30 uses the existence of transfers on M_{-1}, via its reference to Theorem 5.26;
3. Theorem 5.31 uses Corollary 5.30 which cannot be applied in codimension 1.

By constrast, the proof of Tom Bachmann reduces Theorem 5.30 to the case of $\mathbb{P}^1_S$ and then uses a well-defined pushforward for the projection $\mathbb{P}^1_S \to S$.

2.3.2. *Gersten resolutions.* — Assume that the base field k is perfect. It is well-known from Colliot-Thélène, Hoobler, and Kahn (1997) that the strict $\mathbb{A}^1$-invariance of $F = \underline{\pi}_n^{\mathbb{A}^1}(\mathcal{X})$, $n > 1$, implies that F admits a Gersten resolution.(30) In particular, both Zariski and Nisnevich cohomologies of a smooth k-scheme X with coefficients in F can be computed in terms of the associated Gersten complex.(31) In the case of sheaves of the form $F = \underline{\pi}_n^{\mathbb{A}^1}(\mathcal{X})$, the Gersten complex of a smooth k-scheme X with coefficients in F even takes the following simpler form and is called the *Rost–Schmid complex* after Morel (2012, Chap. 5):

$$\bigoplus_{x\in X^{(0)}} F(\kappa(x)) \to \bigoplus_{x\in X^{(1)}} F_{-1}(\kappa(x))\{\nu_x\} \to \cdots \to \bigoplus_{x\in X^{(d)}} F_{-d}(\kappa(x))\{\nu_x\} \tag{2.27.a}$$

with the following notation:

1. d is the dimension of X;
2. $X^{(i)}$ denotes the set of points of codimension i in X;
3. F_{-1} is *Voevodsky's (-1)-construction*, or *simply contraction the contraction of* F: $F_{-1}(X)$ is the cokernel of the split monomorphism $F(\mathbb{G}_m \times X) \xrightarrow{s_1^*} F(X)$, and F_{-n} is the n-th iterated application of the (-1)-construction;
4. we have used notation (2.26.a) for the sheaf F_{-n}, with $\kappa(x)$ being the residue field of x in X;
5. ν_x is the determinant of the conormal sheaf of the regular closed immersion $x \to X_{(x)} = \mathrm{Spec}(\mathcal{O}_{X,x})$;
6. given an invertible line bundle $\mathcal{L}$ over X, $\mathcal{L}^\times$ being the subsheaf complementary of the zero section, we have adopted the following notation after Morel:
$$F_{-n}(X)\{\mathcal{L}\} = F_{-n}(X) \otimes_{\mathbb{Z}[\mathbb{G}_m(X)]} \mathbb{Z}[\mathcal{L}^\times(X)]$$
using the natural action of $\mathbb{G}_m$ on F_{-n} and on $\mathcal{L}^\times$.

In fact, the core argument of the previous theorem is to show that a strongly $\mathbb{A}^1$-invariant sheaf of *abelian groups* F admits a Gersten resolution. This in turn implies the $\mathbb{A}^1$-invariance of all its cohomology sheaves.

For a systematic treatment of Rost–Schmid complexes, in the style of cycle modules defined by Rost (1996), we refer the reader to Feld (2020).

Example 2.28. — For the time of writing, the structure of the sheaf of $\mathbb{A}^1$-connected components $\underline{\pi}_0^{\mathbb{A}^1}(\mathcal{X})$ remains mysterious. We know it is not $\mathbb{A}^1$-invariant because of a counter-example due to Ayoub (2023). However, for an h-group $\mathcal{X}$ (same definition as

(30) Note that the correct terminology should be "Gersten property" as it will be recalled in the next footnote that the Gersten resolution of F, if it exists, is uniquely determined up to unique is omorphism.
(31) Another way of stating this is that the restriction $F|_{X_{\mathrm{Zar}}}$ to the small Zariski site of X is Cohen-Macaulay in the sense of Hartshorne (1966, Def. p. 238). Then the Gersten complex restricted to X_{Zar} is the associated Cousin complex. In particular, this complex is unique up to unique isomorphism thanks to Hartshorne (1966, Prop. 2.3). These considerations can be extended to the Nisnevich topology. See Déglise, Feld, and Jin (2022, §4.3), and also Druzhinin, Kolderup, and Østvær (2024) for further developments.

in topology), the sheaf $\underline{\pi}_0^{\mathbb{A}^1}(\mathcal{X})$ is unramified and $\mathbb{A}^1$-invariant, as proved by Choudhury (2014, Th. 4.18).[(32)]

Moreover, several fundamental computations in this setting were first established in the seminal work of Asok and Morel (2011). Let X be a smooth proper k-scheme over a perfect field k. Applying *loc. cit.* Theorem 6.2.1 and Proposition 6.2.6 (taking into account Remark 2.2.3), one gets a canonical bijection for any function field E/k:

$$X(E)/R \xrightarrow{\sim} \underline{\pi}_0^{\mathbb{A}^1}(X)(E) \tag{2.28.a}$$

where $(-/R)$ denotes the quotient by Manin's R-equivalence relation. This is enough to deduce that if the left-hand side is trivial for any E/k,[(33)] then X is $\mathbb{A}^1$-connected — apply Proposition 2.2.7 of *op. cit.*

Moreover, the isomorphism (2.28.a) was extended to the case where E/k is an arbitrary function field of characteristic 0, and $X = G$ is a semisimple, simply connected, isotropic, and absolutely almost simple algebraic group: see Balwe, Hogadi, and Sawant, 2023, Th. 4.2.[(34)]

Example 2.29. — Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of k-spaces. Following Morel (2012, Definition 7.1, Lemma 7.2), one says that f is an $\mathbb{A}^1$-covering if it has the right lifting property with respect to weak $\mathbb{A}^1$-equivalences.[(35)] For a strongly $\mathbb{A}^1$-invariant sheaf of groups $\mathcal{G}$, any Nisnevich-local $\mathcal{G}$-torsor is an $\mathbb{A}^1$-covering: *op. cit.* Lemma 7.5. An important theorem of Morel, *op. cit.* Theorem 7.8, shows that any pointed $\mathbb{A}^1$-connected k-space $\mathcal{X}$ admits a *universal* $\mathbb{A}^1$*-covering*

$$\tilde{\mathcal{X}} \to \mathcal{X}$$

where $\tilde{\mathcal{X}}$ is simply $\mathbb{A}^1$-connected. As in topology, one formally deduces that

$$\underline{\pi}_1^{\mathbb{A}^1}(\mathcal{X}) = \underline{\mathrm{Aut}}_{\mathcal{X}}(\tilde{\mathcal{X}})$$

where the right-hand side is the sheaf of automorphisms of $\mathbb{A}^1$-covers.

As an example, for an integer $n > 1$, the canonical $\mathbb{G}_m$-torsor

$$\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$$

is in fact the universal $\mathbb{A}^1$-covering of $\mathbb{P}^n$. One deduces (*op. cit.* Theorem 7.13) that, for any integer $n > 1$,

$$\underline{\pi}_1^{\mathbb{A}^1}(\mathbb{P}^n) = \mathbb{G}_m.$$

(32) See the proof p. 51 for the unramified property.

(33) in that case, we can say that X is *universally R-trivial.* Recall that this is implied by the property of being retract k-rational or even, universally CH_0-trivial;

(34) Recall that in this case, according to Gille (2009, Th. 7.2), the left hand side of (2.28.a) is also isomorphic to the so called Whitehead group of G over k. In particular, thanks to *loc. cit.* Theorem 8.1, when k is a global field, G is $\mathbb{A}^1$-connected.

(35) Beware that if we want to give a meaningful ∞-categorical formulation, one really has to consider f as a map in the Nisnevich ∞-category $\mathscr{H}^{\mathrm{Nis}}(k)$.

By contrast, the $\mathbb{A}^1$-homotopy sheaf $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1_k)$ is non-abelian. In fact, it is the *free strictly* $\mathbb{A}^1$*-invariant sheaf* of groups generated by the final sheaf of sets $*$ (see *op. cit.* Lemma 7.23). We will give a more precise description in Proposition 2.38 below.

2.4. Morel quadratic degree

2.4.1. *Milnor–Witt K-theory.* — In the theory of motivic complexes, Milnor K-theory[36] $\mathrm{K}^{\mathrm{M}}_*(k)$ of a field k plays a central role, as the extension algebra between Tate twists $\mathbb{1}(n)$. In motivic homotopy, we have seen in Cazanave's Theorem 1.3 that inner products play a central role, at least visible in the naive $\mathbb{A}^1$-endomorphism classes of the sphere $(\mathbb{P}^1_k, \infty)$.

One of Morel's key insights in his analysis of motivic homotopy theory over a field was the introduction of a theory that encompasses both theories into what is now called the *Milnor–Witt K-theory* of the field k; see Morel, 2012, §3.1. Morel's definition of the corresponding $\mathbb{Z}$-graded ring, denoted by $\mathrm{K}^{\mathrm{MW}}_*(k)$, is by generators and relation:

– <u>Generators</u>: symbols $[u]$ of degree $+1$ for a unit $u \in k^\times$, and the Hopf symbol η of degree -1.

  Let us write: $[u_1, \dots, u_n] = [u_1] \cdots [u_n]$, $\langle u \rangle = 1 + \eta.[u]$, $\epsilon = -\langle -1 \rangle$, $h = 1 - \epsilon$.

– <u>Relations</u>:

(MW1) $[u, 1-u] = 0$, $u \neq 0, 1$;
(MW2) $[uv] = [u] + [v] + \eta[u, v]$;
(MW3) $\eta[u] = [u]\eta$;
(MW4) $\eta h = 0$.

Morel was inspired by the Milnor conjecture, linking Milnor K-theory and the Witt ring. This definition indeed realizes the synthesis between these two theories, as seen by the following computations:

$$\mathrm{K}^{\mathrm{MW}}_*(k)/(\eta) = \mathrm{K}^{\mathrm{M}}_*(k)$$
$$\mathrm{K}^{\mathrm{MW}}_*(k)[\eta^{-1}] = \mathrm{W}(k)[\eta, \eta^{-1}].$$

See *loc. cit.*[37] Moreover, one also deduces

$$\mathrm{K}^{\mathrm{MW}}_0(k) = \mathrm{GW}(k)$$

and in degree 0, the projection modulo η induces the rank map:

$$\mathrm{GW}(k) = \mathrm{K}^{\mathrm{MW}}_0(k) \to \mathrm{K}^{\mathrm{MW}}_0(k)/(\eta) = \mathrm{K}^{\mathrm{M}}_0(k) = \mathbb{Z}. \tag{2.29.a}$$

The following computation is one of the major results of Morel (2012, Cor. 6.43). Recall that the group scheme $\mathbb{G}_m$ is one of our motivic spheres, seen as a pointed k-scheme by the unit element.

[36] Recall that this is the tensor $\mathbb{Z}$-graded algebra of the $\mathbb{Z}$-module $k^\times$ modulo the *Steinberg relation* $\{u, 1-u\} = u \otimes (1-u) = 0$.

[37] The first computation is easy, and the second one follows from the presentation of $\mathrm{W}(k)$ by generators and relation given in 1.2.3.

Theorem 2.30. — *Consider the above notation. Let $n, i, m, j \in \mathbb{N}$ be integers such that $n \geq 2$. Then there are canonical isomorphisms:*

$$[S^m \wedge \mathbb{G}_m^{\wedge j}, S^n \wedge \mathbb{G}_m^{\wedge i}]_\bullet^{\mathbb{A}^1} \simeq \begin{cases} 0 & m < n \text{ or } (m = n, j > 0, i = 0), \\ \mathbb{Z} & m = n, j = i = 0, \\ \mathrm{K}_{i-j}^{\mathrm{MW}}(k) & (m = n, i > 0). \end{cases}$$

The proof makes use of the *motivic version of the Hurewicz theorem* (see Remark 3.7).

Example 2.31. — In fact, one can explicitly identify the generators of the Milnor–Witt ring with geometrically defined morphisms via the isomorphism appearing in the preceding theorem. First, a unit $u \in k^\times$ defines a morphism of k-schemes $\gamma_u \colon \mathrm{Spec}(k) \to \mathbb{G}_m$, whose weak $\mathbb{A}^1$-homotopy class corresponds to the element $[u] \in \mathrm{K}_1^{\mathrm{MW}}(k)$.

More interestingly, the element $\eta \in \mathrm{K}_{-1}^{\mathrm{MW}}(k)$ is sent to the class of the obvious map

$$\eta \colon \mathbb{A}_k^2 - \{0\} \to \mathbb{P}_k^1.$$

We call it the *algebraic Hopf map* and denote it (abusively) by η. In fact, one can observe that when $k = \mathbb{C}$, the topological realization of η is precisely the classical Hopf map $S^3 \to S^2$.

Another meaningful element for motivic homotopy is the class $\rho \coloneqq [-1] \in \mathrm{K}_1^{\mathrm{MW}}(k)$. We refer the reader to Theorem 3.15 for an illustration.

Remark 2.32. — Let us anticipate what follows by stating how this computation relates to motivic cohomology. Taking realization in the category of motivic complexes, one gets a canonical map:

$$[S^m \wedge \mathbb{G}_m^{\wedge j}, S^n \wedge \mathbb{G}_m^{\wedge i}]_\bullet^{\mathbb{A}^1} \to \mathbf{H}_{\mathrm{M}}^{n+i-m-j, i-j}(k)$$

where the right-hand side is the motivic cohomology of the field k. In particular, when $n = m$, we get a map

$$\mathrm{K}_{i-j}^{\mathrm{MW}}(k) \to \mathrm{K}_{i-j}^{\mathrm{M}}(k)$$

which is in fact the projection modulo η, and in particular, the rank map (2.29.a) when $i = j$. In other words, motivic complexes do not see the quadratic phenomena of motivic homotopy.

Note also that Morel's computations do not say anything about the range $m > n$, $i = j$, which in fact corresponds to the range of the Beilinson–Soulé vanishing conjecture in motivic cohomology.

One of the main applications indicated by Morel is a notion of degree in motivic homotopy analogous to the Brouwer degree. We will call it the *Morel degree*.

Corollary 2.33. — *Consider an integer $n \geq 2$. Then there exists isomorphisms of abelian groups:*

$$[(\mathbb{P}_k^1)^{\wedge n}, (\mathbb{P}_k^1)^{\wedge n}]_\bullet^{\mathbb{A}^1} \simeq [\mathbb{A}_k^n - \{0\}, \mathbb{A}_k^n - \{0\}]_\bullet^{\mathbb{A}^1} \simeq \mathrm{GW}(k).$$

Indeed, the abelian group structure on the first two sets comes from Example 2.20: it is proved there that both k-spaces $(\mathbb{P}^1_k)^{\wedge n}$ *and* $\mathbb{A}^n - \{0\}$ *are at least a* 2*-simplicial suspension of another k-space, except for* $\mathbb{A}^2 - \{0\}$ *which is an h-space.*

Example 2.34. — In particular, each pointed endomorphism f of the k-space $\mathbb{A}^n - \{0\}$ admits a quadratic degree $\widetilde{\deg}(f) \in \mathrm{GW}(k)$. As an example, consider a unit $u \in k^\times$ and the associated pointed endomorphism $f_u\colon (t_1, \dots, t_n) \mapsto (ut_1, t_2, \dots, t_n)$ of $\mathbb{A}^n_k - \{0\}$. Then $\widetilde{\deg}(f_u) = \langle \overline{u} \rangle$ with the notation of Section 1.2.3.

Let us consider a complex embedding $\sigma\colon k \to \mathbb{C}$. Then a general pointed endomorphism f as above induces a pointed continuous map

$$\rho_\sigma(f)\colon S^{2n-1} \to S^{2n-1}$$

to which is associated a Brouwer degree. One gets:

$$\mathrm{rk}(\widetilde{\deg}(f)) = \deg(\rho_\sigma(f)).$$

The case of the projective line can also be considered, but it requires more care. This is the following major theorem, obtained by the joint efforts of Morel and Cazanave. We refer the reader to Morel (2012, Theorem 7.36) and Cazanave (2012, Theorems 3.22, 4.6) for the proofs. In addition, it allows to compute both the naive pointed $\mathbb{A}^1$-homotopy classes (Definition 1.2) and the pointed $\mathbb{A}^1$-homotopy classes (Definition 2.14) in the particular case of endomorphisms of the projective line.

Theorem 2.35. — *The map (1.2.a) (page 7) induces an isomorphism of rings:*

$$[\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathbb{A}^1}_\bullet \xrightarrow{\sim} \mathrm{GW}(k) \times_{Q(k)} k^\times$$

where, on the left-hand side, the addition comes from the fact $\mathbb{P}^1_k = S^1 \wedge \mathbb{G}_m$*, and the multiplication is induced by the composition of pointed maps.*

In particular, the canonical map

$$[\mathbb{P}^1_k, \mathbb{P}^1_k]^N_\bullet \to [\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathbb{A}^1}_\bullet$$

can be interpreted as the canonical morphism to the group completion of the left-hand side, with its canonical additive monoid structure.

Example 2.36. — When $k = \mathbb{C}$, the rank map gives the following computation:

$$[\mathbb{P}^1_k, \mathbb{P}^1_k]^N_\bullet = \mathbb{N} \times \mathbb{C}^\times;$$
$$[\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathbb{A}^1}_\bullet = \mathbb{Z} \times \mathbb{C}^\times.$$

When $k = \mathbb{R}$, the signature map gives:

$$[\mathbb{P}^1_k, \mathbb{P}^1_k]^N_\bullet = (\mathbb{N} \times \mathbb{N}) \times \mathbb{R}^\times;$$
$$[\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathbb{A}^1}_\bullet = (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^\times.$$

In fact, whatever the base field k is, naive $\mathbb{A}^1$-homotopy classes and weak $\mathbb{A}^1$-homotopy classes of endomorphisms of the pointed k-space $\mathbb{P}^1_k$ *do not coincide.*

2.4.2. *Unramified Milnor–Witt K-theory and $\mathbb{A}^1$-homotopy sheaves.* — Recall that the *n-th unramified Milnor K-theory* of a smooth connected k-scheme X with function field E is defined as the kernel:

$$\underline{\mathrm{K}}_n^{\mathrm{M}}(X) = \mathrm{Ker}\left(\mathrm{K}_n^{\mathrm{M}}(E) \xrightarrow{d} \oplus_{x \in X^{(1)}} \mathrm{K}_{n-1}^{\mathrm{M}}(\kappa(x))\right)$$

where d is the sum of the residue maps associated with the discrete valuation v_x on E of a codimension 1 point $x \in X$. This defines an unramified and strictly $\mathbb{A}^1$-invariant (see e.g. Section 2.3.1) abelian Nisnevich sheaf $\underline{\mathrm{K}}_n^{\mathrm{M}}$ on Sm_k (see Déglise, 2006, Prop. 6.9 applied to the cycle module $\mathrm{K}_*^{\mathrm{M}}$).

Similarly, one can define following Morel the *n-th unramified Milnor–Witt K-theory* of a smooth k-scheme X as the kernel:

$$\underline{\mathrm{K}}_n^{\mathrm{MW}}(X) = \mathrm{Ker}\left(\mathrm{K}_n^{\mathrm{MW}}(E) \xrightarrow{d} \oplus_{x \in X^{(1)}} \mathrm{K}_{n-1}^{\mathrm{MW}}(\kappa(x))\{\nu_x\}\right)$$

where d is again the sum of some residue maps — we have used the notation of Section 2.3.2. Again $\underline{\mathrm{K}}_n^{\mathrm{MW}}$ is an abelian unramified and strictly $\mathbb{A}^1$-invariant Nisnevich sheaf on Sm_k (see Morel, 2012, §3.2, or apply Feld, 2021b, Theorem 4.1.7 to the Milnor–Witt cycle module $\mathrm{K}_*^{\mathrm{MW}}$). These considerations allow one to state the previous computations in terms of homotopy sheaves. Here is a fundamental example.

Example 2.37. — For $n \geq 2$, we can now give more meaning to the assertion that $\mathbb{A}^n - \{0\}$ is a motivic sphere, suggested in Section 1.1. In fact, Theorem 2.30 implies it is $(n-1)$-$\mathbb{A}^1$-connected in the sense of Definition 2.25, which is reflected by its complex realization. Moreover, one can deduce from Corollary 2.33 (or see directly Morel, 2012, Rem. 6.42) that its first non-trivial homotopy sheaf is

$$\underline{\pi}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) = \underline{\mathrm{K}}_n^{\mathrm{MW}}.$$

The same result hold for $(\mathbb{P}^1)^{\wedge,n}$.

We end up this section with a beautiful computation, again due to Morel, that we state separately.

Proposition 2.38. — *There exists a weak $\mathbb{A}^1$-equivalence of k-spaces:*

$$B\mathbb{G}_m \simeq \mathbb{P}_k^\infty = \varinjlim_{n \in \mathbb{N}} \mathbb{P}_k^n.$$

Moreover, the obvious sequence of k-spaces:

$$(\mathbb{A}^2 - \{0\}) \to \mathbb{P}_k^1 \xrightarrow{p} \mathbb{P}_k^\infty$$

is an $\mathbb{A}^1$-fiber sequence, in the sense that the first k-space is the homotopy fiber of the map p. Applying the functor $\underline{\pi}_1^{\mathbb{A}^1}$, one deduces a short exact sequence of strongly $\mathbb{A}^1$-invariant sheaves of groups:

$$0 \to \underline{\mathrm{K}}_2^{\mathrm{MW}} \to \underline{\pi}_1^{\mathbb{A}^1}(\mathbb{P}_k^1) \to \mathbb{G}_m \to 0.$$

This is in fact a central extension.

We have applied the previous example for the first sheaf and Example 2.26 for the third one. For the rest, see Morel, 2012, Theorem 7.29 taking into account Lemma 7.23 and the remark that follows the proof.

2.5. K-theory and vector bundles

We have already seen with the case of the Picard group how classifying spaces can be used in motivic homotopy. In the case of vector bundles of higher rank, the situation is more complex. Essentially by definition, it is clear that the (Nisnevich) classifying space BGL of the infinite general linear group classifies vector bundles in the simplicial homotopy category, as in topology. This basic fact can be extended to motivic homotopy in several directions. We will start by looking at the higher K-theory groups as defined by Quillen. Here the situation is exceptionally nice, and we have the following fundamental result of Morel and Voevodsky. See Morel and Voevodsky, 1999, §4, Th. 3.13.[(38)]

Theorem 2.39. — *For any smooth k-scheme X and any pair of non-negative integers (n, i), there exists an isomorphism, natural in X with respect to pullbacks:*

$$[S^n \wedge (\mathbb{P}^1_k)^{\wedge i} \wedge X_+, \mathbb{Z} \times \mathrm{BGL}]^{\mathbb{A}^1}_{\bullet} \simeq \mathrm{K}_n(X)$$

where $\mathbb{Z} \times \mathrm{BGL}$ is the product of the discrete k-space $\mathbb{Z}$ with the classifying space of the infinite general linear group, with base point given by the canonical base point of $\{0\} \times \mathrm{BGL}$.

This result uses many of the good properties of Quillen's higher K-theory: its presentation via the Q-construction, its $\mathbb{A}^1$-invariance over regular schemes also due to Quillen, Thomason-Trobaugh's Nisnevich descent theorem and lastly its $\mathbb{P}^1$-periodicity property (see *loc. cit.* for details).

On the other hand, the output is truly remarkable as K-theory is representable by a very simple k-space, $\mathbb{Z} \times \mathrm{BGL}$, which is in some sense the *group completion of the h-monoid* $\sqcup_n \mathrm{BGL}_n$ (see *loc. cit.* for more details).

Moreover, one directly reads off the above computation the following form of the classical (Bott) $\mathbb{P}^1$-periodicity property of algebraic K-theory:

$$\beta : \Omega_{\mathbb{P}^1_k}(\mathbb{Z} \times \mathrm{BGL}) \simeq \mathbb{Z} \times \mathrm{BGL}$$

where the $\mathbb{P}^1$-loop space was defined in Example 2.23. In particular, $\mathbb{Z} \times \mathrm{BGL}$ has a structure of infinite $\mathbb{P}^1$-loop space, which also gives a structure of infinite simplicial loop space. This gives the commutative h-group structure on the motivic pointed k-space $\mathbb{Z} \times \mathrm{BGL}$, which is compatible with the abelian group structure on K-theory via the above isomorphism.

---

(38) Beware that there is a gap in the proof of Morel and Voevodsky that was found by Schlichting and Tripathi (2015, Remark 8.5). It can be fixed by applying Theorem 8.2 of *op. cit.*

Example 2.40. — In his PhD thesis, Riou (2010) used the above representability result to deduce that all operations (λ-operations, Adams operations, products) on the functor K_0 lift to operations on the k-space $\mathbb{Z} \times \mathrm{BGL}$ seen in the $\mathbb{A}^1$-homotopy category $\mathrm{H}(k)$.

2.5.1. *Symmetric vector bundles and classifying spaces.* — From what we have already seen, inner products and quadratic invariants are central in motivic homotopy theory, in particular via the Grothendieck–Witt group of a field. In fact, it is classical that symmetric vector bundles over a field, or more generally a scheme, can be viewed as torsors under the infinite orthogonal group O. However this interpretation is only true étale locally — indeed Nisnevich torsors over a field are trivial. Thus the (Nisnevich) classifying space BO that was introduced in Example 2.2 is not adapted for this study. However, it is possible to define another k-space

$$\mathrm{B}_{\text{ét}}\mathrm{O} = L_{\text{ét}}\mathrm{BO}$$

that represents étale local torsors in the simplicial Nisnevich category: it is the localization of BO with respect to étale hypercovers.[(39)] By construction, we have for any smooth k-scheme X an isomorphism in the homotopy category $\mathrm{H}^{\mathrm{Nis}}(k)$:

$$[X, \mathrm{B}_{\text{ét}}\mathrm{O}]^{\mathrm{Nis}} = H^1_{\text{ét}}(X, \mathrm{O}).$$

Note that $\mathrm{GW}(k)$ is the group completion of the latter pointed set in the particular case $X = \mathrm{Spec}(k)$, with its natural monoid structure.

2.5.2. *Hermitian K-theory.* — In fact, one can extend the Grothendieck–Witt groups of a field along lines similar to those used in K-theory. This is a very rich topic, which was started independently by Karoubi (hermitian K-theory) and Ranicki (L-theory). In the next statement, we will use the construction that was finally done by Hornbostel and Schlichting, at the price of assuming that $\mathrm{char}(k) \neq 2$.[(40)] According to Schlichting (2017), Definition 9.1 (see also Proposition 9.3), given a line bundle L/X, one defines bigraded higher Grothendieck–Witt groups $\mathrm{GW}^{[i]}_n(X, L)$ which are contravariantly functorial — when $L = \mathcal{O}_X$, we just omit it from the notation. They satisfy analogous properties to Quillen's K-theory (contravariant, Nisnevich descent, an appropriate localization property). Here are some important distinctive features:

- *Periodicity*: there exist isomorphisms: $\mathrm{GW}^{[i+4]}_n(X) \simeq \mathrm{GW}^{[i]}_n(X)$.
- When X is affine, the abelian group $\mathrm{GW}^{[0]}_0(X)$ (resp. $\mathrm{GW}^{[2]}_0(X)$) is the Grothendieck group of vector bundles[(41)] with a non-degenerate symmetric (resp. symplectic) bilinear form.[(42)]

[(39)]The associated hypercomplete étale sheaf, viewed in the Nisnevich ∞-topos, in the terminology of Lurie (2009).

[(40)]In fact, there is now a general construction valid even in characteristic 2 (Calmès, Harpaz, and Nardin, 2024). See also Example 3.4(4).

[(41)]equivalently, finite rank locally free $\mathcal{O}_X$-module

[(42)]When X is not affine, one has to take care about the so-called metabolic forms. In general, $\mathrm{GW}^{[0]}_0(X)$ (resp. $\mathrm{GW}^{[2]}_0(X)$) does coincide with the definition of Knebusch (1977).

In particular, $\mathrm{GW}_0^{[0]}(k)$ really is the Grothendieck–Witt group that we have introduced earlier. Given the two previous definitions, one can state the quadratic analogue of the previous theorem which is due to Schlichting and Tripathi (2015).

Theorem 2.41. — *Let k be a field of characteristic different from 2. Then for any smooth k-scheme X and any pair of non-negative integers (n, i), there exists an isomorphism, natural in X with respect to pullbacks:*

$$[S^n \wedge (\mathbb{P}^1_k)^{\wedge i} \wedge X_+, \mathbb{Z} \times \mathrm{B_{\acute{e}t}O}]_\bullet^{\mathbb{A}^1} \simeq \mathrm{GW}_n^{[-i]}(X)$$

where $\mathbb{Z} \times \mathrm{B_{\acute{e}t}O}$ is defined as previously.

The proof is similar to that of the preceding theorem. Note that in this setting, the periodicity theorem now takes the form:

$$\Omega^4_{\mathbb{P}^1_k}(\mathbb{Z} \times \mathrm{B_{\acute{e}t}O}) \simeq \mathbb{Z} \times \mathrm{B_{\acute{e}t}O}$$

which aligns well with the classical topological case.

Remark 2.42. —
1. In addition, to both these representability theorems, one also has a geometric presentation of the k-space $\mathbb{Z} \times \mathrm{BGL}$ (resp. $\mathbb{Z} \times \mathrm{B_{\acute{e}t}O}$) by the infinite grassmannian (resp. the infinite orthogonal grassmannian). See the references already mentioned in both cases.
2. Though we now have a definition of hermitian K-theory which has all the expected properties in characteristic 2, thanks to Calmès, Harpaz, and Nardin (2024), the extension of the previous theorem is not clear at the moment.

2.5.3. *Motivic obstruction theory.* — If one restricts to a smooth affine k-scheme X, the two previous statements can be improved: the pointed set of isomorphism classes of rank n vector bundles over X is represented in the stable homotopy category by the k-space BGL_n. See Morel, 2012, Th. 8.1 for $n \neq 2$ and Asok, Hoyois, and Wendt, 2020 for a generalization to torsors over appropriate reductive algebraic goups.

Using this fact, Morel (2012) started a motivic homotopical study of algebraic vector bundles, modeled on the topological case. He set up a general obstruction theory, based on an appropriate notion of Postnikov tower on the motivic homotopy category (*loc. cit.* Appendix B), and used it (in the case of SL_n) to define an Euler class for vector bundles with trivial determinant which refines the usual top Chern class; *loc. cit.* Th. 8.14.

As proposed by Morel, this class was later compared by Asok and Fasel to a previous definition due to Barge and Morel (2000) and related to the so-called Chow-Witt groups, thoroughly studied in Fasel (2020) and subsequent works. This result also started a series of works by the two first named authors based on Morel's obstruction theory, which culminated in the proof of a conjecture of Murthy in dimension 4:

> Over an algebraically closed field of characteristic not 2, a rank 3 vector bundle over an algebraic smooth affine 4-fold splits off a trivial line bundle if and only if its third Chern class vanishes.

See Asok and Fasel, 2015, Theorem 2. In fact, the method used in proving this conjecture critically rests on the determination of the second non-trivial motivic homotopy sheaf $\underline{\pi}_n^{\mathbb{A}^1}(\mathbb{A}^n - \{0\})$ of the sphere $\mathbb{A}^n - \{0\}$, going one degree further than Example 2.37; *loc. cit.* Theorem 3.

The authors also highlight a plausible computation of $\underline{\pi}_n^{\mathbb{A}^1}(\mathbb{A}^n - \{0\})$, the *Asok–Fasel conjecture*, that would solve Murthy's conjecture in general. For more detail, the reader is advised to consult the report of Asok and Fasel (2023). We will see an unconditional stable version of this conjecture in 3.25.

3. Stable motivic homotopy

3.1. The Dold–Kan correspondance

In classical topology, the relations between homotopy and homology are governed by two main results:

- The *Dold–Kan correspondence*: it provides an explicit equivalence of categories between that of simplicial abelian groups and homologically non-negative complexes of abelian groups. This allows us to define an adjunction of homotopy/∞ categories:
$$\mathbb{Z}\colon \mathscr{H} \leftrightarrows \mathscr{D}_{\geq 0}(\mathrm{Ab}) : K$$
where the right-hand side category is the subcategory of the derived ∞-category of abelian groups made by complexes whose homology is non-negative. The left-adjoint functor, that we denoted by $\mathbb{Z}$ as the *(derived) free abelianisation functor*, is really the functor which to a space X associates the complex of singular chains $C_*(X, \mathbb{Z})$. Its right adjoint K has the classical property that for any abelian group A, and any integer $n \geq 0$,
$$K(A, n) = K(A[n])$$
where the left-hand side is the *n-th Eilenberg–MacLane space associated with* A, and, on the right-hand side, $A[n]$ denotes the complex with only one non-trivial term equal to A in homological degree n.[43]
- The *Hurewicz theorem*: for a simply connected pointed space X, it says that X is n-connected if and only if the complex $\mathbb{Z}(X) = C_*(X, \mathbb{Z})$ is concentrated in homological degree $> n$. That is,
$$(\forall i \leq n, \pi_i(X) = 0) \Leftrightarrow (\forall i \leq n, H_i(X, \mathbb{Z}) = 0)$$

This picture is beautiful, but incomplete. First, the homotopy category of the right-hand side is not triangulated; equivalently, the underlying ∞-category is not stable. Each Eilenberg–MacLane space $K(\mathbb{Z}, n)$ only represents a single homology group:

$$[X_+, K(\mathbb{Z}, n)] \simeq H^n(X, \mathbb{Z}). \tag{3.0.a}$$

[43] Note that $K(A, 0)$ is simply equal to A with the discrete topology.

Nevertheless, one can assemble the Eilenberg–MacLane spaces into a *spectrum*, using the natural suspension maps:[44]

$$\sigma_n \colon S^1 \wedge K(\mathbb{Z}, n) \to K(\mathbb{Z}, n+1).$$

This is the so-called Eilenberg–MacLane spectrum $H\mathbb{Z} = (\mathrm{K}(\mathbb{Z}, n), \sigma_n)_{n \geq 0}$. Spectra of this kind are the objects of an ∞-category $\mathscr{SH}$, called the *stable homotopy category*,[45] which allows us to complete the picture drawn by the Dold–Kan correspondence as follows:

(3.0.b)
$$\begin{array}{ccc} \mathscr{H} & \underset{K}{\overset{\mathbb{Z}}{\rightleftarrows}} & \mathscr{D}_{\geq 0}(\mathrm{Ab}) \\ {\scriptstyle \Omega^\infty}\uparrow\downarrow{\scriptstyle \Sigma^\infty} & & {\scriptstyle \tau_{\geq 0}}\uparrow\downarrow{\scriptstyle \iota} \\ \mathscr{SH} & \underset{H}{\overset{\mathbb{Z}}{\rightleftarrows}} & \mathscr{D}(\mathrm{Ab}). \end{array}$$

The adjoint pair $(\Sigma^\infty, \Omega^\infty)$ is made of the infinite suspension spectrum and infinite loop space functors. On the right-hand side, $\tau_{\geq 0}$ is the truncation functor in *homological notations*, and ι the obvious inclusion.

One of the fundamental ideas of motivic homotopy, which is certainly the main driving insight of Voevodsky, is that this picture should exist in algebraic geometry as well. Moreover, the derived ∞-category of abelian groups would be replaced by the derived ∞-category of motivic complexes whose existence was conjectured by Beilinson. Finally, motivic cohomology would assume the universal role of singular cohomology.

3.2. $\mathbb{P}^1$-stabilization

3.2.1. *Stabilization.* — The classical stable homotopy category can be obtained along classical lines with an explicit model category.[46] Here the ∞-categorical framework is much more efficient. The ∞-category $\mathscr{SH}$ actually satisfies two universal properties:

1. It is the universal symmetric monoidal ∞-category which is the target of a symmetric monoidal functor with source $\mathscr{H}$ which sends the sphere S^1 to a $\otimes$-invertible object.
2. It is the universal ∞-category with an exact functor with source $\mathscr{H}_\bullet$ and whose target is stable as an ∞-category.

The notion of stability in ∞-categories is one of the great advantages of this formalism. It replaces the more classical notion of triangulated categories in homological algebra. Strikingly, it is actually a property and not an additional structure. Explicitly, an ∞-category $\mathscr{C}$ is *stable* if it admits all finite limits and colimits. In addition, a square

(44)For example, they come out of the universal property given by the functorial isomorphism (3.0.a), which gives a unique weak equivalence (isomorphism in ∞-categorical terms): $\Omega K(\mathbb{Z}, n+1) = K(\mathbb{Z}, n)$.
(45)We refer the reader (for example) to Lurie (2017, §1.4) for the construction of this ∞-category.
(46)In fact, there are two concurrent model categories in order to get a symmetric monoidal model category: S-modules after Elmendorf, Kriz and May and symmetric spectra after Hovey, Smith and Shipley.

is cartesian if and only if it is cocartesian. These two properties suffice to ensure that the homotopy category $\mathrm{Ho}\,\mathscr{C}$ admits a canonical structure of triangulated category. We refer the reader to the beautiful and efficient presentation of Lurie (2017, §1.1).[47]

Let us quickly comment on those two constructions. The first one comes from the classical construction of spectra, recall at the beginning. The general construction, which consists in adding a $\otimes$-inverse of an object in a presentable monoidal ∞-category has been first written down in Robalo (2015, §2.1), to which we refer. The second one has been obtained in Lurie (2017, §1.4.2): and *pointed ∞-category with finite limits* admits a universal stabilization (see *loc. cit.* for the precise formulation).

As announced earlier, we will follow the same path in motivic homotopy theory. The main difference is that we have several possible spheres (see Section 1.1) to choose from when considering the analogue of construction (1). The chosen sphere, guided by Beilinson's conjectures on motivic complexes, is the projective line $\mathbb{P}^1_k$.

DEFINITION 3.1. — *The motivic stable homotopy ∞-category $\mathscr{SH}(k)$ over k is the universal ∞-category obtained from $\mathscr{H}(k)$ by tensor-inverting the k-space $\mathbb{P}^1_k$. This construction is called the* $\mathbb{P}^1$-stabilization. *The objects of $\mathscr{SH}(k)$ are called* motivic spectra.

In particular, we have a pair of adjoint functors

$$\Sigma^\infty\colon\ \mathscr{H}_\bullet(k) \leftrightarrows \mathscr{SH}(k) : \Omega^\infty$$

where Σ^∞ is a monoidal functor such that $\Sigma^\infty\,\mathbb{P}^1$ is $\otimes$-invertible.

According to the decomposition of Example 2.20, both spheres S^1 and $\mathbb{G}_m$ become $\otimes$-invertible in $\mathscr{SH}(k)$. In particular, the ∞-category $\mathscr{SH}(k)$ is stable. We use two conventions for the various powers of spheres:

$$S^{n,i} = \mathbb{1}(i)[n] = (\Sigma^\infty S^1)^{\otimes n-i} \otimes (\Sigma^\infty \mathbb{G}_m)^{\otimes i}. \tag{3.1.a}$$

According to the construction of $\mathscr{SH}(k)$ as the countable limit over the $\mathbb{P}^1$-loop space functor $\Omega_{\mathbb{P}^1}$ of Example 2.23, one deduces for any k-spaces $\mathcal{X}$, $\mathcal{Y}$, the following computation:

$$[\Sigma^\infty \mathcal{X}_+, \Sigma^\infty \mathcal{Y}_+]_{\mathrm{SH}(k)} \coloneqq \varinjlim_n\, [(\mathbb{P}^1_k)^{\wedge n} \wedge \mathcal{X}_+, (\mathbb{P}^1_k)^{\wedge n} \wedge \Sigma^\infty \mathcal{Y}_+]^{\mathbb{A}^1}_\bullet \tag{3.1.b}$$

where the left-hand side stands for morphisms in the homotopy category $\mathrm{SH}(k) = \mathrm{Ho}\,\mathscr{SH}(k)$. We simply drop the subscript when the context is clear. A morphism of pointed k-spaces $f\colon \mathcal{X} \to \mathcal{Y}$ will be called a *stable weak $\mathbb{A}^1$-equivalence* if it becomes an isomorphism after application of the functor Σ^∞. According to the previous computation, this is equivalent to ask that there exists an integer $n > 0$ such that $(\mathbb{P}^1_k)^{\wedge n} \wedge f$ is a weak $\mathbb{A}^1$-equivalence.

Example 3.2. — 1. Wickelgren (2016) showed that, over the base field $\mathbb{Q}$, $\mathbb{G}_m \vee \mathbb{G}_m$ and $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ are stably weakly $\mathbb{A}^1$-equivalent but not weakly $\mathbb{A}^1$-equivalent.

[47] See Proposition 1.1.3.4 of *op. cit.* for the definition we have adopted in this paragraph.

2. A pointed smooth k-scheme X is *stably* $\mathbb{A}^1$*-contractible* if the structural map $X \to \mathrm{Spec}(k)$ is a stable weak $\mathbb{A}^1$-equivalence. Hoyois, Krishna, and Østvær (2016) proved that Koras–Russel threefolds of the second kind, for integer positive integers m, n, α_1, α_2, such that $\alpha_1, \alpha_2 \geq 2$, $n\alpha_1$ and α_2 are coprime, and a unit $a \in k^\times$:
$$K_{m,n,\alpha_i,a} \coloneqq \{(x^n + y^{\alpha_1})^m z = t^{\alpha_2} + ax = 0\} \subset \mathbb{A}^4_{\mathbb{C}}$$
are stably weakly $\mathbb{A}^1$-contractible. It is not known whether they are weakly $\mathbb{A}^1$-contractible or not.

According to the decomposition $\mathbb{P}^1_k = S^1 \wedge \mathbb{G}_m$, both the spheres S^1 and $\mathbb{G}_m$ become invertible in $\mathscr{SH}(k)$. This implies that the ∞-category $\mathscr{SH}(k)$ is stable, and one can define on a motivic spectrum $\mathbf{E}$ shifts and twists by integers n and i:
$$\mathbf{E}(i)[n] \coloneqq \mathbf{E} \otimes (\Sigma^\infty S^1)^{\otimes n-i} \otimes (\Sigma^\infty \mathbb{G}_m)^{\otimes i}.$$

Definition 3.3. — *Let* $\mathbf{E}$ *be a motivic spectrum over* k*. One defines the cohomology of a smooth* k*-scheme* X *in bidegree* (n, i) *represented by* $\mathbf{E}$ *by the formula:*
$$\mathbf{E}^{n,i}(X) = [\Sigma^\infty X_+, \mathbf{E}(i)[n]] .$$
We say that $\mathbf{E}$ *is a* ring spectrum *(resp.* E_∞-ring spectrum*) if* $\mathbf{E}$ *admits a commutative monoid structure in the homotopy category* $\mathrm{Ho}\,\mathscr{SH}(k)$ *(resp. is a commutative monoid in the monoidal* ∞*-category* $\mathscr{SH}(k)$*). In that case, one defines a product on the cohomology theory represented by* $\mathbf{E}$ *as usual.*

All ring spectra appearing in the sequel have in fact a structure of E_∞-ring spectra. In particular, we will say abusively ring spectra for E_∞-ring spectra.

Example 3.4. — 1. Classical cohomology theories in algebraic geometry are all representable by ring spectra in $\mathscr{SH}(k)$: in characteristic 0, integral Betti cohomology (see also below), algebraic de Rham cohomology in positive characteristic $p > 0$, rigid cohomology and in all characteristics integral ℓ-adic étale cohomology ($\ell \in k^\times$). With rational coefficients, all these theories are in fact instances of a mixed Weil cohomology as defined by Cisinski and Déglise (2012). Their representability is proved in *loc. cit.* Prop. 2.1.6, Using the $\mathbb{A}^1$-derived category of 3.2.3.

2. Motivic cohomology of smooth k-schemes with integral coefficients is represented by the so-called *Eilenberg–MacLane motivic spectrum* $\mathbf{H}_{\mathrm{M}}\mathbb{Z}$ defined by Voevodsky (1998, §6.1). In particular, one gets a computation in terms of Bloch's higher Chow groups:
$$\mathbf{H}^{n,i}_{\mathrm{M}}(X) = \mathrm{CH}^i(X, 2i - n).$$
The construction of Beilinson's conjectural motivic cohomology theory was in fact Voevodsky's main motivation for introducing motivic homotopy theory.[48]

[48] Given an arbitrary ring R, one simply gets the R-linear Eilenberg–MacLane motivic spectrum $\mathbf{H}_{\mathrm{M}}R$ by applying Voevodsky's construction with R-linear finite correspondences, obtained by naively tensoring with R over $\mathbb{Z}$ — using that the groups of finite correspondences are free abelian groups.

3. Algebraic K-theory (resp. higher Grothendieck–Witt groups) is representable by a ring spectrum **KGL** (resp. **GW**) according to Bachmann and Hoyois (2021, Th. 15.22) (resp. Calmès, Harpaz, and Nardin, 2024).[(49)] According to the periodicity isomorphisms, one has the following canonical identifications:

$$\mathbf{KGL}^{n,i}(X) = \mathrm{K}_{2i-n}(X)$$
$$\mathbf{GW}^{n,i}(X) = \mathrm{GW}^{[i]}_{2i-n}(X).$$

4. Algebraic cobordism is representable by the Thom spectrum **MGL**, as first defined by Voevodsky (1998) (see Bachmann and Hoyois (2021, Th. 16.19) for the E_∞-structure). Levine and Morel (2007) provided a more concrete notion of algebraic cobordism which was presented in this seminar by Loeser (2003). In characteristic 0, it was shown by Levine that the latter theory is isomorphic to the $\mathbb{Z}$-graded part of Voevodsky's theory: $\mathbf{MGL}^{2*,*}(-)$. Levine's argument was conditional on the Hopkins-Morel theorem, which was proved by Hoyois (2015); see also *loc. cit.* Cor. 8.15.

Remark 3.5. — One also find the name *hermitian K-theory* and the notation **KQ** for the ring spectrum GW (see Section 3.4.3). Both notations have their advantages.

3.2.2. *Topological realizations.* — We consider again the notation of Section 2.2.2. Then one deduces the following realization:

– Given a complex embedding $\sigma\colon k \to \mathbb{C}$, one obtains the σ-Betti realization $\rho_\sigma\colon \mathscr{SH}(k) \to \mathscr{SH}$, which sends $\Sigma^\infty X_+$ to $\Sigma^\infty X(\mathbb{C})_+$. This is obvious according to *loc. cit.*, as the (unstable) realization functor maps the motivic sphere $\mathbb{P}^1_k$ to S^2. We refer the interested reader to Ayoub (2010) for Betti realizations over a base complex scheme.

– Let us recall that one defines the genuine stable homotopy category of $\mathbb{Z}/2$-equivariant homotopy category by $\otimes$-inverting both the simplicial sphere S^1 and the sphere S^1 with the antipodal action of $\mathbb{Z}/2$, usually denoted by S^σ.[(50)]

  Given a real embedding $\sigma\colon k \to \mathbb{R}$, from the unstable realization attached to σ, one deduces the σ-Betti equivariant realization $\rho_\sigma^{\mathbb{Z}/2}\colon \mathscr{SH}(k) \to \mathscr{SH}_{\mathbb{Z}/2}$, which sends $\Sigma^\infty X_+$ to $\Sigma^\infty X(\mathbb{C})_+$ with the action of $\mathbb{Z}/2$ given by the complex conjugation. We refer the reader to Heller and Ormsby (2018, §4.4) for the explicit construction.

  After taking homotopy fixed points (with respect to $\mathbb{Z}/2$), one deduces the σ-Betti realization $\rho_\sigma\colon \mathscr{SH}(k) \to \mathscr{SH}$, which sends $\Sigma^\infty X_+$ to $\Sigma^\infty X(\mathbb{R})_+$.

(49)The first proof of this representability result (regardless of the E_∞-ring structure) was given by Riou (2010, §5.2), based on Theorem 2.39. For hermitian K-theory, the representability result was first proved by Hornbostel (2005), in characteristic not 2 and without the ring structure.

(50)The adjective genuine here refers to the fact one has $\otimes$-inverted both the spheres S^1 and S^σ.

Example 3.6. — In both the complex and real cases, ρ_σ is monoidal and admits a right adjoint that we will denote by $\rho_{\sigma*}$. When σ is a complex embedding, and fixing an arbitrary (commutative) ring R, one deduces a motivic ring spectrum:[(51)]

$$\mathbf{H}_{\mathrm{B}}^{\sigma} R \coloneqq \rho_{\sigma*}(\mathrm{H}R)$$

which, by the adjunction property, represents the the R-linear Betti cohomology over k attached to the embedding σ. When $\sigma = \mathrm{Id}_{\mathbb{C}}$, we simply denote by $\mathbf{H}_{\mathrm{B}} R$ this ring spectrum.

3.2.3. *The $\mathbb{A}^1$-derived category.* — The Dold–Kan correspondence recalled in Section 3.1 easily extends to the motivic setting. One first considers $\mathscr{D}(Sh(k,\mathbb{Z}))$, the derived ∞-category of abelian Nisnevich sheaves on Sm_k.[(52)] Then one follows the recipe used to construct the stable motivic homotopy category, replacing X_+ by the abelian sheaf $\mathbb{Z}(X)$ represented by a smooth k-scheme X: one considers its $\mathbb{A}^1$-localization as in Definition 2.14, and then its $\mathbb{P}^1$-stabilization as in the preceding definition. The resulting category is denoted by $\mathscr{D}_{\mathbb{A}^1}(k)$ and called, after Morel, the ($\mathbb{P}^1$-stable) $\mathbb{A}^1$-derived category over k. The Dold–Kan correspondence formally extends as an adjunction:

$$\mathbb{Z}_{\mathbb{A}^1} \colon \mathscr{SH}(k) \leftrightarrows \mathscr{D}_{\mathbb{A}^1}(k) : \underline{\mathrm{H}}_{\mathbb{A}^1} .$$

Remark 3.7. — One of the main theorems of Morel (2012) that we have not yet discussed, is the motivic analogue of the Hurewicz theorem, which discusses the properties of the composite functor $\mathscr{H}_\bullet(k) \xrightarrow{\Sigma^\infty} \mathscr{SH}(k) \xrightarrow{\mathbb{Z}_{\mathbb{A}^1}} \mathscr{D}_{\mathbb{A}^1}(k)$. To avoid inflating indefinitely this presentation, we refer the reader to *loc. cit.* Section 6.3.

3.2.4. *Motivic complexes.* — Let k be a perfect field. Voevodsky's theory of motivic complexes was exposed in the Bourbaki seminar by Friedlander (1997). For our needs, we need to consider a slightly bigger category, the "big" derived ∞-category $\mathscr{DM}(k)$ of mixed motives over k. As above, it is obtained by applying the $\mathbb{A}^1$-localization and $\mathbb{P}^1$-stabilization procedure to the the monoidal derived ∞-category of the abelian category of sheaves with transfers over k (*loc. cit.* Section 2).[(53)]

Then one can refine the previous Dold–Kan correspondence and introduce the following *motivic realization functor*:

$$M \colon \mathscr{SH}(k) \xrightarrow{\mathbb{Z}_{\mathbb{A}^1}} \mathscr{D}_{\mathbb{A}^1}(k) \to \mathscr{DM}(k).$$

[(51)] the E_∞-structure comes for free from the fact the (∞-)functor $\rho_{\sigma*}$ is weakly monoidal, as a right adjoint of a monoidal (∞-)functor;

[(52)] This can be simply obtained as the localization of the nerve of the category of complexes of such sheaves with respect to quasi-isomorphisms, but then the monoidal structure is not obvious. A classical way of constructing this monoidal ∞-category is by using model structures (see e.g. Cisinski and Déglise, 2009). A more elegant procedure is to identify $\mathscr{D}(\mathrm{Sh}(k,\mathbb{Z}))$ with the commutative monoid objects of the Nisnevich ∞-topos associated the smooth site Sm_k, equipped with its cartesian monoidal structure. Then the desired monoidal structure comes from Lurie (2017, Ex. 3.2.4.4).

[(53)] This procedure can be realized via an explicit monoidal model structure, according to Röndigs and Østvær (2008, Th. 11) or Cisinski and Déglise (2009, Def. 11.1.1).

The last map is derived from the functor which "adds transfers" to an abelian Nisnevich sheaf.[(54)] The functor M is monoidal, and sends the infinite suspension $\mathbb{P}^1$-spectrum $\Sigma^\infty X_+$ to the motivic complex $M(X)$.

The following beautiful theorem, due to Bachmann (2018b, Th. 25) is an avatar of the Hurewicz theorem, which in some sense strengthen the results of Morel recalled in the above remark.

THEOREM 3.8. — *Let k be a perfect field with finite 2-cohomological dimension. Then the motivic realization functor $M\colon \mathscr{SH}(k) \to \mathscr{DM}(k)$ is conservative when restricted to compact motivic spectra, i.e., it detects isomorphisms between such objects.*

Classically in this context, a motivic spectrum $\mathbf{E}$ is compact if the functor $[\mathbf{E}, -]$ commutes with coproducts. A nice feature of the Nisnevich topology is that this condition is equivalent to ask that $\mathbf{E}$ is in the subcategory of $\mathscr{SH}(k)$ generated by extensions and finite sums of motivic spectra of the form $\Sigma^\infty X_+(i)[n]$. One says that $\mathbf{E}$ is *constructible* or *geometric*.

Remark 3.9. — Formally, the monoidal functor M admits a right adjoint $\underline{\mathrm{H}}_M\colon \mathscr{DM}(k) \to \mathscr{SH}(k)$. It follows from the adjunction property that $\underline{\mathrm{H}}_M(\mathbb{1})$ is a motivic spectrum which represents motivic cohomology. For an arbitrary ring R, one gets the R-linear version $\mathbf{H}_{\mathrm{M}}R$ by using the ∞-category of R-linear motivic complexes. In fact, coming back to the underlying model categories, it is exactly the motivic spectrum described by Voevodsky (1998, §6.1). The advantage of this presentation is that, as a right adjoint of a monoidal functor, $\underline{\mathrm{H}}_M$ is weakly monoidal. This immediately implies that $\mathbf{H}_{\mathrm{M}}\mathbb{Z} = \underline{\mathrm{H}}_M(\mathbb{1})$ is an E_∞-ring spectrum.

Moreover, this allowed Röndigs and Østvær (2008, see Th. 58, Th. 68) to build a monoidal functor $\mathbf{H}_{\mathrm{M}}\mathbb{Z}-\mathrm{mod} \to \mathscr{DM}(k)$ with, as source, the category of modules over the motivic ring spectrum, and to deduce that it is an equivalence of ∞-categories when k is of characteristic 0, or after tensoring with $\mathbb{Q}$.

3.3. Motivic stable stems and Morel degree

3.3.1. *Stable stems.* — According to the classical terminology, due to Freudenthal (1937) who used the German word *n-Stamm*,[(55)] one defines for an integer $n \geq 0$ the *n-stem* as the n-th extension group of the sphere spectrum:

$$\pi_n^{\mathrm{S}} := [\Sigma^\infty S^n, \Sigma^\infty S^0]_{\mathrm{SH}} = \varinjlim_r [S^{n+r}, S^r]_{\mathrm{H}_\bullet} = \pi_{n+r}(S^r) \text{ for } r \geq n+2.$$

The last isomorphism follows from Freudenthal suspension theorem, *loc. cit.* According to well-known facts from algebraic topology, π_*^{S} is a non-negatively graded ring and $\pi_0^{\mathrm{S}} = \mathbb{Z}$. Moreover, Serre's finiteness theorem implies that, for $n > 0$, π_n^{S} is finite.

[(54)]see Röndigs and Østvær (2008, §2.2.1) or Cisinski and Déglise (2019, §11.2.16).
[(55)]see the first paragraph of *loc. cit.*

Definition 3.10. — *For any pair of integers* $(n, i) \in \mathbb{Z}^2$*, and any field* k*, the* motivic (n,i)-stable stem *over* k *is*

$$\pi^k_{n,i} := [S^{n,i}, S^{0,0}]_{\mathrm{SH}(k)}.$$

The motivic stable stem is a rich invariant of fields. It can be connected to other important invariants. First, the constant simplicial sheaf functor induces a morphism of $\mathbb{Z}$-graded ring

$$\pi^{\mathrm{S}}_* \to \pi^k_{*,0}. \tag{3.10.a}$$

Moreover, the motivic realization functor induces a morphism of bigraded rings:

$$\pi^k_{*,*} \to \mathbf{H}^{-*,-*}_{\mathrm{M}}(k) \tag{3.10.b}$$

inverting all indices on the right, as we have used homological (cohomological) conventions for the left-hand (resp. right-hand) side.

The determination of the motivic stable stem can be reduced to computations in the (unstable) motivic homotopy category according to formula (3.1.b). Therefore, as a corollary of Morel's fundamental computation, one knows at least some portion of these groups.

Theorem 3.11. — *For any field* k *and all integers* $n \geq i$*, one gets:*

$$\pi^k_{n,i} = \begin{cases} \mathrm{K}^{\mathrm{MW}}_{-n}(k) & \text{if } n = i, \\ 0 & \text{if } n < i. \end{cases}$$

This result should be considered as the analogue of the description of the negatively graded part of the stable stem. In degree 0, one gets the fundamental identification: $\pi^k_{0,0} = \mathrm{GW}(k)$.

The morphism (3.10.a) in degree 0 is the obvious canonical map $\mathbb{Z} \to \mathrm{GW}(k)$. In particular, it is not an isomorphism for non-quadratically closed fields.

In fact, one should be careful that the constant sheaf functor, left adjoint to the evaluation at the base field k, which is a fiber functor of the Nisnevich site Sm_k, does induce a fully faithful functor $\mathscr{H} \to \mathscr{H}(k)$.[56] But the $\mathbb{P}^1$-stabilization procedure has introduced a non-trivial phenomena and the induced functor $c\colon \mathscr{SH} \to \mathscr{SH}(k)$ is not fully faithful in general, as seen by the previous example. However, one has the following remarkable result of Levine (2014).

Theorem 3.12. — *When* k *is an algebraically closed field of characteristic* 0*, the map* $\pi^{\mathrm{S}}_* \to \pi^k_{*,0}$ *of* (3.10.a) *is an isomorphism.*

One deduces that under the above assumption on k, the above functor c is in fact fully faithful: this easily follows as $\mathscr{SH}$ is generated by the sphere spectrum. This theorem is analogous to Suslin's rigidity theorem in algebraic K-theory, and in fact, it uses some version of rigidity for Suslin homology, due to Suslin and Voevodsky.[57] The

(56) We leave this as an exercise to the reader.

(57) See Section 4.2 for more information on this point.

core argument of the proof consists in analyzing two slice spectral sequences, and most notably, proving their strong convergence.

3.3.2. *Morel's plus-minus decomposition.* — Identifying $\pi^k_{0,0}$ with $\mathrm{GW}(k)$ and using notation from Sections 1.2.1 and 2.4.1, we consider $\epsilon = -\langle -1\rangle$ as an endomorphism of the unit $\mathbb{1}$ of $\mathscr{SH}(k)$. Then it follows that ϵ is an idempotent automorphism of $\mathbb{1}$. After inverting 2 in $\mathscr{SH}(k)$, one deduces, two complementary orthogonal projectors of the object $\mathbb{1}[1/2]$:

$$p_+ = \frac{1-\epsilon}{2},\ p_- = \frac{1+\epsilon}{2}.$$

This implies that any motivic spectrum $\mathbf{E}$ on which 2 is invertible decomposes as

$$\mathbf{E} = \mathbf{E}^+ \oplus \mathbf{E}^-, \tag{3.12.a}$$

where $\mathbf{E}^+$ (resp. $\mathbf{E}^-$) is the image of p_+ (resp. p_-) and called the plus-part (resp. minus-part) of $\mathbf{E}$. By construction, the action of ϵ on $\mathbf{E}^+$ (resp. $\mathbf{E}^-$) becomes $(+1)$ (resp. (-1)). The following result gives a beautiful interpretation of the rational motivic stable stem.

Theorem 3.13. — *For an arbitrary field k and all integers $(n,i)\in\mathbb{Z}^2$, the decomposition* (3.12.a) *induces the following identifications:*

$$\begin{aligned}
\pi^k_{n,i}\otimes\mathbb{Q} &= \left(\pi^{k+}_{n,i}\otimes\mathbb{Q}\right),\oplus\left(\pi^{k-}_{n,i}\otimes\mathbb{Q}\right),\\
\pi^{k+}_{n,i}\otimes\mathbb{Q} &\simeq \mathbf{H}_{\mathrm{M}}^{-n,-i}(k)_{\mathbb{Q}} = K^{(-i)}_{n-2i}(k)_{\mathbb{Q}}\\
\pi^{k-}_{n,i}\otimes\mathbb{Q} &\simeq \begin{cases} W(k)_{\mathbb{Q}} & \text{if } n=i,\\ 0 & \text{if } n\neq i.\end{cases}
\end{aligned}$$

The first isomorphism is induced by the map (3.10.b)*, while the second one is induced by that of the Theorem 3.11. One deduces that $\pi^{k-}_{n,i}$ is torsion whenever $n\neq i$, or in all cases if (-1) is a sum of squares in k.*[58]

For the plus part, we refer the reader to Cisinski and Déglise (2019, Theorem 16.2.13). This result was first announced by Morel, and indeed, the proof of *loc. cit.* critically rests on Morel's previous theorem, as well as on the homotopy t-structure (see Section 3.4). It also uses the rational motivic spectrum **KGL** and its decomposition under the Adams-operation, as was obtained by Riou (2010). The final argument is a description of the projector on the plus-part as taking the homotopy cofiber with respect to η.

The computation of the minus-part was obtained by Ananyevskiy, Levine, and Panin (2017, Theorem 5). It also uses the homotopy t-structure, and the description on the projector on the minus-part by the so-called operation of inverting η.

(58) e.g. if k is quadratically closed or a field of positive characteristic.

3.3.3. *The action of the algebraic Hopf map.* — Motivated by the preceding sketch of proof, we show how Morel's plus-minus decomposition is related to the algebraic Hopf map. Consider a motivic spectrum $\mathbf{E} = \mathbf{E}[1/2]$ as in Section 3.3.2. Let us consider the algebraic Hopf map η, geometrically defined in Section 2.31, as a morphism $\eta\colon S^{1,1} = \mathbb{1}(1)[1] \to S^{0,0} = \mathbb{1}$ of motivic spectra. Let us formulate some relations which hold in the Milnor–Witt ring of k:

$$\epsilon.\eta = \eta,$$
$$\rho.\eta = -1 - \epsilon, \text{ where } \rho = [-1].$$

One deduces that the action of η on $\mathbf{E}^+$ (resp. E^-) becomes trivial (resp. invertible). Moreover, one gets the following (homotopy) exact sequence in $\mathscr{SH}(k)$ (obtained by tensoring the analogous sequence for $\mathbf{E} = \mathbb{1}[1/2]$):

$$\mathbf{E}(1)[1] \xrightarrow{\eta} \mathbf{E} \to \mathbf{E}^+.$$

The minus part is obtained by formal inversion of η

$$\mathbf{E}^- = \mathbf{E}[\eta^{-1}],$$

where one defines $\mathbf{E}[\eta^{-1}]$ as the homotopy colimit of the following tower:

$$\mathbf{E} \xrightarrow{\eta} \mathbf{E}(-1)[-1] \xrightarrow{\eta} \mathbf{E}(-2)[-2] \dots$$

Remark 3.14. — In fact, the previous theorem can be extended as a computation of the whole rational motivic stable ∞-category $\mathscr{SH}(k) \otimes \mathbb{Q}$. Indeed, the plus-minus decomposition extends at the ∞-categorical level, and one can identify its plus-part with the ∞-category of rational mixed motivic complexes $\mathscr{DM}(k) \otimes \mathbb{Q}$ and its minus-part as the modules over the *rational unramified Witt sheaf*. These computations are motivic extensions of the fact that the stable Dold–Kan adjunction $(\mathbb{Z}, H)$ of (3.0.b) induces an equivalence after rationalization.

We refer the interested reader to Cisinski and Déglise (2019, Theorem 16.2.13) and Ananyevskiy, Levine, and Panin (2017, Theorem 7). Let us further illustrate these techniques with the following theorem of Bachmann (2018a, Theorem 35 and Proposition 36).

Theorem 3.15. — *Consider the endomorphism $\rho\colon \mathbb{1} \to \mathbb{1}(1)[1]$ as defined above, associated with the unit $(-1) \in k^\times$.*

The localization procedure with respect to ρ, described above, allows to define the stable motivic ∞-category $\mathrm{SH}(k)[\rho^{-1}]$ of ρ-inverted objects. Then the real realization functor induces an equivalence of monoidal ∞-categories:

$$\mathscr{SH}(\mathbb{R})[\rho^{-1}] \to \mathscr{SH}\,.$$

Further, for an arbitrary field k, one obtains an equivalence

$$\mathscr{SH}(k)[\rho^{-1}] \to \mathscr{SH}(k_{\text{rét}}),$$

where the right-hand side is the S^1-stable ∞-category associated with Scheiderer real-étale site of k.

As a corollary (*loc. cit.* Cor. 42), one obtains the following description of the ρ-inverted motivic stable stem of a real closed field k, as a bigraded ring:

$$\pi^k_{**}[\rho^{-1}] \simeq \pi^{\mathrm{S}}_*[\rho^{-1}]\,.$$

Here the letter ρ on the right-hand side stands for a variable of bidegree $(-1,-1)$.

3.4. Stable motivic homotopy sheaves

Following Morel, as in the unstable case, we adopt the following definition.

Definition 3.16. — *Let $\mathbf{E}$ be a motivic spectrum over k. For any integer $n \in \mathbb{Z}$, the n-th* motivic stable homotopy sheaf *of $\mathbf{E}$ is the $\mathbb{Z}$-graded (Nisnevich) sheaf $\underline{\pi}^{\mathbb{A}^1}_n(\mathbf{E})$ on Sm_k associated with the following presheaf*

$$V \mapsto \Big[\Sigma^\infty V_+, \mathbf{E}(r)[r-n]\Big], r \in \mathbb{Z}.$$

If we want to refer to the $\mathbb{Z}$-grading of this kind of sheaves, one uses the notation $\underline{\pi}^{\mathbb{A}^1}_n(\mathbf{E})_r$. This numbering can seem awkward but it will be justified here below.

Example 3.17. —
1. Recall from Section 1.3.3 that a separated function field F/k defines a fiber functor of the smooth site on Sm_k. By a base change argument, one obtains the following computation:
$$\underline{\pi}^{\mathbb{A}^1}_n(\mathbb{1})_r(F) = \pi^F_{n-r,-r}.$$
In particular, one gets $\underline{\pi}^{\mathbb{A}^1}_0(\mathbb{1})_r(F) = \mathrm{K}^{\mathrm{MW}}_r(F)$, according to Theorem 3.11.
2. Let $\mathbf{E}$ be a motivic spectrum, and $\mathbf{E}^{**}$ be the associated bigraded cohomology theory. Then, the motivic sheaves $\underline{\pi}^{\mathbb{A}^1}_n(\mathbf{E})$ have already famously been considered by Bloch and Ogus (1974), in their extension of the Gersten conjecture. More precisely, one has the relation
$$\underline{\pi}^{\mathbb{A}^1}_n(\mathbf{E})_r = \underline{\mathbf{E}}^{r-n,r}$$
where we have denoted by $\underline{\mathbf{E}}^{n-r,r}$ what is usually called the *unramified cohomology* associated with $\mathbf{E}$ (denoted with curly letters in the first page *loc. cit.*). As an example, one deduces that, over any perfect field k
$$\underline{\pi}^{\mathbb{A}^1}_0(\mathbf{H}_{\mathrm{M}}\mathbb{Z}) = \underline{\mathrm{K}}^{\mathrm{M}}_*,$$
where the right-hand side is the unramified Milnor K-theory sheaf (see Paragraph 2.4.2).

Remark 3.18. — Consider the notations of second point in the preceding example. In general, with the help of the six functors formalism, $\mathbf{E}^{**}$ can indeed be extended to a Poincaré duality theory with support in the sense of Bloch and Ogus (1974), except that one has to modify properties (1.3.3) (Fundamental class) and (1.3.4) (Poincaré duality), in order to take into account twists by Thom spaces (defined in Example 2.20). We refer the reader to Déglise, Jin, and Khan (2021) for this extension — summarized in the notion of (twisted) bivariant theory. We have already seen the need for Thom spaces, when considering particular forms of the Gersten resolution of an unstable motivic homotopy sheaf in Section 2.3.2.

The following result, analogous to Theorem 2.27, is also due to Morel (2005, Corollary 6.2.9), using Hogadi and Kulkarni (2020) when the base field k is finite.

THEOREM 3.19. — *Let k be a perfect field. Then for any motivic spectrum $\mathbf{E}$ over k and any integer $n \in \mathbb{Z}$, the $\mathbb{Z}$-graded sheaf $\underline{\pi}_n^{\mathbb{A}^1}(\mathbf{E})$ is unramified and strictly $\mathbb{A}^1$-invariant (as in Theorem 2.27).*

The key point of the proof is the so called *stable $\mathbb{A}^1$-connectivity theorem*, which states that the $\mathbb{A}^1$-localization functor on the ∞-category of S^1-spectra associated with the ∞-category of (Nisnevich) sheaves on Sm_k (*aka* the stabilization of the Nisnevich ∞-topos on Sm_k) respects connectivity in the sense of the canonical (Postnikov) t-structure. See Morel, 2005, Theorem 6.1.8.

3.4.1. *Homotopy modules.* — The $\mathbb{Z}$-graduation of a stable motivic homotopy sheaf $F_* = \underline{\pi}_n^{\mathbb{A}^1}(\mathbf{E})_*$ is not arbitrary. First, note that one gets a tautological split homotopy exact sequence of motivic spectra:

$$\Sigma^\infty \operatorname{Spec}(k)_+ \xrightarrow{s_{1*}} \Sigma^\infty(\mathbb{G}_m)_+ \to \Sigma^\infty \mathbb{G}_m = \mathbb{1}(1)[1].$$

By definition of Voevodsky's (-1)-construction, Section 2.3.2(2), this yields a canonical isomorphism for any integer $n \in \mathbb{Z}$:

$$\epsilon_n \colon (F_n)_{-1} \to F_{n-1}. \tag{3.19.a}$$

DEFINITION 3.20. — *A* homotopy module *over k is a $\mathbb{Z}$-graded Nisnevich sheaf F_* on Sm_k which is strictly $\mathbb{A}^1$-invariant and equipped with an isomorphism ϵ_* of the above form. Morphisms of homotopy modules are natural transformations of $\mathbb{Z}$-graded sheaves, homogeneous of degree 0, compatible with the structural isomorphisms ϵ_*.*

We let $\mathrm{HM}(k)$ be the category of homotopy modules over k. Given a homotopy module F_* and an integer $i \in \mathbb{Z}$, one defines the i-twisted homotopy modules $F_*\{i\}$ such that $(F_*\{i\})_r = F_{r+i}$.

Example 3.21. — Over a perfect field k, the fundamental result of Feld (2021b) defines an equivalence of categories which allows us to describe homotopy modules as certain $\mathbb{Z}$-graded functors M_* on function fields over k called *Milnor–Witt cycle modules*, inspired by the theory cycle modules due to Rost (1996).[59]

To such a functor M_* and any smooth k-scheme X, Feld attaches an explicit complex $C^*(X, M)$ of $\mathbb{Z}$-graded abelian concentrated in non-negative cohomological degrees. It

[59] And later developments due to his PhD student Manfred Schmid on the so-called Rost-Schmid complex.

has the form described by (2.27.a). He then shows that the zero cohomology of this complex, which for a connected X can be described as:

$$\underline{M}_*(X) = \mathrm{Ker}\left(M_*(\kappa(X)) \to \oplus_{x \in X^{(1)}} M_*(\kappa(x))\{\nu_x\}\right),$$

does define a homotopy module over k. Examples are given by the Milnor–Witt functor $\mathrm{K}_*^{\mathrm{M}}$ defined in Section 2.4.1, as well as Milnor K-theory $\mathrm{K}_*^{\mathrm{M}}$, extending the considerations of Paragraph 2.4.2.

When k is a perfect field, stable motivic sheaves are particular instances of homotopy modules. In fact, one easily deduces from the preceding theorem and the stable $\mathbb{A}^1$-connectivity theorem the following result of Morel (2004, §5.2).

Theorem 3.22. — *Assume k is perfect.*

There exists a unique t-structure on $\mathrm{SH}(k)$*, called the* homotopy t-structure*, whose homologically non-negative (resp. negative) objects are the motivic spectra* $\mathbf{E}$ *over k such that*

$$\underline{\pi}_n^{\mathbb{A}^1}(\mathbf{E}) = 0 \text{ if } n < 0 \quad (\text{resp. } n \geq 0).$$

This t-structure is compatible with the monoidal structure on $\mathrm{SH}(k)$.[60]

The canonical functor $\underline{\pi}_0^{\mathbb{A}^1}\colon \mathrm{SH}(k) \to \mathrm{HM}(k)$ *induces an equivalence of categories* $\mathrm{SH}(k)^{\heartsuit} \to \mathrm{HM}(k)$*. We will denote by* $\underline{\mathrm{H}}\colon \mathrm{HM}(k) \to \mathrm{SH}(k)$ *the ∞-functor obtained by composition of the reciprocal equivalence and the natural inclusion of the heart.*

This implies that $\mathrm{HM}(k)$ is a monoidal Grothendieck abelian category. The formula for the tensor product is: $F_* \otimes^H G_* = \underline{\pi}_0^{\mathbb{A}^1}\left(\underline{\mathrm{H}}(F_*) \otimes \underline{\mathrm{H}}(G_*)\right)$.

Remark 3.23. — The results of this theorem can be extended to the case when k is a non perfect field of positive characteristic p, up to inverting p in $\mathscr{SH}(k)$. One can reduce to the perfect case by a limit argument and by using Lemma B.3 proved by Levine, Yang, Zhao, and Riou (2019).

Example 3.24. — All this theory shows that the motivic stable stem admits a strong structure. First, as a functor on function fields over k, it can be organized into a Milnor–Witt cycles module in the sense of Example 3.21. Moreover, this uniquely corresponds to a homotopy module. This is valid for any motivic spectrum in place of the motivic sphere spectrum $\mathbb{1} = \Sigma^\infty S^0$. This connects with the unstable computations in Example 2.37.

1. We deduce from Theorem 3.11 the following fundamental result:

$$\underline{\pi}_n^{\mathbb{A}^1}(\mathbb{1}) = \begin{cases} 0 & \text{if } n < 0, \\ \underline{\mathrm{K}}_*^{\mathrm{MW}} & \text{if } n = 0. \end{cases}$$

[60] The unit object $\mathbb{1}$ is non-negative and the tensor product respect non-negative objects.

It follows that $\underline{\mathrm{K}}_*^{\mathrm{MW}}$ is the unit of the monoidal structure on $\mathrm{HM}(k)$. It is therefore a (commutative) monoid object, and every object of $\mathrm{HM}(k)$ admits a canonical $\underline{\mathrm{K}}_*^{\mathrm{MW}}$-module structure.

2. One deduces from elementary vanishing in motivic cohomology that $\mathbf{H}_{\mathrm{M}}\mathbb{Z}$ is a non-negative spectrum for the homotopy t-structure.[(61)] We have already mentioned that $\underline{\pi}_0^{\mathbb{A}^1}(\mathbf{H}_{\mathrm{M}}\mathbb{Z}) = \underline{\mathrm{K}}_*^{\mathrm{M}}$. Moreover, the unit $o\colon \mathbb{1} \to \mathbf{H}_{\mathrm{M}}\mathbb{Z}$ of the ring spectrum $\mathbf{H}_{\mathrm{M}}\mathbb{Z}$ induces the canonical projection map
$$\underline{\mathrm{K}}_*^{\mathrm{MW}} = \underline{\pi}_0^{\mathbb{A}^1}(\mathbb{1}) \xrightarrow{o_*} \underline{\pi}_0^{\mathbb{A}^1}(\mathbf{H}_{\mathrm{M}}\mathbb{Z}) = \underline{\mathrm{K}}_*^{\mathrm{M}} = \underline{\mathrm{K}}_*^{\mathrm{MW}}/(\eta)$$
explaining Remark 2.32.
More generally, it can be seen that the right adjoint $\mathbf{H}_{\mathrm{M}}\colon \mathscr{DM}(k) \to \mathscr{SH}(k)$ is t-exact for Voevodsky's homotopy t-structure on the left-hand side, and the induced functor on the heart is fully faithful with essential image the homotopy module on which η acts trivially. Using Feld's equivalence of categories, as stated in Example 3.21, the category of homotopy modules with a trivial action of η can be identified with that of cycle modules, as defined by Rost (1996).[(62)]

3. The previous result can be extended to other oriented ring spectra. The unit of the cobordism ring spectrum $o\colon \mathbb{1} \to \mathbf{MGL}$ does induce an isomorphism on the 0-th stable motivic sheaves: $\underline{\pi}_0^{\mathbb{A}^1}(\mathbf{MGL}) \simeq \underline{\mathrm{K}}_*^{\mathrm{M}}$. In characteristic 0, this result was extended by Yakerson (2021, Theorem 3.6.3) to the special-linear cobordism ring spectrum $\mathbf{MSL}$ showing that:
$$\underline{\pi}_0^{\mathbb{A}^1}(\mathbf{MSL}) \simeq \underline{\mathrm{K}}_*^{\mathrm{MW}}.$$

4. The case of algebraic K-theory is different, as $\mathbf{KGL}$ is a $S^{2,1}$-periodic motivic ring spectrum. In particular, it is unbounded with respect to the homotopy t-structure and one has for any integer $i \in \mathbb{Z}$
$$\underline{\pi}_i^{\mathbb{A}^1}(\mathbf{KGL}) = \underline{\mathrm{K}}_*\{i\},$$
where the right-hand side is the unramified K-theory sheaf.

5. Similarly, higher Grothendieck–Witt groups are $S^{8,4}$-periodic and one gets in particular:
$$\underline{\pi}_{i+4}^{\mathbb{A}^1}(\mathbf{GW}) = \underline{\pi}_i^{\mathbb{A}^1}(\mathbf{GW})\{4\}.$$

3.4.2. *Slice filtration.* — Note that the situation described in point 4 of the above example contrasts with the topological context, where the complex K-theory spectrum in degree 0 coincides with the Eilenberg–MacLane spectrum, therefore leading to the (topological) Atiyah–Hirzebruch spectral sequence.

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(61) This follows from the fact homotopy modules are unramified as $\mathbb{Z}$-graded sheaves and from the vanishing $\mathbf{H}_{\mathrm{M}}^{n,i}(k,\mathbb{Z}) = 0$ if $n > i$, which is proved by Suslin and Voevodsky (2000, Lemma 3.2(2)).
(62) See Déglise (2013) for all these assertions.

This fact led Voevodsky to introduce the *slice filtration* on a motivic spectrum $\mathbf{E}$, an avatar of the cellular filtration in the motivic world except that cells are given by the motivic sphere $\mathbb{P}^1_k$:[(63)]

$$\ldots \to f_n\mathbf{E} \to \ldots \to f_1\mathbf{E} \to f_0\mathbf{E} \to f_{-1}\mathbf{E} \to \ldots$$

He defined the n-th *slice* $s_n(\mathbf{E})$ of $\mathbf{E}$ as the homotopy cofiber of $f_{n+1}\mathbf{E} \to f_n\mathbf{E}$.

Then, Voevodsky (2002, Conjectures 7, 10) postulated that the 0-slice of both **KGL** and the sphere spectrum $\mathbb{1}$ are given by the the Eilenberg–MacLane motivic spectrum:

$$s_0(\mathbb{1}) = \mathbf{H}_{\mathrm{M}},\ \ s_n(\mathbf{KGL}) = \mathbf{H}_{\mathrm{M}}(n)[2n].$$

The first conjecture was proved by Voevodsky in characteristic 0. Then, Levine (2008) proved both conjectures over a perfect infinite base field k.[(64)] This lead to a new definition of the motivic version of the Atiyah–Hirzebruch spectral sequence[(65)] for a smooth k-scheme X: it can be described as the spectral sequence associated to the slice filtration on **KGL**, as conjectured by Voevodsky and proved in Levine (2008). See also Hoyois (2015, Th. 8.5, 8.7) for another approach.

Note that another construction of the motivic Atiyah–Hirzebruch spectral sequence was already done by Friedlander and Suslin (2002). As of now, the agreement of the latter with the slice spectral sequence of the K-theory spectrum **KGL** is not known. Also, the convergence of the slice spectral sequence in general is a difficult matter, solved in Levine (2013). See also Hoyois (2015, Th. 8.12).

3.4.3. *Morel's π_1-conjecture and beyond.* — Before stating the last main computation of this section, let us recall that Spitzweck and Østvær (2012) considered the interaction between Morel's homotopy t-structure and Voevodsky's slice filtration. This led them to introduce an important ring spectrum, called the *very effective hermitian K-theory.*[(66)] It is defined as:

$$\mathbf{kq} = f_0(\tau_{\geq 0}\mathbf{KQ})$$

where $\tau_{\geq 0}$ is the truncation functor with respect to Morel's homotopy t-structure and f_0 the first stage of the slice filtration. It admits a canonical structure of commutative algebra in the ∞-category $\mathscr{SH}(k)$ (i.e., it is an E_∞-spectrum). It is proposed in *loc. cit.* that **kq** plays a role analogous to the connective cover of real topological K-theory in algebraic topology.

The following result of Röndigs, Spitzweck, and Østvær (2024) emerged from a far-reaching conjecture of Morel, and was the motivation for many works in motivic homotopy theory. Let us also add that it is the stable analogue of the Asok-Fasel's conjecture mentioned in Section 2.5.3.

Theorem 3.25. — *Assume the characteristic exponent c of k is different from* 2.

(63) beware that this tower is possibly infinite in both directions;
(64) The assumption that k is infinite can be removed thanks to Hogadi and Kulkarni (2020).
(65) whose existence was conjectured by Beĭlinson (1987, §5, B.);
(66) Beware about Remark 3.5 concerning the terminology used here.

1. *The unit* $u\colon \mathbb{1} \to \mathbf{kq}$ *of the very effective hermitian K-theory induces a short exact sequence of homotopy modules after inverting* e*:*

$$0 \to \underline{\mathrm{K}}_*^{\mathrm{M}}/24\{2\} \to \underline{\pi}_1^{\mathbb{A}^1}(\mathbb{1}) \xrightarrow{u_*} \underline{\pi}_1^{\mathbb{A}^1}(\mathbf{kq}) \to 0$$

Looking at the zero-th graded part and evaluating at k*, the sequence becomes the following explicit computation of the motivic stable stem, which was conjectured by Morel:*

$$0 \to \mathrm{K}_2^{\mathrm{M}}(k)/24 \to \pi_{1,0}^k \to k^\times/2 \oplus \mathbb{Z}/2 \to 0.$$

2. *After application of the second motivic homotopy sheaf,* u *induces the following short exact sequence of homotopy modules after inverting* e*:*

$$0 \to \underline{\pi}_1^{\mathbb{A}^1}(\mathbf{H}_{\mathrm{M}}\mathbb{Z})_*/24\{2\} \oplus \underline{\mathrm{K}}_*^{\mathrm{M}}/2\{4\} \to \underline{\pi}_2^{\mathbb{A}^1}(\mathbb{1}) \xrightarrow{u_*} \underline{\pi}_2^{\mathbb{A}^1}(\mathbf{kq}).$$

Looking at the first graded part and evaluating at k*, one further obtains the split short exact sequence, after inverting* e*:*

$$0 \to \mathrm{K}_3^{\mathrm{M}}(k)/2 \to \pi_{3,1}^k \to \mu_{24}(k) \to 0.$$

Based on fundamental results on the slice and very effective slice filtrations, such as Levine's convergence result for the former, the proof consists of a very involved computation of the slice spectral sequences for both the sphere and very effective hermitian K-theory spectra.

4. Application to stable stems

4.1. Computing the stable stems: a short review

Since the initial introduction of the fundamental group by Poincaré (1895), and its subsequent extension to higher homotopy groups by his successors, the computations of homotopy groups has been a driving force in homotopy theory. The case of higher-dimensional spheres remains a central problem to this day.

While Poincaré initiated some of the first computations (via coverings and polyhedral decompositions of varieties), the main advancements in the first half of the 20th century were the following ones:

- Hopf (1931) constructed the *Hopf map*, that we will denote by η^{top}, and carried out the first computation of higher homotopy groups: $\pi_3(S^2) = \mathbb{Z}.\eta^{\mathrm{top}}$.
- Hurewicz (1935) proved the theorem that now bears his name, allowing the computation of higher homotopy groups in terms of homology groups. As mentioned by Hurewicz in *op. cit.*, this result implies:

$$\pi_i(S^n) = \begin{cases} 0 & \text{if } i < n, \\ \mathbb{Z} & \text{if } i = n. \end{cases}$$

- For an integer $i \geq 0$, Freudenthal (1937) established the theorem that bears his name, proving that the suspension map is an isomorphism in the following cases:

$$\pi_{i+n}(S^n) \xrightarrow{\sim} \pi_{i+n+1}(S^{n+1}) \text{ if } n > i+1. \tag{4.0.a}$$

  This implies the existence of the *i-stem*, which we have already denoted by π_i^{S} in Section 3.3.1.

A new era was inaugurated by the PhD thesis of Serre (1951), which introduced fibrations (notably the path fibration) and spectral sequences in this area. It provided a systematic method for computing homotopy groups of spheres and demonstrated the finiteness of most of these groups, particularly the non-zero stable stems.[(67)]

From this background, a wealth of techniques and theories have emerged, partly motivated by the problem of unveiling the stable stem. Let us indicate some landmark results, and provide relevant details for this note thereafter.

- Motivated by the Kervaire invariant problem and the aim of improving the computability of Serre's spectral sequences, Adams (1958) introduced the Adams spectral sequence, which uses the homological algebra of the Steenrod algebra to compute the p-torsion in the stable stems.
- Novikov (1967) extended Adams's construction to any (appropriate) spectrum in place of the Eilenberg–MacLane spectrum $\mathbf{H}\mathbb{F}_p$ and advocated for the use of the complex cobordism spectrum $\mathbf{MU}$.
- The final point we wish to highlight is not a single work but rather a collection of results and a unifying philosophy, now referred to as *chromatic homotopy theory*.[(68)] Numerous contributors have shaped this area, with cornerstone by Quillen (1969). A pioneering work, rooted in Morava's contributions — including his localization theorem and the introduction of Morava K-theories — was made by Miller, Ravenel, and Wilson (1977), where the term "chromatic" was introduced.

In order to introduce the reader to the techniques used in the sequel, we will now review the main spectral sequences that were mentioned above. We will later recall a general construction to obtain all of them, in Example 4.14.

4.1.1. *The Adams spectral sequence.* — Let us fix a prime p, which will be implicit in all subsequent notation. The Adams spectral sequence at the prime p is a very efficient tool to compute the p-adic completion $\hat{\pi}_*^{\mathrm{S}}$ of the stable stems which, according to Serre's fundamental theorem, are given by the formula:

$$\hat{\pi}_i^{\mathrm{S}} = \pi_i^{\mathrm{S}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \begin{cases} \mathbb{Z}_p \ (p\text{-adic integers}) & i = 0, \\ (\pi_i^S) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = (\pi_i^S)[p^\infty] = (\pi_i^S)_{p-\mathrm{tor}} & i > 0, \\ 0 & i < 0. \end{cases}$$

(67) Around the same time, J. H. C. Whitehead (1950) and G. W. Whitehead (1953), building on Blakers and Massey's notion of triads, introduced the *EHP sequence*, which also provides a general inductive method for computing homotopy groups of spheres.

(68) See also the Bourbaki talks n° 728, 1005, 1029.

The Adams spectral sequence[69] is multiplicative and takes the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A_{\mathrm{cl}}}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \hat{\pi}^{\mathrm{S}}_{t-s}. \tag{4.0.b}$$

Let us clarify the notation.

- The grading follows the conventions from algebraic topology: differentials on the E_r-page have bidegree $(r, r-1)$:
$$d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+r-1}.$$
- A_{cl} is the classical Steenrod algebra at the prime p, made by the cohomological stable operations in (singular) $\mathbb{F}_p$-cohomology. This is a $\mathbb{Z}$-graded Hopf algebra over $\mathbb{F}_p$, which in degree n can be defined as:
$$A^n_{\mathrm{cl}} = [\mathbf{H}\mathbb{F}_p, S^n \wedge \mathbf{H}\mathbb{F}_p]_{\mathrm{SH}} = [\mathbf{H}\mathbb{F}_p, \mathbf{H}\mathbb{F}_p[n]]_{\mathrm{SH}}.$$
- $\mathrm{Ext}^{s,t}_{A_{\mathrm{cl}}}$ denote the s-th extension group computed in the category of $\mathbb{Z}$-graded A_{cl}-modules, and the integer t refers to the internal r-th grading.
- The filtration on the abutment has the following form:
$$\hat{\pi}^{\mathrm{S}}_* = F^1\hat{\pi}^{\mathrm{S}}_* \supset F^2\hat{\pi}^{\mathrm{S}}_* \supset \cdots$$
See Bousfield (1979, page 275) for an explicit formula. Moreover, the E_2-page is concentrated in the following region:[70]
$$E_2^{s,t} = 0 \text{ if } \begin{cases} s < 0, \text{or } t < s, \\ \text{or } 0 < s < t < 2s - 3. \end{cases}$$
This implies that the above filtration is finite in each degree, that the spectral sequence converges (strongly) and that, for $r > \max\left(s, \frac{1}{2}(t - 3s + 2)\right)$,
$$E_\infty^{s,t} = E_r^{s,t}.$$

4.1.2. *Computing with the Adams spectral sequence.* — Since its introduction, the above spectral sequence has been a successful tool for computing the stable stem, by focusing on p-primary parts for each prime p. There are three major challenges to overcome in this computation:

Step 1. *Determining the E_2-term.* It is customary to represent $E_2^{s,t}$ in a diagram called an *Adams chart*, where s (the *Adams filtration*) is plotted on the vertical axis, and $f = t - s$ (the *stem*) on the horizontal axis.

Step 2. *Understanding enough of the differentials on each page to deduce the E_∞-term.* An element a in a given page $E_r^{s,t}$ that induces a non-trivial element in the subquotient $E_\infty^{s,t}$ is called a *permanent cycle.*

Step 3. *Solving the extension problem.* It allows one to reconstruct the whole filtered $\mathbb{F}_p$-vector space from its associated graded (under a finite filtration).

[69] Historically, the construction and results are all due to Adams. A classical reference is Adams (1974, p. III.15). The following review is largely based on the excellent account made by Ravenel (1986).
[70] These vanishing conditions can be improved in many ways. See Adams (1966, Th. 1.1) to begin with.

The goal of this section is in particular to explain how motivic homotopy theory has introduced new tools that allows improving calculations at each stage (especially steps 2 and 3). Since the main difficulty lies in the first and only even prime $p = 2$, we will focus on this case. To give the reader a sense of the progress achieved, we will state a few general results. According to Ravenel (1986, Th. 3.4.1), the first four nontrivial rows in the Adams chart are given by the $\mathbb{F}_2$-algebras with generators and relations as follows:

– $E_2^{0,*} = \mathbb{F}_2[0]$,
– $E_2^{1,*} = \mathbb{F}_2\langle h_i, i \geq 0\rangle$, where $\deg(h_i) = 2^i$,
– $E_2^{2,*} = \mathbb{F}_2\langle h_i h_j\rangle/(h_i h_j - h_j h_i, h_i h_{i+1})$,
– $E_2^{3,*} = \mathbb{F}_2\langle h_i h_j h_k, 0 \leq i < j < k; c_i, i \geq 0\rangle/(h_i h_{i+2}^2, h_i^2 h_{i+2})$, $\deg(c_i) = 11.2^i$.

The family of elements $(h_i)_{i\geq 0}$ in $Ext^1_{A_{\mathrm{cl}}}(\mathbb{F}_p, \mathbb{F}_p)$ corresponds to the family of generators $(\mathrm{Sq}_{2^i})_{i\geq 0}$ of the Steenrod algebra (at the prime 2). The element c_i is detected by a Massey product $c_i \in \langle h_{i+1}, h_i, h_{i+2}^2\rangle$, which is well-defined thanks to the relations satisfied by the family of elements $(h_i)_{i\geq 0}$.

Example 4.1. — As a basic example of resolution in Step 2 above, one knows that h_0, h_1, h_2, h_3 are all permanent cycles, and detect respectively $2 \in \hat{\pi}_0^{\mathrm{S}}$, and the Hopf elements $\eta_{\mathrm{cl}} \in \hat{\pi}_1^{\mathrm{S}} = \mathbb{Z}/2$, $\nu \in \hat{\pi}_3^{\mathrm{S}} = \mathbb{Z}/8$, $\sigma \in \hat{\pi}_7^{\mathrm{S}} = \mathbb{Z}/16$. It is a celebrated theorem of Adams (1960) that none of the others h_i are permanent — in fact, $d_2^{1,2^i} h_i = h_0 h_{i-1}^2 \neq 0$ for $i > 3$.

Remark 4.2. — The resolution of the above Step 1 involves the use of other kinds of algebraic spectral sequences, which converge to the E_2-term to be computed. There are many such spectral sequences: Cartan–Serre, May, algebraic Adams–Novikov, chromatic, etc. These tools are also used in the computations of Isaksen, Wang, and Xu (2023), and will briefly appear at the end of this lecture (see Theorem 4.26). We refer the reader to Ravenel (1986) (Chap. 3, §2 for May spectral sequences, Chap. 5 for the chromatic spectral sequence), as well as to Miller (1981) for interactions between these various spectral sequences.

4.1.3. *The Adams–Novikov spectral sequences.* — One of the reasons to consider the so-called Adams–Novikov spectral sequences is that they allow determining certain differentials in the Adams spectral sequence. This is based on the fact that one can replace the ring spectrum $\mathbf{H}\mathbb{F}_p$ in the preceding construction by an arbitrary ring spectrum $\mathbf{E}$ (see next section). The resulting construction is functorial in $\mathbf{E}$.

Let us clarify the premises of chromatic homotopy theory. The theory of characteristic classes can be expressed in stable homotopy by a very simple structure: a (complex) orientation on a given ring spectrum $\mathbf{E}$ is a class $c \in \tilde{\mathbf{E}}^2(\mathbb{CP}^\infty) = \mathbf{E}^2(BU)$ in the associated reduced cohomology of the infinite projective space which restricts to $1 \in \tilde{\mathbf{E}}^2(\mathbb{CP}^1) \simeq \mathbf{E}^0(*)$. The beauty of stable homotopy is that this single class determines higher Chern classes satisfying all classical properties, except that the first Chern class of a tensor product of two line bundles is not always given by addition, but in general is expressed via a well-defined (commutative) formal group law $F_{\mathbf{E}}(x, y)$, with coefficients in the base ring $\mathbf{E}_* = \mathbf{E}_*(*)$, and depending only on the oriented ring spectrum $(\mathbf{E}, c)$.

A fundamental observation made by Quillen (1969) is that the complex cobordism ring spectrum $\mathbf{MU}$ admits a canonical orientation $c_{\mathbf{MU}}$ such that $(\mathbf{MU}, c_{\mathbf{MU}})$ is the universal oriented ring spectrum and the associated formal group $\big(\mathbf{MU}_*, F_{\mathbf{MU}}(x, y)\big)$ is the universal formal group law, defined by Lazard.[(71)] As $\mathbb{F}_p$-linear singular cohomology is oriented,[(72)] it inherits a ring map $c \colon \mathbf{MU} \to \mathbf{H}\mathbb{F}_p$. This yields the (first) Adams–Novikov spectral sequence:

$$E_{2,\mathbf{MU}}^{s,t} = \mathrm{Ext}^{s,t}_{\mathbf{MU}_*\mathbf{MU}}(\mathbf{MU}_*, \mathbf{MU}_*) \Rightarrow \pi^{\mathrm{S}}_{t-s}, \tag{4.2.a}$$

a multiplicative spectral sequence which converges to the integral stable stem. In contrast to the Adams spectral sequence, we have used the graded $\mathbb{Z}$-algebra $\mathbf{MU}_*\mathbf{MU}$, which is the $\mathbb{Z}$-dual of the algebra of stable operations $\mathbf{MU}^*\mathbf{MU}$ on complex cobordism. The pair $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$ forms what is called a (graded) *Hopf algebroid* over the ring $R = \mathbb{Z}$: a groupoid in the category of (graded) R-algebras.[(73)] The E_2-term is given by the extension groups in the category of graded comodules over this graded Hopf algebroid, and as before, the index t refers to the internal grading.

One can get a more useful spectral sequence by working in the p-local stable homotopy category $\mathrm{SH}_{(p)}$, obtained by inverting all primes except p. Indeed, Quillen showed that the p-local ring spectrum $\mathbf{MU}_{(p)}$ splits into a direct sum of tensor products of a ring spectrum $\mathbf{BP}$ called the Brown–Perterson spectrum. This decomposition reflects the structure of the category of formal group laws over $\mathbb{Z}_{(p)}$, as $\mathbf{BP}$ is complex oriented and the associated formal group law $(\mathbf{BP}_*, F_{\mathbf{BP}})$ is the universal p-typical one.[(74)] Applying the general construction to $\mathbf{BP}$, one gets the (second) Adams–Novikov spectral sequence:

$$E_{2,\mathbf{BP}}^{s,t} = \mathrm{Ext}^{s,t}_{\mathbf{BP}_*\mathbf{BP}}(\mathbf{BP}_*, \mathbf{BP}_*) \Rightarrow \pi^{\mathrm{S}}_{t-s} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}\,, \tag{4.2.b}$$

with similar properties to the first one, but converging to the p-local stable stem. In fact, the above E_2-term is precisely given by the p-component of (4.2.a), according to Ravenel (1986, Theorem 1.4.2).

Since the formal group law of $\mathbf{H}\mathbb{F}_p$ is additive, therefore p-typical, one deduces that the ring map c corresponding to the orientation of $\mathbf{H}\mathbb{F}_p$ induces a ring map $c' \colon \mathbf{BP} \to \mathbf{H}\mathbb{F}_p$. The map c' induces a morphism of multiplicative spectral sequences $E_{r,\mathbf{BP}}^{s,t} \to E_r^{s,t}$. One

(71) In particular, the ring of coefficients $\mathbf{MU}_*$ is isomorphic to the Lazard ring $\mathbb{L} = \mathbb{Z}[x_1, x_2, \ldots]$. See Ravenel (1986, Theorems 4.1.6 and A2.1.10).

(72) Naturally, the associated formal group law is the additive one.

(73) We refer the reader to Ravenel (1986, Appendix 1) for more information, which attributes the terminology to Haynes Miller.

(74) This structure is based on the Cartier isomorphism for p-local formal group laws. We refer the reader to Ravenel (1986, Appendix 2) for a concise exposition aimed at applications in homotopy theory. A more systematic treatment can be found in the classical monograph of Hazewinkel (2012). Recall in particular that $\mathbf{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, with $\deg(v_i) = 2(p^i - 1)$.

can then combine information between both spectral sequences to obtain computations of the stable stem.[75]

Remark 4.3. — As explained in a celebrated course by Hopkins (1999), the Hopf algebroid $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$ is an affine presentation of the stack $\mathcal{M}_{FG}$ of formal group laws with strict isomorphisms. One can define a line bundle ω on it by assigning to a formal group law over a ring R its space of invariant differential forms. Then the E_2-term of the Adams–Novikov spectral sequence(4.2.a) is concentrated in even degree t and can be computed as:

$$E_{2,\mathbf{MU}}^{s,2t} = H^s\Big(\mathcal{M}_{FG}, \omega^{\otimes t}\Big).$$

This beautiful formula (proved in Goerss, 2008, §3.2, (3.5)) allows one to express chromatic homotopy theory as arising from the stratification of the p-localization of the stack $\mathcal{M}_{FG}$ induced by the height of p-typical formal group laws.

4.1.4. *Spectral sequences in motivic and classical stable homotopy: a first link.* — To provide the reader with an initial sense of the relevance of the motivic perspective in comparison to previous computational tools, we conclude this subsection with a striking comparison between spectral sequences. Fix an algebraically closed field k of characteristic 0. As recalled in Theorem 3.12, the motivic stable stems of k in weight 0 agree with the classical stable stems. Furthermore, as discussed in Section 3.4.2, every motivic spectrum admits a canonical slice filtration. When applied to the motivic sphere spectrum over k, one deduces the *slice spectral sequence for the motivic sphere spectrum in weight* 0:

$$E_{1,\text{slice}}^{p,q} = [s_q(\mathbb{1}), \mathbb{1}[p+q]] \Rightarrow \pi^k_{-p-q,0} \simeq \pi^{\mathrm{S}}_{-p-q}\,.$$

The main result of Levine (2015) is the following comparison theorem.

Theorem 4.4. — *Consider the above assumptions. Then, up to the following reindexing, the above slice spectral sequence is isomorphic to the Adams–Novikov spectral sequence:*

$$E_{r,\text{slice}}^{p,q} \simeq E_{2r+1,\mathbf{MU}}^{3p+q,2p}.$$

In particular, both induced filtrations on π^{S}_{-p-q} *coincide.*

This result unveils a surprising link between the algebro-geometric content of motivic spectra and the basic structure of classical spectra.[76]

(75) See Ravenel (1978, §4) for a thorough study up to degree 18.

(76) Such a connection — though not stated in this form — was already anticipated by Voevodsky (2002), at the very end of Section 6.

4.2. Motivic cohomology with torsion coefficients

We fix a prime ℓ invertible in k and state in this paragraph the known results, all due to Voevodsky, about the motivic Eilenberg–MacLane spectrum $\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell$ with $\mathbb{F}_\ell$-coefficients (see Example 3.4(2), Remark 3.9). To comply with Dugger and Isaksen (2010) and Isaksen, Wang, and Xu (2023), we let $\mathbf{M}_\ell$ be the bigraded $\mathbb{F}_\ell$-algebra such that $\mathbf{M}_\ell^{n,i} = \mathbf{H}_{\mathrm{M}}^{n,i}(k, \mathbb{F}_\ell)$.

We also write $\mathbf{H}_{\text{ét}}\mu_\ell$ for the motivic ring spectrum which represents étale cohomology with coefficients in the $\mathbb{Z}$-graded $\mathbb{F}_\ell$-torsion sheaf $\mu_\ell^{\otimes,*}$. According to the rigidity theorem of Suslin and Voevodsky (2000, Corollary 6.4.2), it can be identified with the étale sheafification $a_{\text{ét}}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell)$ of $\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell$.[(77)] This yields a canonical morphism of motivic ring spectra:

$$\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell \xrightarrow{\gamma_{\text{ét}}} \mathbf{H}_{\text{ét}}\mu_\ell. \tag{4.4.a}$$

Given that result, it is notable that the Beilinson–Lichtenbaum conjecture, proved by Voevodsky (2011),[(78)] is equivalent to the following formulation stated purely in terms of the homotopy t-structure on $\mathscr{SH}(k)$, as defined in Theorem 3.22. In the case where k is not perfect of characteristic p, the formulation still makes sense and remains valid by appealing to Remark 3.23.

Theorem 4.5. — *Consider the above notation. Then the map (4.4.a) induces an isomorphism of motivic ring spectra over k:*

$$\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell \to \tau_{\geq 0}\Big(\mathbf{H}_{\text{ét}}\mu_\ell\Big),$$

using the (homological) truncation functor associated with the homotopy t-structure on $\mathscr{SH}(k)$ of Theorem 3.22.

We leave it to the reader to verify the equivalence of this statement with the classical formulation of the Beilinson–Lichtenbaum conjecture given in Suslin and Voevodsky (2000, § 3, Conjecture (Be4)).[(79)]

4.2.1. *Periodicity in Galois cohomology.* — We will deduce from the preceding theorem a description of the stable motivic homotopy sheaves of $\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell$ (Definition 3.20). According to the preceding result, let us consider étale unramified μ_q-cohomology, a classical invariant in arithmetic, organized as a homotopy module (Definition 3.20). For any integer $i \in \mathbb{Z}$, we let $\mathcal{H}^i_{\text{ét}}(\mu_\ell)_*$ be the homotopy module over k whose n-th graded part is given by the sheaf

$$\mathcal{H}^i_{\text{ét}}(\mu_\ell)_n = \mathcal{H}^{i+n}_{\text{ét}}(-, \mu_\ell^{\otimes n}),$$

obtained by considering the Zariski, or equivalently Nisnevich, sheafification of the corresponding étale cohomology presheaf on Sm_k. It is worth noting that the homotopy

(77) The cohomology represented by $a_{\text{ét}}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell)$ is usually called the *Lichtenbaum motivic cohomology* with $\mathbb{F}_\ell$-coefficients.

(78) See also the review of Riou (2014).

(79) Hint: use the properties of the homotopy t-structure as stated in Section 3.4.

module $\mathcal{H}^i_{\text{ét}}(\mu_\ell)_*$ is equivalent to the Rost cycle module defined by the following Galois cohomology functor on function fields E/k:

$$E \mapsto H^{i+*}(G_E, \mu_q^*).$$

This follows from the equivalence of categories mentioned in 3.24(2), the fact that η acts trivially on $\mathbf{H}_{\text{ét}}\mu_\ell$ and the identification of étale cohomology of fields with Galois cohomology.

By construction, we also have the relation $\underline{\pi}_i^{\mathbb{A}^1}(\mathbf{H}_{\text{ét}}\mu_\ell) = \mathcal{H}^{-i}_{\text{ét}}(\mu_\ell)_*$. The ring structure of $\mathbf{H}_{\text{ét}}\mu_\ell$, as well as the classical cup product in Galois cohomology, corresponds to the fact that $\mathcal{H}^*_{\text{ét}}(\mu_\ell)_*$ is a commutative monoid object in the category of homotopy modules. We will simply say that it is an algebra. It is striking that the stable motivic homotopy sheaves $\underline{\pi}_i^{\mathbb{A}^1}(\mathbf{H}_{\text{ét}}\mu_\ell)$ satisfy a periodicity property analogous to the v_i-periodicity observed in the stable stem.

Let us first recall that, for any integer q, the action of the absolute Galois group G_k on the Galois module $\mu_\ell^{\otimes,q}$ is given by the q-th power χ^q of the cyclotomic character $\chi\colon G_k \to \mathbb{F}_\ell^\times$. This implies that there exists a canonical isomorphism $\mu_\ell^{\otimes,\ell-1} \simeq \mathbb{F}_\ell$ of Galois modules (or étale sheaves), which can be immediately translated into a periodicity isomorphism in the algebra $H^*(G_E, \mu_q^*)$ for any extension field E/k. To get a motivic formulation that will be shortly stated, we apply the above theorem to deduce a canonical isomorphism:

$$\mathbf{M}_\ell^{0,0} \simeq H^0(G_k, \mathbb{F}_\ell) \simeq H^0(G_k, \mu_\ell^{\otimes,\ell-1}) \simeq \mathbf{M}_\ell^{0,\ell-1}$$

and we denote by $\tau' \in \mathbf{M}_\ell^{0,\ell-1}$ the image of 1 under this isomorphism.

We can improve this periodicity if k admits an ℓ-th root of unity. Then the choice of $\zeta_\ell \in \mu_\ell(k)$ induces an isomorphism $\mu_\ell \simeq \mathbb{F}_\ell$, and therefore a periodicity in $H^*(G_E, \mu_q^*)$ as above. Moreover, let us recall that one gets the following exact sequence in motivic cohomology:[80]

$$0 \to \mathbf{H}_{\mathrm{M}}^{0,1}(k, \mathbb{F}_\ell) \to \mathbf{H}_{\mathrm{M}}^{1,1}(k, \mathbb{Z}) = k^\times \xrightarrow{\ell} k^\times = \mathbf{H}_{\mathrm{M}}^{1,1}(k, \mathbb{Z}) \to \mathbf{H}_{\mathrm{M}}^{1,1}(k, \mathbb{F}_\ell) \to 0.$$

It induces a canonical isomorphism $\mathbf{M}_\ell^{0,1}(k) \simeq \mu_\ell(k)$. Then we denote generically by $\tau \in \mathbf{M}_\ell^{0,1}(k)$ the element which corresponds to ζ_ℓ via this isomorphism.[81]

This study, paired with the previous theorem, gives the following corollary formulated in terms of homotopy modules.

Corollary 4.6. — *Consider the above notation. Then there exists canonical isomorphisms of commutative monoids in the monoidal abelian category of homotopy modules:*

$$\underline{\pi}_*^{\mathbb{A}^1}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell) \simeq \oplus_{i=0}^{l-2}\mathcal{H}^{-i}_{\text{ét}}(\mu_\ell)_*[\tau']$$
$$\underline{\pi}_*^{\mathbb{A}^1}(\mathbf{H}_{\text{ét}}\mu_\ell) \simeq \oplus_{i=0}^{l-2}\mathcal{H}^{-i}_{\text{ét}}(\mu_\ell)_*[\tau', \tau'^{-1}]$$

[80] One uses the exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \to \mathbb{F}_\ell$, viewed for example in motivic complexes, and the known computations of motivic cohomology with twists 0 and 1.
[81] Observe that according to these definitions, $\tau' = \tau^{\ell-1}$.

where the left-hand side is seen as a polynomial algebra (resp. Laurent polynomial algebra) in the variable τ', also seen as an element of the respective left-hand sides.

If we assume that k contains an ℓ-th root of unity and let τ be given as above, then one deduces canonical isomorphisms of algebras in homotopy modules:

$$\underline{\pi}_*^{\mathbb{A}^1}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell) \simeq \underline{\mathrm{K}}_*^{\mathrm{M}}/\ell[\tau]$$
$$\underline{\pi}_*^{\mathbb{A}^1}(\mathbf{H}_{\text{ét}}\mu_\ell) \simeq \underline{\mathrm{K}}_*^{\mathrm{M}}/\ell[\tau, \tau^{-1}]$$

with the same description of the right-hand sides as previously, but with respect to τ.

In both cases, the first isomorphism is induced by (4.4.a) and follows from the periodicity properties studied earlier together with the preceding theorem; the second isomorphism is a reformulation of these periodicity properties.

Remark 4.7. — 1. Since we are mixing cohomological bidegrees and homological bidegrees for the homotopy t-structure, we make explicit the effect of multiplication by τ' (respectively τ) in terms of the $\mathbb{G}_m$-twist operation on homotopy modules (see Definition 3.20):

$$\underline{\pi}_i^{\mathbb{A}^1}(\mathbf{M}_\ell).\tau' = \underline{\pi}_{i+l-1}^{\mathbb{A}^1}(\mathbf{M}_\ell)\{l-1\}$$
$$\text{resp. } \underline{\pi}_i^{\mathbb{A}^1}(\mathbf{M}_\ell).\tau = \underline{\pi}_{i+1}^{\mathbb{A}^1}(\mathbf{M}_\ell)\{1\}.$$

2. When k does not contain an ℓ-th root of unity, the stated periodicity is the best possible as for all $q \geq 0$, one gets:

$$\underline{\pi}_q^{\mathbb{A}^1}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell)_q(k) = \mathbf{M}_\ell^{0,q} \simeq H^0(G_k, \mu_\ell^{\otimes, q}) = \begin{cases} \mathbb{F}_\ell & \text{if } q = 0 \bmod \ell - 1, \\ 0 & \text{otherwise.} \end{cases}$$

One deduces from the previous corollary the following result.

Corollary 4.8. — *Under the assumptions of the Theorem 4.5, the morphism of motivic ring spectra over k, induced by* (4.4.a)*,*

$$\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell[\tau'^{-1}] \to \mathbf{H}_{\text{ét}}\mu_\ell$$

is an isomorphism, where the left-hand side is obtained by internally inverting τ' (see Section 4.3.3).

This statement has a long history, starting with the pioneering work of Thomason (1985) where an analogous result for algebraic K-theory was established. The above motivic version was in fact proved independently of the validity of the Beilinson–Lichtenbaum conjecture (which is stronger) by Levine (2000, Theorem 6.2).

Note that this corollary consists of two distinct assertions: first, an étale descent theorem for the τ'-inverted theory; second, a computation of étale motivic cohomology in terms of classical étale cohomology. The latter is, as mentioned before Theorem 4.5, a rigidity statement (that also has a long history), and was established for motivic cohomology by Suslin and Voevodsky (2000, Corollary 6.4.2). It has since been generalized in various contexts, the latest statement in motivic homotopy theory being Bachmann

(2021). The descent statement was likewise extended in motivic homotopy theory by Elmanto, Levine, Spitzweck, and Østvær (2022) (for arbitrary **MGL**-modules).

The following corollary plays a central role in Isaksen, Wang, and Xu (2023).

COROLLARY 4.9. — *Let k be an algebraically closed field of characteristic prime to ℓ. Then the $\mathbb{F}_\ell$-linear motivic cohomology ring of coefficients over k is the polynomial algebra:*

$$\mathbf{M}_\ell = \mathbb{F}_\ell[\tau]$$

where τ is an element of homological *bidegree $(0,-1)$ defined by the choice of an ℓ-th root of unity in k.*

Indeed, as k is separably closed, one obviously obtains an isomorphism of graded algebras $\mathrm{K}^{\mathrm{M}}_*(k)/\ell = \mathbb{F}_\ell$, concentrated in degree 0.

4.2.2. *The motivic Steenrod algebra.* — The last ingredient that we will need is the motivic Steenrod algebra modulo 2 over the field $\mathbb{C}$. More generally, the mod ℓ motivic Steenrod algebra over a field k of characteristic prime to ℓ is defined as the $\mathbb{F}_\ell$-algebra of stable cohomology operations on $\mathbb{F}_\ell$-linear motivic cohomology:

$$\mathcal{A}^{**}(k,\mathbb{F}_\ell) \coloneqq (\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell)^{**}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell),$$

using the topological notation, keeping in mind the bidegree grading specific to the motivic context.

Instead of recalling all the details, we will give some key references to the reader. These operations were first computed by Voevodsky (2003) over a perfect field.[(82)] Nice accounts, with corrections and complements, were given by Riou (2012) and Hoyois, Kelly, and Østvær (2017).

As in the topological situation, explained in the previous section, it is easier to work with the dual motivic Steenrod algebra modulo ℓ, defined as follows:[(83)]

$$\mathcal{A}_{p,q}(k,\mathbb{F}_\ell) = (\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell)_{p,q}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell) = [\mathbb{1}(q)[p], \mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell \otimes \mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell]_{\mathrm{SH}(k)}.$$

For future reference, we will simply write $A = (\mathbf{H}_{\mathrm{M}}\mathbb{F}_2)_{**}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_2)$ taken over the field $\mathbb{C}$. We now recall the explicit presentation of this bigraded commutative $\mathbb{F}_2$-algebra which also carries with a bigraded $\mathbf{M}_2$-algebra structure:

$$A = \mathbf{M}_2[\tau_0,\tau_1,\dots,\xi_1,\xi_2,\dots]/(\tau_i^2 - \tau\xi_{i+1}) \tag{4.9.a}$$

where the generators are the motivic analogue of the Milnor basis:

- τ_i has bidegree $(2^{i+1}-1, 2^i-1)$ and is dual to Sq^{2^i-1},
- ξ_i has bidegree $(2^{i+1}-2, 2^i-1)$ and is dual to $\mathrm{Sq}^{2^i}\,\mathrm{Sq}^{2^{i-1}}\cdots\mathrm{Sq}^1$.

(82) This assumption can easily be removed as the $\mathbb{F}_\ell$-linear motivic cohomology spectrum is invariant under purely inseparable extensions.

(83) While this bigraded $\mathbb{F}_\ell$-algebra is $\mathbb{F}_\ell$-dual to its cohomological counterpart, its algebraic structure involves the Cartan relations which are simpler than the Adem relations which describe the multiplication of $\mathcal{A}^k_\ell$. This is perfectly analogous to the situation in topology.

We refer the reader to Hoyois, Kelly, and Østvær (2017, Theorem 5.6) for a precise and comprehensive description of all the algebraic structures of the Hopf algebroid $(\mathbf{M}_2, A)$.

4.2.3. *Complex realization.* — We conclude by studying the complex case $k = \mathbb{C}$, using the complex realization $\rho_{\mathbb{C}}\colon \mathscr{SH}(\mathbb{C}) \to \mathscr{SH}$ of Section 3.2.2. Fixing an ℓ-th root ζ_ℓ of unity in $\mathbb{C}$, we get the class $\tau \in \mathbf{M}_\ell^{0,1}$ as above.

Recall that we have in Example 3.2.2 defined the $\mathbb{F}_\ell$-linear Betti motivic spectrum: $\mathbf{H}_{\mathrm{B}}\mathbb{F}_\ell := \rho_{\mathbb{C}*}(\mathbf{H}\mathbb{F}_\ell)$. Note that, apart from the fact that it has coefficients in a field of positive characteristic, it satisfies all the axioms of a Mixed Weil cohomology theory, as defined in Cisinski and Déglise (2012). In particular, many of the results of *loc. cit.* apply *mutatis mutandis* to the resulting motivic ring spectrum $\mathbf{H}_{\mathrm{B}}\mathbb{F}_\ell$, except that one needs to modify the discussion using the K-theory spectrum **KGL** at the end of §2.3.[84] More importantly, $\mathbf{H}_{\mathrm{B}}\mathbb{F}_\ell$ is $(0,1)$-periodic and in fact, one can define a canonical *Tate-twisting* $\mathbb{F}_\ell$-vector space attached to this cohomology. Using the notation of *loc. cit.*, 2.1.3, 2.1.5, one puts:

$$\mathbb{F}_\ell(1) := \tilde{\mathrm{H}}^1_{\mathrm{B}}(\mathbb{G}_m, \mathbb{F}_\ell)$$

using the reduced Betti cohomology of the sphere $\mathbb{G}_m$. According to the axioms of a mixed Weil theory, this is a 1-dimensional $\mathbb{F}_\ell$-vector space so that we can define its i-th tensor power $\mathbb{F}_\ell(i)$ for any integer i. Moreover, for any smooth complex scheme X, any pair of integers $(n,i) \in \mathbb{Z}^2$, one gets a canonical isomorphism:

$$\mathbf{H}^{n,i}_{\mathrm{B}}(X, \mathbb{F}_\ell) \simeq \mathbf{H}^n_{\mathrm{B}}(X, \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} \mathbb{F}_\ell(i).$$

See *loc. cit.* §2.1.7. In fact, in the case of Betti cohomology, Tate-twists admit a canonical trivialization:

$$\mathbb{F}_\ell(1) = \tilde{\mathrm{H}}^1(\mathbb{G}_m(\mathbb{C}), \mathbb{F}_\ell) \simeq \mathrm{H}^0(S^0, \mathbb{F}_\ell) = \mathbb{F}_\ell.$$

We will still denote by $c \in \mathbf{H}^{0,1}_{\mathrm{B}}(\mathbb{C}, \mathbb{F}_\ell)$ the class defined by this isomorphism,[85] so that the $(0,1)$-periodicity of Betti cohomology is induced by multiplication by c.

According to the description of the motivic Eilenberg–MacLane spectrum using symmetric powers (see Voevodsky, 1998, §6.1),[86] one deduces a canonical isomorphism: $\rho_{\mathbb{C}}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell) \simeq \mathbf{H}\mathbb{F}_\ell$. This allows one to apply the adjunction $(\rho_{\mathbb{C}}, \rho_{\mathbb{C}*})$ to get the following morphism of ring spectra:[87]

$$\gamma_{\mathrm{B}}\colon \mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell \to \rho_{\mathbb{C}*}\rho_{\mathbb{C}}(\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell) \simeq \rho_{\mathbb{C}*}(\mathbf{H}\mathbb{F}_\ell) = \mathbf{H}_{\mathrm{B}}\mathbb{F}_\ell.$$

The functoriality properties of the induced map on the associated cohomologies and the construction of τ imply that $\gamma_{\mathrm{B}*}(\tau) = c$.

[84] Indeed, Theorem 2.3.23 of *loc. cit.* fails in the $\mathbb{F}_\ell$-linear context because of the existence of nontrivial Steenrod operations. This is why we use a different argument to get the map γ_{B} below.
[85] According to *loc. cit.* 2.2.6 and 2.2.8, it induces an orientation of the motivic ring spectrum $\mathbf{H}_{\mathrm{B}}\mathbb{F}_\ell$, which is unique as seen from the preceding isomorphism;
[86] this works well since we are over a field of characteristic 0,
[87] There are of course several other ways to build this map.

Finally, the comparison of $\mathbb{F}_\ell$-linear étale cohomology with Betti cohomology induces a commutative diagram of motivic ring spectra:

$$\begin{array}{ccccc} \mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell & \xrightarrow{\gamma_{\text{ét}}} & \mathbf{H}_{\text{ét}}\mu_\ell & \xrightarrow[\sim]{\tau} & \mathbf{H}_{\text{ét}}\mathbb{F}_\ell \\ \| & & & & \downarrow\wr \\ \mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell & & \xrightarrow{\gamma_{\mathrm{B}}} & & \mathbf{H}_{\mathrm{B}}\mathbb{F}_\ell \end{array} \tag{4.9.b}$$

where the right-hand side map is the isomorphism, obtained by the analytification map. The commutativity of the diagram can be obtained by using the analytification functor of Ayoub (2010, §2). Therefore, one can restate all the previous results in terms of complex realization as follows.

Corollary 4.10. — *Over the field $k = \mathbb{C}$, the map γ_{B} induces isomorphisms of motivic ring spectra:*

$$\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell \simeq \tau_{\geq 0}\mathbf{H}_{\mathrm{B}}\mathbb{F}_\ell,$$
$$\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell[\tau^{-1}] \simeq \mathbf{H}_{\mathrm{B}}\mathbb{F}_\ell.$$

The following result was first proved by Dugger and Isaksen (2010) (see in particular Corollary 2.9).

Corollary 4.11. — *Consider the notation introduced above. Then the complex realization functor $\rho_{\mathbb{C}}$ induces an isomorphism of graded Hopf algebroid (see Section 4.1.3) over the graded ring $\mathbf{M}_\ell[\tau^{-1}] \simeq \mathbb{F}_\ell[\tau, \tau^{-1}]$:*

$$(\mathbf{M}_\ell, \mathcal{A}_{**}(\mathbb{C}, \mathbb{F}_\ell))[\tau^{-1}] \simeq (\mathbb{F}_\ell, A_{\mathrm{cl}}) \otimes_{\mathbb{F}_\ell} \mathbf{M}_\ell[\tau^{-1}].$$

The left-hand side is obtained by inversion of the element τ, while the right-hand side is obtained by scalar extensions.

In fact, *op. cit.* is slightly less precise, and considers only the case $\ell = 2$, which is the most interesting one from a computational perspective.

Remark 4.12. — This corollary is the first manifestation of the principle that the weights in (torsion) motivic homotopy theory add a supplementary dimension to the classical invariants. As an example, one should be aware that the cohomological operations on Betti cohomology modulo ℓ are obtained from the Steenrod algebra by scalar extension along the Laurent polynomial $\mathbb{F}_2$-algebra $\mathbb{F}_2[c, c^{-1}]$. In particular, besides the Steenrod operations, we have the operation corresponding to multiplication by the scalar $c \in \mathbf{H}^{0,1}_{\mathrm{B}}(\mathbb{C}, \mathbb{F}_\ell)$, which expresses the $(0, 1)$-periodicity of Betti cohomology.

4.3. The motivic Adams spectral sequence

4.3.1. *Bousfield nilpotent completions and Adams towers.* — We will now recall a general construction due to Bousfield (1979, §5), from which we adopt the notation. The aim is to define generalized Adams spectral sequences, compute their E_2-term as suitable extension groups, compute their abutment in terms of appropriate resolutions, and to elucidate their convergence properties. Nowadays, the construction can be made in an arbitrary stable presentable (symmetric) monoidal ∞-category $\mathscr{C}$, which encompasses both the cases of SH and SH(k).[88]

To be consistent with the previous notation, we let $[X,Y]_{\mathscr{C}}$ be the morphisms between two objects of $\mathscr{C}$ computed in its homotopy category. Given an integer $s \in \mathbb{Z}$, we define: $\pi_s(X,Y) = [X[s],Y]$.

We fix an object $\mathbf{E}$ in $\mathscr{C}$ with a (not necessarily commutative) monoid structure[89] $u\colon \mathbb{1} \to \mathbf{E}$ and $\mu\colon \mathbf{E}\otimes\mathbf{E} \to \mathbf{E}$ in the associated homotopy category $\mathrm{Ho}\,\mathscr{C}$. We slightly abuse notation and denote by $u\colon \mathbb{1} \to \mathbf{E}$ an arbitrary representative, as an element of $\mathscr{C}_1$, of the homotopy class u. We then consider the homotopy fiber $\overline{\mathbf{E}}$ of the map u and put $\overline{\mathbf{E}}^s = \overline{\mathbf{E}}^{\otimes s}$. One deduces homotopy exact sequences:

$$\begin{array}{l} \overline{\mathbf{E}} \xrightarrow{f_0=\epsilon} \mathbb{1} \xrightarrow{\quad u \quad} \mathbf{E}, \\ \overline{\mathbf{E}}^{s+1} \xrightarrow{f_{s+1}=\epsilon\otimes\overline{\mathbf{E}}^s} \overline{\mathbf{E}}^s \longrightarrow \mathbf{E}\otimes\overline{\mathbf{E}}^s, \end{array}$$

the second one being obtained from the first one by left-tensoring with $\overline{\mathbf{E}}^s$. We deduce a decreasing *tower* of objects over $\mathbb{1}$:

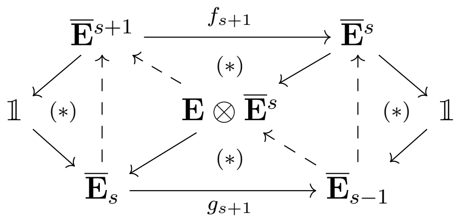
$$\cdots \to \overline{\mathbf{E}}^{s+1} \xrightarrow{f_{s+1}} \overline{\mathbf{E}}^s \to \cdots \to \overline{\mathbf{E}} \xrightarrow{f_0} \mathbb{1}$$

or, in other words, an ∞-functor $\overline{\mathbf{E}}^\bullet\colon \mathbb{Z}^{\mathrm{op}}_{\geq 0} \to (\mathscr{C}/\mathbb{1})$, with values in the indicated comma ∞-category (see Cisinski, 2016, Définition 10.1 for comma ∞-categories). Let us also define by $\overline{\mathbf{E}}_s$ the homotopy cofiber which fits into the homotopy exact sequence:

$$\overline{\mathbf{E}}^{s+1} \to \mathbb{1} \to \overline{\mathbf{E}}_s.$$

Applying the octahedron axiom (see Lurie, 2017, Theorem 1.1.2.14), to the preceding homotopy exact sequences, one deduces an octahedral diagram, in planar form:

(4.12.a)



[88] The reader can also consult Mathew, Naumann, and Noel (2017, Part 1), Bachmann and Østvær (2022, §2), or Mantovani (2024).

[89] It is useful to avoid the commutativity assumptions. Indeed, it is easier to construct an A_∞-structure rather than an E_∞-structure on ring spectra, in the classical stable homotopy category. For example, it is a famous result of Lawson (2018) that there cannot exist an E_∞-ring structure on $\mathbf{BP}$.

where a triangle containing $(*)$ is a homotopy exact sequence, a dashed arrow is a boundary map of such a homotopy exact sequence (therefore the target is implicitly suspended once), and all other triangles are commutative (including the ones obtained by identifying the two objects $\mathbb{1}$ to a single vertex). We have therefore obtained a tower $\overline{\mathbf{E}}_\bullet\colon \mathbb{Z}_{\geq 0}^{\mathrm{op}} \to \mathbb{1}/\mathscr{C}$ of objects under $\mathbb{1}$.(90)

We can then apply any homological functor(91) $H\colon \operatorname{Ho}\mathscr{C} \to \mathcal{A}$ to the diagram (4.12.a), yielding two exact couples that fit into a Rees system in the terminology of Eckmann and Hilton (1966). According to Theorem 7.10 of *loc. cit.*, both exact couples give rise to the same spectral sequence. We apply this construction to the homological functor $[\mathbb{1}, -]_{\mathscr{C}}$.(92)

Definition 4.13. — *Consider the above assumptions. One defines the nilpotent* **E**-*completion of an arbitrary object* **X** *as the following homotopy limit:*

$$\hat{\mathbf{X}}^{\mathbf{E}} := \lim_{n\geq 0} \left(\mathbf{X} \otimes \overline{\mathbf{E}}_n\right).$$

The spectral sequence associated to the exact couples obtained from (4.12.a) *by replacing the entries* $\mathbb{1}$ *with* $\hat{\mathbb{1}}^{\mathbf{E}}$ *and then applying the homological functor* $[\mathbb{1}, -]_{\mathscr{C}}$*:*

$$E_{1,\mathbf{E}}^{s,t} = \pi_{t-s}(\mathbb{1}, \mathbf{E} \otimes \overline{\mathbf{E}}^s) \Rightarrow \pi_{t-s}(\mathbb{1}, \hat{\mathbb{1}}^{\mathbf{E}})$$

is called the **E**-Adams spectral sequence.(93)

Example 4.14. — 1. Applying the above construction with $\mathscr{C} = \mathscr{SH}$, and with the associative algebra $\mathbf{E} = \mathbf{H}\mathbb{F}_p, \mathbf{MU}, \mathbf{BP}$ respectively, one obtains the three spectral sequences which appear in the previous section.

2. We will apply the above construction in the motivic case $\mathscr{C} = \mathscr{SH}(k)$. Note that in the motivic case, there is an additional grading coming from the Tate twist. In particular, instead of applying the above construction to a given ring spectrum **E**, we will apply it to the graded object $\mathbf{E}(*)$. This implies that we will be working with the same object as above, but equipped with an additional grading, which will automatically be compatible with products.

   The main example for us will be $\mathbf{E} = \mathbf{H}_{\mathrm{M}}\mathbb{F}_2$ over the field $k = \mathbb{C}$. However, it is possible to consider the two cases $\mathbf{E} = \mathbf{H}\mathbb{F}_\ell, \mathbf{MGL}$. In addition, the analog of the Brown-Peterson spectrum exists in motivic homotopy, a ring spectrum denoted by **BPGL**. See Hu, Kriz, and Ormsby (2011), or more generally Naumann, Spitzweck, and Østvær (2009).

Remark 4.15. — The convergence of the **E**-Adams spectral sequence is a delicate matter. If $E_1^{s,t}$ vanishes for $s > t$ (which will follow in all our applications), then one gets a

(90) According to Hopkins (1999), one should call it the *normalized A-Adams resolution of* $\mathbb{1}$. By tensoring with an arbitrary object **X**, we obtain the *normalized* **E**-*Adams resolution of* **X**.

(91) i.e., $\mathcal{A}$ is an abelian category and H sends a homotopy exact sequence to a long exact sequence;

(92) The general construction applies the functor $[\mathbf{X}, (-) \otimes \mathbf{Y}]_{\mathscr{C}}$ for arbitrary objects **X**, **Y**.

(93) Following the topological conventions, the differentials d_1^{**} are of bidegree $(1, 0)$.

weak form of convergence as explained in Bousfield (1979), beginning of §6, and called *conditional convergence*. According to *loc. cit.* Proposition 6.3, the strong convergence[94] is equivalent to the vanishing: $\lim_r^1\left(E_r^{s,t}\right) = 0$.

In the applications, the spectral sequence will in fact be concentrated in a bounded region from the second page onward. This implies that for any (s,t), the sequence $(E_r^{s,t})_{r\geq 1}$ stabilizes and that the filtration on the abutment is finite. (See the argument in Section 4.1.1.)

4.3.2. *The cobar construction.* — Let us consider the notation of the above definition, and assume in addition $\mathbf{E}$ is an A_∞-object in $\mathscr{C}$, or equivalently an associated algebra object as defined by Lurie (2017, §4.1.1, Definition 4.1.1.6).

The next step is to compute the E_2-term of the preceding spectral sequence. As in topology, we associate to $\mathbf{E}$ a homology theory on $\mathscr{C}$ which to an object X of $\mathscr{C}$ associates the graded $\mathbf{E}_*$-module $\mathbf{E}_*X = \pi_*(\mathbb{1}, \mathbf{E} \otimes X)$, where $\mathbf{E}_* = \pi_*(\mathbb{1}, \mathbf{E})$ is the associated *ring of coefficients*. It follows as in Section 4.1.3, that the pair $(\mathbf{E}_*, \mathbf{E}_*\mathbf{E})$ is a *Hopf algebroid*.

To go further, we again follow the classical approach from topology highlighted by Hopkins (1999, §5). Using the A_∞-structure on $\mathbf{E}$, one defines a cosimplicial diagram $\mathrm{CB}^\bullet(\mathbf{E})\colon \Delta \to \mathscr{C}$:

$$\mathbf{E} \rightrightarrows \mathbf{E} \otimes \mathbf{E} \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E} \cdots\!\!\rightarrow \cdots$$

such that $\mathrm{CB}^n(\mathbf{E}) = \mathbf{E}^{\otimes n+1}$; see Mathew, Naumann, and Noel (2017, Construction 2.7). This is called the *cobar construction* on $\mathbf{E}$.[95] According to a classical procedure (originally formalized by Bousfield and Kan (1972)), one can extract a tower $\mathbb{Z}_{\geq 0}^{op} \to \mathscr{C}$ by taking partial limits:[96]

$$\mathrm{Tot}_n \, \mathrm{CB}^\bullet(\mathbf{E}) = \varprojlim_{i\leq n} \mathrm{CB}^i(\mathbf{E}).$$

Let us state explicitly the following fundamental comparison result, proved in this form in *op. cit.* Proposition 2.14.

Proposition 4.16. — *Under the above assumptions, there exists a canonical equivalence of towers:* $\overline{\mathbf{E}}_\bullet \simeq \mathrm{Tot}_\bullet \, \mathrm{CB}^\bullet(\mathbf{E})$.[97]

The tower $\mathrm{Tot}_\bullet \, \mathrm{CB}^\bullet(\mathbf{E})$ *will be called the* tower $\mathbf{E}$-Adams resolution *attached to* $\mathbf{E}$.[98]

[94] By "strong convergence", we mean that the filtration on the abutment is exhaustive, separated (Hausdorff) and complete;

[95] Note from *loc. cit.* that it admits augmentation by $\mathbb{1}$.

[96] In fact, the elegant Theorem 2.8 of *op. cit.*, attributed to Lurie, asserts that the general ∞-functor

$$\mathrm{Tot}_\bullet\colon \mathscr{F}\mathrm{un}(\Delta, \mathscr{C}) \to \mathscr{F}\mathrm{un}(\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathscr{C})$$

is an equivalence of stable ∞-categories. This is a stable ∞-categorical version of the Dold-Kan equivalence.

[97] We have neglected the natural augmentation with source $\mathbb{1}$.

[98] Here, it is understood that this is a resolution of $\mathbb{1}$.

One deduces that for any integer $t \in \mathbb{Z}$, the complexes $E_1^{*,t}$ from the $\mathbf{E}$-Adams spectral sequence take the following form

$$\cdots 0 \to \mathbf{E}_t \to \mathbf{E}_{t-1}\mathbf{E} \to \mathbf{E}_{t-2}(\mathbf{E}^{\otimes 2}) \to \cdots \to \mathbf{E}_{t-s}(\mathbf{E}^{\otimes s}) \to \cdots$$

where $\mathbf{E}_*$ sits in (cohomological) degree $s = 0$, and the differentials are obtained as the alternating sum of the maps coming from the cobar-resolution of $\mathbf{E}$. Using the exterior pairing $\mathbf{E} * (X) \otimes_{\mathbf{E}_*} \mathbf{E} * (Y) \to \mathbf{E}_*(X \otimes Y)$, one deduces the following corollary.

Corollary 4.17. — *Consider the above notation. Assume that* $\mathbf{E}$ *satisfies the following* weak Künneth property*:*

$$\forall s > 1, \text{ the exterior pairing induces an isomorphism } (\mathbf{E}_*\mathbf{E})^{\otimes_{\mathbf{E}_*} s} \to \mathbf{E}_*(\mathbf{E}^{\otimes s}).$$

Then one can compute the E_2-term of the $\mathbf{E}$-Adams spectral sequence as

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathbf{E}_*\mathbf{E}}(\mathbf{E}_*, \mathbf{E}_*)$$

the Ext*-group being computed in the category of comodules over the Hopf algebroid* $(\mathbf{E}_*, \mathbf{E}_*\mathbf{E})$.

In our examples from topology, the weak form of the Künneth property follows from the assumption that $\mathbf{E}_*\mathbf{E}$ is flat over $\mathbf{E}_*$; see Hopkins (1999, Proposition 5.7).

4.3.3. *Localization and completion.* — We finally recall the general tools to compute the abutment of the previous Adams spectral sequence, still following Bousfield (1979). According to the general philosophy developed by Bousfield and Kan (1972), this involves the ∞-categorical analogue of completion or localization in classical algebra. To recall these basic definitions, we will now assume that $\mathscr{C}$ is a presentable monoidal stable ∞-category.

We consider a 1-morphism $f : L \to \mathbb{1}$,[99] which will play the role of elements in the (motivic) stable stem, and X be object of $\mathscr{C}$.

1. For an integer $n > 0$, one defines the n-th (homotopy) quotient X/f^n of X by f as the homotopy cofiber of the map $X \otimes L^{\otimes n} \to X$.
2. One defines the completion of X at f as the following homotopy limit:[100]
$$\hat{X}_f := \mathrm{holim}_n \left(X/f^n\right).$$
3. Assuming that L is $\otimes$-invertible, one defines the localization of X at f as the following homotopy limit:
$$X[f^{-1}] := \mathrm{hocolim}\left(X \xrightarrow{f} X \otimes L^{-1} \xrightarrow{f} X \otimes L^{-2} \to \cdots\right)$$
where we have denoted by L^{-n} the n-th tensor power of the $\otimes$-inverse of L, and simply by f the morphism induced by f after the obvious tensor product.

[99] Usually, it will be a homotopy class but the constructions will not depend on a choice of representative;
[100] This is where we use the assumption that $\mathscr{C}$ is presentable;

These constructions can be extended to an n-tuple $I = (f_i \colon L_i \to \mathbb{1})$ in an obvious way (left to the reader). We will (loosely) say that I is an ideal of $\mathbb{1}$, and use the notation $\mathbf{X}/I$, $\hat{\mathbf{X}}_I$, $\mathbf{X}[I^{-1}]$.

In order to state the next result, we will need to assume the existence of a t-structure t on $\mathscr{C}$.(101) We still use homological conventions, write $A \geq 0$ for non-negative objects and let $\underline{\pi}_0$ be the associated (co)homological functor. We will assume that $\mathscr{C}$ is *left-complete*.(102) We will also assume that t is *compatible with the tensor structure* in the usual sense (see footnote 60, page 46).

We can now state the following pretty generalization of Bousfield (1979, Theorems 6.5, 6.6), due to Bachmann and Østvær (2022) and Mantovani (2024).

Theorem 4.18. — *Consider the above notation. We let* $\mathbf{E}$ *be an* E_∞*-algebra in* $\mathscr{C}$ *such that* $\mathbf{E} \geq 0$ *and* $\mathbf{X}$ *be a connective object with respect to* t.(103) *Let* $I = (f_i \colon L_i \to \mathbb{1})$ *be an ideal of* $\mathbb{1}$ *as above.*

1. *Assume that for all* i, L_i *is non-negative, strongly dualizable with a non-negative strong dual, and* $\underline{\pi}_0(\mathbf{E}) \simeq \underline{\pi}_0\left(\mathbb{1}/I\right)$. *Then:* $\hat{\mathbf{X}}_{\mathbf{E}} = \hat{\mathbf{X}}_I$.
2. *Assume that for all* i, *the tensor product with* L_i *is a* t*-exact equivalence of* ∞*-categories and* $\underline{\pi}_0(\mathbf{E}) \simeq \underline{\pi}_0\left(\mathbb{1}[I^{-1}]\right)$. *Then:* $\hat{\mathbf{X}}_{\mathbf{E}} = \mathbf{X}[I^{-1}]$.

We refer the reader to Bachmann and Østvær (2022, Theorem 2.1+2.2),(104) or to Mantovani (2024).

Example 4.19. —

1. Let $\mathbf{X}$ be a connective spectrum with respect to the canonical t-structure on $\mathscr{SH}$. Then, for a prime $\ell \in k^\times$, one has:
$$\hat{\mathbf{X}}_{\mathbf{H}\mathbb{F}_\ell} = \hat{\mathbf{X}}_\ell, \hat{\mathbf{X}}_{\mathrm{H}\mathbb{Z}} = \hat{\mathbf{X}}_{\mathbf{MU}} = \mathbf{X}, \hat{\mathbf{X}}_{\mathbf{BP}} = \mathbf{X}[\ell^{-1}].$$
2. Let $\mathbf{X}$ be a connective motivic spectrum with respect to the homotopy t-structure on $\mathscr{SH}(k)$. Then, by applying the above theorem and Example 3.24, one deduces:
$$\hat{\mathbf{X}}_{\mathbf{MGL}} = \hat{\mathbf{X}}_{\mathbf{H}_{\mathrm{M}}\mathbb{Z}} = \hat{\mathbf{X}}_\eta, \hat{\mathbf{X}}_{\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell} = \hat{\mathbf{X}}_{(\ell,\eta)}, \hat{\mathbf{X}}_{\mathbf{BPGL}} = \mathbf{X}[\ell^{-1}].$$
3. Consider $\mathbf{X}$ as in the previous point. Assume in addition that (-1) is a sum of squares in k and that $\ell \neq 2$. Then: $\hat{\mathbf{X}}_{(\ell,\eta)} = \hat{\mathbf{X}}_\ell$. See Mantovani (2024, Lemma 3.3.1).
4. Assume that (-1) is a sum of squares in k and that k has finite 2-cohomological dimension. Then the $\mathbf{H}_{\mathrm{M}}\mathbb{F}_2$-nilpotent completion of the motivic sphere spectrum agrees with its 2-completion:
$$\hat{\mathbb{1}}_{\mathbf{H}_{\mathrm{M}}\mathbb{F}_2} = \hat{\mathbb{1}}_2.$$

(101) Or, equivalently, its homotopy category $\mathrm{Ho}\,\mathscr{C}$ following standard usage in the literature.
(102) i.e., for any object X in $\mathscr{C}$, the canonical map $X \longrightarrow \operatorname{colim}_{n \to -\infty}\left(\tau_{\geq n}(X)\right)$ is an isomorphism.
(103) In other words, homologically bounded below.
(104) The second point can easily be deduced from *loc. cit.* Theorem 2.1.

This fact was first proved, over algebraically closed fields, by Hu, Kriz, and Ormsby (2011). The more general case is proved in Mantovani (2024, Lemma 3.3.2) (based on *loc. cit.*).

Remark 4.20. — We do not enter into the details, but it is important to note that another corollary of the above theorem is that, under the stated hypothesis, the nilpotent $\mathbf{E}$-localization also coincides with the (left Bousfield) localization with respect to $\mathbf{E}$, which consists in inverting maps inducing isomorphisms in $\mathbf{E}_*$-homology — beware that in motivic homotopy, one has to consider the bigraded homology theory $\mathbf{E}_{**}$. The resulting localization functor L_E, following the general procedure described in Section 1.4.3, has good properties — for example, it respects E_∞-ring spectra.

Given all these preparations, we can now state the existence and form of the motivic Adams spectral sequence, first introduced by Morel (1999), and more thoroughly studied by Hu, Kriz, and Ormsby (2011) and Dugger and Isaksen (2010). We state it only over the base field $\mathbb{C}$ and modulo 2 as this is our main focus. We leave to the reader the formulation of the other cases.

PROPOSITION 4.21. — *We work over the base field $k = \mathbb{C}$, and modulo the prime $\ell = 2$. Then there exists a* weight graded motivic Adams spectral sequence*, with a trigraded multiplicative structure, of the following form:*

$$E_2^{s,t,w} = \mathrm{Ext}_A^{s,t,w}(\mathbf{M}_2, \mathbf{M}_2) \Rightarrow \pi_{t-s,w}\left(\hat{\mathbb{1}}\right) =: \hat{\pi}^{\mathbb{C}}_{t-s,w}$$

with differentials on the r-th page of tri-degree $(r, r-1, 0)$.

- *the* Ext *group is computed in the category of bigraded comodules over the Hopf algebroid $(A, \mathbf{M}_2)$ (as defined in 4.2), s being the degree of the extension group, and (t, w) being the internal degree of that category;*
- *on the abutment, $\hat{\mathbb{1}}$ denotes the 2-completion (defined in 4.3.3) of the motivic sphere spectrum, or equivalently its nilpotent $\mathbf{H}_{\mathrm{M}}\mathbb{F}_2$-completion according to the preceding example.*

In addition, the spectral sequence is strongly convergent, and the filtration on the abutment is finite in each motivic stem *$f = (t - s)$.*

Indeed, this is simply the $\mathbf{H}_{\mathrm{M}}\mathbb{F}_2$-Adams spectral sequence constructed above. To get the correct form of the E_2-term, one needs the appropriate weak Künneth property to apply Corollary 4.17. This is proved in Dugger and Isaksen (2010, Proposition 7.5), but it follows more generally from the fact $\mathbf{H}_{\mathrm{M}}\mathbb{F}_\ell$ is cellular according to Hoyois (2015) using Dugger and Isaksen (2005). The last assertion follows from Dugger and Isaksen (2010, Corollary 7.15), which establishes vanishing properties for the motivic Adams spectral sequence analogous to that of the Adams spectral sequence.

Remark 4.22. — As in the topological case, Isaksen, Wang, and Xu (2023) (and all the other papers in this topic) uses a different grading convention to depict the r-th page of the motivic Adams spectral sequence. More precisely, the group $E_r^{s,t,w}$ is displayed on a plane, following the conventions of the Adams charts, as described in Step 1 of Section 4.1.2:

- the lines are indexed by the integer s, which is called the *Adams degree*. In particular, the picture is concentrated in degree $s \geq 0$;
- the columns are indexed by the integer $f = t - s$, which is called the *stem*.
- the picture has to be thought of as a projection of a three-dimensional picture, along the axis represented by the index w, called the *weight*.

To summarize, a group of grading (s, t, w) is pictured in the point of coordinates $(f = t - s, s)$. Then the differentials on the r-th page have tri-degree $(1, r-1, 0)$. Moreover, when displaying the E_r-term for r big enough, each column represents the gradings of the abelian group $\hat{\pi}^{\mathbb{C}}_{f,*}$.

There is a deep interplay between the motivic Adams spectral sequence and the (classical) Adams spectral sequence. Roughly speaking, one gets back the latter from the former by inverting the motivic element τ. More precisely, building on Corollary 4.11, Dugger and Isaksen proved the following remarkable result.[105]

Theorem 4.23. — *Keep the assumptions of the preceding proposition. Then the complex realization functor induces an isomorphism of weight-graded* $\mathbf{M}_2[\tau^{-1}]$*-linear spectral sequences:*

$$E^{***}_{*,\mathbf{H}_{\mathrm{M}}\mathbb{F}_2}[\tau^{-1}] \xrightarrow{\sim} \mathbf{E}^{**}_{*,\mathbf{H}\mathbb{F}_2} \otimes_{\mathbb{F}_2} \mathbf{M}_2[\tau^{-1}]$$

where we have indicated the ring spectra for more clarity.

In fact, one can derive a direct proof using the construction of $\mathbf{E}_*$-Adams spectral sequences from the cobar resolution and Corollary 4.10.

As both spectral sequences are strongly convergent, one deduces the following corollary.

Corollary 4.24. — *Consider the above assumptions. Then the element* τ *lifts to a (homological) bidegree* $(0, -1)$ *element in the* $\hat{\pi}^{\mathbb{C}}_{**}$ *motivic stable homotopy groups of the* 2*-completed motivic sphere, that we continue to denote by* τ*. Moreover, the complex realization functor induces an isomorphism of bigraded* $\mathbf{M}_2[\tau^{-1}]$*-algebras:*

$$\hat{\pi}^{\mathbb{C}}_{**}[\tau^{-1}] \simeq \pi^{\mathrm{S}}_{*} \otimes_{\mathbb{Z}} \mathbb{Z}_2[\tau, \tau^{-1}]$$

where on the left-hand side, one has localized with respect to the element τ*, and the isomorphism maps the motivic class* τ *to the indeterminate* τ *on the right-hand side.*

In particular, we have obtained a map

$$\tau\colon \hat{\mathbb{1}}(-1) \to \hat{\mathbb{1}} \qquad (4.24.a)$$

which lifts the previously defined map $\tau\colon \mathbf{H}_{\mathrm{M}}\mathbb{F}_2(-1) \to \mathbf{H}_{\mathrm{M}}\mathbb{F}_2$. This is a key player of the computations of Isaksen, Wang, and Xu (2023). The reader may appreciate the

[105] This is stated explicitly in Isaksen (2019, Proposition 3.0.2).

direct and elegant construction of this lift due to Hu, Kriz, and Ormsby (2011, Remark following Lemma 23) and based on Morel's Theorem 3.11.

Remark 4.25. — One can check that the isomorphism of the above corollary is compatible with the one constructed by Levine in Theorem 3.12 for $k = \mathbb{C}$, using the canonical map $\pi^{\mathbb{C}}_{**} \to \hat{\pi}^{\mathbb{C}}_{**}$.

4.4. Deforming homotopy theories via the motivic class τ

4.4.1. *The special fiber of τ.* — We learned in the end of the previous section that inverting the motivic element τ — which in particular kills all τ-torsion elements — allows one to recover the classical Adams spectral sequence from the motivic Adams spectral sequence.

On the other hand, the pages of the motivic Adams spectral sequence may contain non-trivial τ-torsion classes, and they sometimes hide extensions that still are visible in the abutment. To better understand this phenomenon, Isaksen (2019, Chapter 5), introduces the *cofiber of* τ denoted by $C\tau$ and defined as the following homotopy cofiber in the ∞-category of 2-complete motivic spectra[(106)]

$$\hat{\mathbb{1}}(-1) \xrightarrow{\tau} \hat{\mathbb{1}} \xrightarrow{i} C\tau \xrightarrow{\partial} \hat{\mathbb{1}}(-1)[1]. \tag{4.25.a}$$

This procedure of killing homotopy classes is very classical. A remarkable result of Gheorghe (2018) is that, in this particular case, the cofiber $C\tau$ not only acquires a ring structure in homotopy category, but can also be equipped with an E_∞-ring structure compatible with the canonical map i. This allows one to define the ∞-category $C\tau-$mod of modules over $C\tau$, which can be thought of as *τ-torsion 2-complete motivic spectra over* $\mathbb{C}$. Although preliminary computations of Isaksen (2019, Proposition 6.2.5) may suggest such a connection, the following beautiful result proved by Gheorghe, Wang, and Xu (2021, Corollary 1.2) reveals another surprising bridge between motivic and classical homotopical invariants.

Theorem 4.26. — *With the above notation, there exists a canonical equivalence of stable ∞-categories*

$$C\tau - \mathrm{mod}^{\mathrm{cell}} \simeq \mathrm{Stable}(\mathbf{BP}_*\mathbf{BP} - \mathrm{comod}^{\mathrm{ev}})$$

where:

- *the left-hand side is made by the cellular $C\tau$-modules, i.e., the full sub-∞-category spanned by colimits of $C\tau$-modules of the form $C(\tau)(q)[p]$ for any pair $(p, q) \in \mathbb{Z}^2$;*
- *the right-hand side is Hovey's stable ∞-category of even comodules over the Hopf algebroid $(\mathbf{BP}_*, \mathbf{BP}_*\mathbf{BP})$, obtained by localizing complexes of such comodules along homotopy isomorphisms.*

(106) therefore, it can be computed by first taking the homotopy cofiber in the motivic homotopy category and then applying the 2-completion functor;

Remark 4.27. — 1. This result can be reformulated in terms of stacks: the ∞-category of cellular $C\tau$-modules is equivalent to the ind-∞-category of perfect complexes on the moduli stack of formal group laws $\mathcal{M}_{FGL}$ over $\mathbb{Z}_2$ (see also Remark 4.3). This interpretation essentially underlies the proof of Corollary 1.2 given by Gheorghe, Wang, and Xu (2021), as discussed after Remark 4.15 therein.

2. The preceding theorem was generalized by Bachmann, Kong, Wang, and Xu (2022) through the construction of the so-called *Chow-t-structure* on the stable motivic homotopy category $\mathscr{SH}(k)$ for any base field k.

   This t-structure can be described as the unique one whose non-negative objects are generated under colimits and extension by the Thom spaces $\mathrm{Th}(v)$ of a virtual vector bundle v over a smooth *and proper* k-scheme X.[107] The authors then relate the invariants associated with this t-structure — such as truncations, its heart, and heart-valued objects — to even comodules over appropriate Hopf algebroids, after inverting the characteristic exponent e of the base field k.

   As an example, they identify the cellular heart of the Chow-t-structure on $\mathrm{SH}(k)[e^{-1}]$ with the e-localized ∞-category of $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$-modules (see Theorem 1.12).

The important point from the computational perspective is the following corollary, which follows from the previous theorem (see Gheorghe, Wang, and Xu (2021), Theorem 1.3 for the statement and Part 2 for details).

Corollary 4.28. — *There is an isomorphism of spectral sequences between the motivic Adams spectral sequence for $C\tau$ and the algebraic Adams–Novikov spectral sequence.*[108]

4.4.2. *Final procedure to compute motivic and classical stable stems.* — The innovative ingredient introduced by Isaksen, Wang, and Xu (2023) for computing stable stems out of motivic stable stems is the following *deformation diagram*

$$\hat{\mathbb{1}}[\tau^{-1}] \longleftarrow \hat{\mathbb{1}} \underset{i}{\overset{\partial}{\leftrightarrows}} C\tau \tag{4.28.a}$$

where the left-hand (resp. right-hand) side is thought as the generic (resp. special) fiber. Concretely, this diagram means two facts:

1. the motivic Adams spectral sequence is constrained along the maps i and ∂ by the algebraic Adams–Novikov spectral sequence, which corresponds according to the previous theorem to the Adams spectral of the special fiber $C\tau$;

[107] The name is likely inspired by the work of Bondarko, who defined the *Chow weight structure* on the category of motivic complexes over k using similar types of generators. See, e.g., Bondarko (2010, §7.1) for effective motivic complexes.

[108] The latter is an algebraic spectral sequence based on resolutions of $\mathbf{BP}_*\mathbf{BP}$-comodules, and computes the E_2-term of the Adams–Novikov spectral sequence. See *loc. cit.* §9.1, and also Remark 4.2 for a broader picture.

2. the generic fiber computes the classical Adams spectral sequence, according to Theorem 4.23.

From a practical point of view, this leads to the following computational strategy, which both draws on and refines the classical approach to computing stable stems described in Section 4.1.2:

Step$^\tau$1. Begin by computing the E_2-term of the motivic Adams spectral sequence. This tri-graded object captures refined information, including torsion in the motivic weight. The computation relies on algebraic tools such as the May spectral sequence, and reduces to an algorithmic problem which can be handled by computer.

Compute the pages of the algebraic Adams–Novikov spectral sequence using similar algebraic methods, that can also be handled by computer.

Step$^\tau$2. The previous step gives the determination of the Adams spectral sequence for the cofiber of τ. Then the right part of the deformation diagram allows one to determine differentials of the motivic Adams spectral sequence, and also additional refined information (notably, *Toda brackets*). Ultimately, this allows one to determine the Adams motivic E_∞-page up to a fixed range.

Step$^\tau$3. Solve the extension problem for determining the motivic stable stem from the information of the E_∞-term (as in Step 3 in the original strategy 4.1.2), using again the information coming from the special fiber. This implies finding the so-called *hidden extensions*(109) and in particular with respect to the motivic element τ.

Step$^\tau$4. The last step is obvious, and uses the left part of the deformation diagram: read off the classical stable stems (and information on the classical Adams differentials) from the computations obtained in the previous step.

This motivic approach has led to a significant breakthrough in the effective computation of stable stems. Thanks to the method introduced by Isaksen, Wang, and Xu (2023), it is now possible to push the determination of the 2-primary component(110) of the stable homotopy groups of spheres from dimension 66 up to dimension 90.(111)

4.4.3. *Towards synthetic homotopy.* — As in topology, Step$^\tau$2&3 cannot be made algorithmic and carried out by a computer. In particular, one of the key technical tools that the authors had to use is the construction of an intermediate motivic ring spectrum over $\mathbb{C}$, called the *motivic modular forms spectrum*. It is the motivic analog of the famous *topological modular forms spectrum* of Ando, Hopkins, Strickland (see e.g. Goerss, 2010). The way they construct this object is both interesting and relevant to conclude this lecture.

(109)as precisely defined in Isaksen, Wang, and Xu (2023), Definition 2.10;
(110)this is by far the most difficult primary part of the stable stems;
(111)Albeit a few remaining uncertainties: four unresolved differentials in the motivic Adams spectral sequence, see Table 9 in *loc. cit.*

One still works, as above, in the stable monoidal ∞-category $\mathscr{SH}(\mathbb{C})^{\wedge}_2$, the 2-completed motivic stable homotopy category over $\mathbb{C}$.[112] As in the previous theorem and following Dugger and Isaksen (2005), one restricts to the full sub-∞-category $\mathscr{SH}^{\mathrm{cell}}(\mathbb{C})^{\wedge}_2$ of cellular 2-complete motivic spectra; that is the full sub-∞-category of $\mathscr{SH}(\mathbb{C})^{\wedge}_2$ spanned by colimits of motivic sphere spectra $\hat{\mathbb{1}}(q)[p]$ for $(p,q) \in \mathbb{Z}^2$.[113]

With the aim of describing the latter ∞-category, Gheorghe, Isaksen, Krause, and Ricka (2022) discovered the idea that one can work directly with the Adams resolutions of spectra, framed in the language of filtered spectra. This leads them to the ∞-category $\mathscr{F}\mathrm{un}(\mathbb{Z}^{\mathrm{op}}, \mathscr{SH}^{\wedge}_2)$ of *filtered* 2*-complete spectra*, which is presentable, stable and monoidal (using the Day convolution product). Via a fully faithful embedding, one can consider the tower Adams resolutions of spectra defined in 4.16 viewed as objects in this ∞-category. The importance of the Adams–Novikov spectral sequence (Section 4.1.3) justifies specifically considering the tower Adams resolution associated with the 2-completion $\widehat{\mathbf{MU}}$ of $\mathbf{MU}$:

$$T_\bullet := \mathrm{Tot}_\bullet \mathrm{CB}^\bullet(\widehat{\mathbf{MU}}).$$

In order to make the next theorem work, one looks at a suitably truncated tower, denoted by $\Gamma_\star(S^0)$ in *loc. cit.*,[114] whose w-th term is given by the following formula:

$$\Gamma_w(S^0) := \mathrm{Tot}_\bullet(\tau_{\geq 2w} \mathrm{CB}^\bullet(\widehat{\mathbf{MU}}))$$

where $\tau_{\geq 2w}$ is the truncation functor associated with the canonical t-structure on $\mathscr{SH}$, and we apply it term-wise to the cosimplicial object $\mathrm{CB}^\bullet(\widehat{\mathbf{MU}})$ (see Definition 3.2 of *loc. cit.*).[115] It is proved in *loc. cit.* that $\Gamma_\star(S^0)$ is actually an E_∞-object in filtered spectra. In particular, one can consider the ∞-category of modules $\Gamma_\star(S^0){-}\mathrm{mod}$. The following result, Theorem 6.12 of *loc. cit.*, sheds light on the relation between classical and motivic invariant that we have met on several occasions in this section.

Theorem 4.29. — *There is an explicit pair of mutually inverse equivalences of stable monoidal ∞-categories:*

$$\Gamma_\star(S^0){-}\mathrm{mod} \underset{\sim}{\overset{\sim}{\rightleftarrows}} \mathscr{SH}^{\mathrm{cell}}(\mathbb{C})^{\wedge}_2.$$

(112) From what we have seen before, it can be described in full generality with the left Bousfield localization of $\mathscr{SH}(\mathbb{C})$ with respect to the motivic spectrum $\mathbb{1}/2$. Under an appropriate finiteness assumption, this is also the Bousfield localization with respect to $\mathbf{H}_{\mathrm{M}}\mathbb{F}_2$; see Remark 4.20.
(113) The author emphasizes that, from a motivic perspective, cellular objects should also be viewed as *ind-Artin–Tate objects*, since we work over the algebraically closed field $\mathbb{C}$. Artin–Tate motives play a key role in the theory of motivic complexes and in our current understanding of L-functions and periods over number fields.
(114) The symbol $\star$ here replaces the symbol $\bullet$ that we have used up to now to suggest the underlying filtration on objects. They serve exactly the same purpose.
(115) The effect of this construction is that the spectral sequence associated to $\Gamma_\bullet(S^0)$ is a truncation of the Adams–Novikov spectral sequence associated to $\mathbf{MU}$. See *loc. cit.* Remark 3.4.

Thanks to this surprising equivalence, the authors were able to define the motivic modular forms spectrum mmf in the right-hand side ∞-category, by transporting a topological construction done in filtered 2-complete spectra. In fact, one can also realize the fundamental deformation diagram (4.28.a) directly in $\Gamma_\star(S^0)-\mathrm{mod}$.

4.4.4. *Epilogue.* — The construction and properties of $\Gamma_\star(S^0)-\mathrm{mod}$ have since been axiomatized under the name *synthetic homotopy theory* by Pstrągowski (2023). Though arising from motivic methods, this framework is now fully topological and expected to play a central role in the future of algebraic topology. This stands as a beautiful example of the deep and productive interactions between algebraic geometry and algebraic topology through motivic homotopy theory.

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