Séminaire BOURBAKI
78e année, 2025–2026, n° 1242

Novembre 2025

ON RAVENEL’S TELESCOPE CONJECTURE AND ITS DISPROOF
[After Burklund–Hahn–Levy–Schlank]

by **Christian Ausoni**

Introduction

Ravenel’s Telescope Conjecture is an attempt to account for the periodicity phenomena observed in the stable homotopy category, with for example applications to the computation of the (stable) homotopy groups of a space or a spectrum in the sense of algebraic topology.

Consider a *pointed space* (X, x_0), where X is a topological space and $x_0 \in X$ is a chosen point called *the base point.* All maps and homotopies that we consider here are supposed to be pointed, meaning that they preserve base points. We denote by S^n the Euclidean sphere of dimension n, with a chosen base-point $*$. *The n-th homotopy group of* (X, x_0) is the set of homotopy classes of pointed maps

$$\pi_n(X, x_0) \coloneqq [(S^n, *), (X, x_0)]_* \,.$$

This is a pointed set, and can be given a natural group structure for $n \geq 1$, which is commutative for $n \geq 2$.

The most basic and fundamental example is provided by the spheres, with the problem of determining all continuous maps

$$S^n \to S^m$$

up to homotopy, or in other words, of computing $\pi_n(S^m, *)$. A first interesting example is given by the following self-map of the circle

$$f_d \colon S^1 \to S^1, \quad z \mapsto z^d$$

where $d \in \mathbb{Z}$ and where $S^1 \subset \mathbb{C}$ is viewed as the subspace of complex numbers of modulus 1. These give all the examples of maps $S^1 \to S^1$ up to homotopy, and are not homotopic for different values of d. In other words, we have an isomorphism

$$\mathbb{Z} \to \pi_1(S^1, *), \quad d \mapsto [f_d] \,.$$

On the other hand, it can be proven, using for example covering spaces, that $\pi_n(S^1, *) = 0$ for all $n > 1$.

Let us from now on omit the base point from the notation. In the 1930's, Hurewicz obtained the first systematic information on the homotopy groups of S^m for $m \geq 2$, see Theorem 5.1. He showed that $\pi_n(S^m) = 0$ for $n < m$, while $\pi_m(S^m) \cong \mathbb{Z}$, with the identity map of S^m as generator. However, at the same time it was discovered (in contrast with the case $m = 1$) then when $m \geq 2$ the groups $\pi_n(S^m)$ are not necessarily trivial for $n > m$. The first example of such a homotopically non-trivial map was constructed by Hopf, with the map

$$\eta \colon S^3 \to S^2$$

that now carries his name. It can be defined by restricting the standard quotient map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ to $S^3 \subset \mathbb{C}^2 \setminus \{0\}$, where $\mathbb{C}P^1$ is the complex projective line homeomorphic to S^2. It was then proven by Serre, see Theorem 5.3, that

$$\pi_3(S^2) \cong \mathbb{Z}$$

with η as generator.

We can define a functor $\Sigma \colon \mathrm{Top}_* \to \mathrm{Top}_*$ from the category of pointed spaces to itself, called the suspension, by the formula

$$\Sigma(X, x_0) = \big((X \times [0,1])/\simeq, *\big),$$

the quotient of $X \times [0,1]$ by the equivalence relation generated by $(x,0) \simeq (x,1) \simeq (x_0, t)$ for all $x \in X$ and $t \in [0,1]$, and with $*$ the class of $(x_0, 0)$. One can show that $\Sigma(S^n)$ is homeomorphic to S^{n+1}, and that we have a homomorphism

$$\sigma \colon \pi_n(X) \to \pi_{n+1}(\Sigma X), \quad [f] \mapsto [\Sigma(f)]\,.$$

The fascinating Suspension Theorem of Freudenthal 5.2 states that

$$\sigma \colon \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism if $k < n-1$, and a surjection for $k = n-1$. For example, starting with $\pi_3(S^2)$, this theorem tells us that the chain of homomorphisms

$$\pi_3(S^2) \twoheadrightarrow \pi_4(S^3) \xrightarrow{\cong} \pi_5(S^4) \xrightarrow{\cong} \pi_6(S^5) \xrightarrow{\cong} \cdots .$$

stabilizes with having only isomorphisms beyond $\pi_4(S^3)$. This motivates the following definition.

Definition 0.1. — *Let X be a pointed space. The k-th stable homotopy group of X is defined as the colimit of groups*

$$\pi_k^s(X) = \mathrm{colim}(\pi_k(X) \xrightarrow{\sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\sigma} \pi_{k+2}(\Sigma^2 X) \xrightarrow{\sigma} \ldots)$$

under the suspension homomorphism σ.

We will denote by $\pi_k(\mathbb{S})$ the group $\pi_k^s(S^0)$, and called it *the k-th homotopy group of the sphere spectrum.* Note that by Freudenthal's Theorem, the natural map

$$\pi_{n+k}(S^n) \to \pi_k(\mathbb{S})$$

is an isomorphism when $n > k+1$. By Hurewicz' Theorem 5.1, we deduce that $\pi_0(\mathbb{S}) \cong \mathbb{Z}$. To compute $\pi_1(\mathbb{S})$, it suffices to compute $\pi_4(S^3)$, which we know by the above discussion to be a quotient of $\pi_3(S^2)$. The kernel is generated by 2η, so that we have

$$\pi_1(\mathbb{S}) \cong \pi_4(S^3) \cong \mathbb{Z}/2\,.$$

Serre proved that for any $k \geq 1$, $\pi_k(\mathbb{S})$ is a finite (abelian) group, see Theorem 5.3 and the discussion thereafter. This group has been completely computed for $k \leq 100$ (roughly). For example, Ravenel (2004, Appendix 3) contains tables of these homotopy groups in low degrees. An open and probably unrealistic challenge of the field is to determine them all.

Nevertheless, some aspects of their global structure are know. A major breakthrough was Adams' discovery of the infinite *periodic α-family* of p-torsion elements, for any prime p. For simplicity, let us suppose that p is an odd prime, the behavior at 2 differing (but Adams treats it also). Let $v_p\colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ denote the p-adic valuation.

THEOREM 0.2 (Adams, 1966, Theorem 1.5). — *Let p be an odd prime. Then for all $k \geq 1$, there exists an element*

$$\alpha_k \in \pi_{2(p-1)k-1}(\mathbb{S})$$

generating a cyclic subgroup of order $p^{v_p(k)+1}$ that is a direct factor. Moreover, α_1 is the p-torsion element of lowest degree in $\pi_*(\mathbb{S})$.

Note that the p-torsion summands detected by the α-family $\{\alpha_k\}_k$ lie in degrees

$$2p-3, \quad 2p-3+(2p-2), \quad 2p-3+\mathbf{2}(2p-2), \quad 2p-3+\mathbf{3}(2p-2), \ldots$$

In other words, they appear periodically with a period of $2(p-1)$. This is an example of v_1 periodicity in the sense of chromatic homotopy, as will be explained below. Shortly later, further periodic families were discovered, with a larger period: Smith (1977) and Toda (1971) constructed an infinite $\{\beta_k\}_k$ family of period $2(p^2-1)$ for $p \geq 5$, and Miller, Ravenel, and Wilson (1977) constructed an infinite $\{\gamma_k\}_k$ family of period $2(p^3-1)$ for $p \geq 7$. These are example of v_2-periodic and of v_3-periodic phenomena, respectively.

The Telescope Conjecture is precisely addressing the decomposition of stable homotopy groups into such periodic families. If we are interested in understanding $\pi_*(\mathbb{S})$, since in positive degrees all groups are finite abelian, we might as well work one prime p at a time, localizing away from p. Then the periodic families (such as the one described above) will superpose each other, mixing the different periods, as with the different wavelengths of light superposing to produce the white. If we look at the global picture, no periodicity is visible. The purpose of *chromatic homotopy theory* is to study stable homotopy by isolating and investigating the v_n-periodic phenomena one n at a time.

Before returning to the construction of the α, β and γ families and discussing higher v_n-periodicity, we briefly introduce spectra.

All constructions in stable homotopy take place in the category Sp of spectra. This category can roughly be thought of as a localization of the category Top_* obtained by inverting the suspension operator Σ. This offers a great simplification, since it is no longer necessary to "suspend enough" or pass to colimits to study stable phenomena. Furthermore the category Sp is better behaved, and has a structure that is comparable to the structure of the derived category $D(\mathbb{Z})$ of $\mathbb{Z}$-modules, but were $\mathbb{Z}$ is replaced by $\mathbb{S}$. The localization functor $\mathrm{Top}_* \to \mathrm{Sp}$ is denoted Σ^∞.

We refer to Lurie (2006, 2017) for the modern construction of Sp, and list below the notions and notations that we will use in the sequel. The category Sp is a cocomplete, stable (∞)-category generated by a single object, the sphere spectrum $\mathbb{S}$. It admits a symmetric monoidal product $\otimes$, often called the *smash product*, which admits $\mathbb{S}$ as a unit, and preserves colimits in each variable. It admits a t-structure that is both right and left complete, and its heart is equivalent to the category of abelian groups under the Eilenberg–Mac Lane functor H.

Its homotopy category ho(Sp) is triangulated, and we usually call its triangles

$$X \to Y \to Z \to \Sigma X$$

cofiber sequences. We will often omit ΣX from the notation. For two spectra X and Y, the group $\mathrm{ho}(\mathrm{Sp})(X, Y)$ is usually denoted $[X, Y]$. The n-th homotopy group of a spectrum X is defined as

$$\pi_n(X) = [\Sigma^n \mathbb{S}, X]\,.$$

Note that these are $\mathbb{Z}$-graded. Here $\pi_*(\mathbb{S})$ precisely coincides with the stable homotopy groups introduced in the discussion above, with $\pi_n(\mathbb{S}) = 0$ for $n < 0$ (by Hurewicz' Theorem).

A spectrum E is said to be *n-connective* for $n \in \mathbb{Z}$ if $\pi_k(E) = 0$ for $k < n$.

If E is a spectrum, the associated (generalized) homology theory $E_*(-)$ is defined defined by

$$E_n(X) = \pi_n(E \otimes X)\,, \quad n \in \mathbb{Z}\,.$$

In particular, $\pi_*(X) = \mathbb{S}_*(X)$ is the homology theory associated to the sphere spectrum $\mathbb{S}$. The homotopy groups $\pi_*(E) = E_*(\mathbb{S})$ of E, usually denoted E_*, are called *the coefficients of E*. The cohomology theory $E^*(-)$ associated to E is defined by

$$E^n(X) = [X, \Sigma^n E]\,.$$

If A is an abelian group, the ordinary (co)-homology with coefficients in A is the theory associated to the Eilenberg–Mac Lane spectrum HA, characterized by $\pi_n(HA) = A$ if $n = 0$, and $\pi_n(HA) = 0$ otherwise. When W is a pointed space and E is a spectrum, we define $E_n(W) := E_n(\Sigma^\infty W)$ and $E^n(W) := E^n(\Sigma^\infty W)$.

Let us now briefly mention the construction of the $\{\alpha_k\}_k$ family. As above, let p be an odd prime. The mod p Moore spectrum is defined as sitting in the cofiber sequence

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{j} V(0) \xrightarrow{\partial} \Sigma\mathbb{S}\,,$$

or in other words $V(0) = \mathbb{S}/p$. Adams showed that there is a self map

$$v_1 \colon \Sigma^{|v_1|} V(0) \to V(0)\,,$$

with $|v_1| = 2(p-1)$, which is periodic in the sense that all iterates (using appropriate suspensions)

$$v_1^k \colon \Sigma^{k|v_1|}V(0) \to V(0)$$

are not null-homotopic. We define a class $\alpha'_k \in \pi_{k|v_1|-1}(\mathbb{S}) \cong [\Sigma^{k|v_1|}\mathbb{S}, \Sigma\mathbb{S}]$ as

$$\alpha'_k \colon \Sigma^{k|v_1|}\mathbb{S} \xrightarrow{j} \Sigma^{k|v_1|}V(0) \xrightarrow{v_1^k} V(0) \xrightarrow{\partial} \Sigma\mathbb{S}\,.$$

The class α_k is then a lift of α'_k over $\Sigma\mathbb{S} \xrightarrow{p^\ell} \Sigma\mathbb{S}$ for ℓ as large as possible.

The $\{\beta_k\}_k$ family is constructed in a similar manner, using a periodic self-map

$$v_2 \colon \Sigma^{|v_2|}V(1) \to V(1)\,,$$

where $V(1) = V(0)/v_1$ and $|v_2| = 2(p^2-1)$, and the $\{\gamma_k\}_k$ family using a periodic self-map

$$v_3 \colon \Sigma^{|v_3|}V(2) \to V(2)\,,$$

where $V(2) = V(1)/v_2$ and $|v_3| = 2(p^3-1)$.

As we will learn from the Periodicity Theorem 2.5, this process continues indefinitely, featuring v_n-torsion and v_n-periodicity phenomena of period $|v_n| = 2(p^n-1)$ for all $n \geq 0$. It starts with v_0-periodicity, where $v_0 = p$. We have seen that the only v_0-periodic family lies in $\pi_0(\mathbb{S}) \cong \mathbb{Z}$, where $p^k \neq 0$ for all $k \geq 1$, since all other groups are finite groups. Reducing mod v_0, the periodic class v_1 appears in $V(0)_*$, allowing for the study of v_1-torsion and v_1-periodicity phenomena, and so on. However, these higher "primes" v_n appear only in $\mathbb{S}$ when reducing modulo the lower primes $(v_0, \ldots, v_{n-1})$.

A powerful tool for proving that the classes α_k, β_k and γ_k constructed above are non-trivial is offered by the Adams–Novikov spectral sequence based on the complex cobordism spectrum MU. Inspired by ideas of Jack Morava (published later (Morava, 1985)), Miller, Ravenel, and Wilson (1977) defined a chromatic filtration on the Adams–Novikov spectral sequence expected to reflect the periodicity phenomena described above. Ravenel (1984) then published a foundational paper where he stated eight conjectures in chromatic homotopy, including the Telescope Conjecture.

The Telescope Conjecture equates two competing notions of v_n*-torsion classes*, which can be used to define two competing chromatic filtrations of the homotopy groups of a spectrum. All of these conjectures by Ravenel – *except the Telescope Conjecture* – were proven true within a few years, following the breakthrough achieved by Devinatz, Hopkins, and Smith (1988).

The Telescope Conjecture was known to hold in height 0 and 1; nevertheless, it was apparently believed to be too optimistic by experts, resisting until 2023, when in groundbreaking work Burklund, Hahn, Levy, and Schlank (BHLS) announced its disproof at all primes and all heights ≥ 2, producing explicit counter-examples provided by algebraic K-theory.

This brings us to the content of the present paper.

- In Section 1, we review the Adams–Novikov spectral sequence and explain how the theory of complex cobordism leads to the *algebraic* chromatic filtration.

- In Section 2, we state the Periodicity Theorem 2.5, discuss telescopes, and present the *geometric* chromatic filtration. We also state a first version of the Telescope Conjecture 2.13.
- In Section 3, we discuss Bousfield localization.
- In Section 4, we state the Telescope Conjecture 4.11 and touch upon its relevance.
- In Section 5, we sketch why the Telescope Conjecture holds at height 0 and 1, illustrating also several of the concepts introduced earlier.
- In Section 6, after a brief introduction to algebraic K-theory and its interaction with chromatic homotopy, we sketch the disproof of the Telescope Conjecture by Burklund, Hahn, Levy, and Schlank (BHLS) in the simplest case of $n = 2$ and $p \geq 7$.

We list below some sources and videos that are not given in the references, and that could be of interest to the reader.

- Doug Ravenel has set up a *Telescope Conjecture Webpage* with an archive of papers and further useful resources, including many of his talks on the topic, see: https://www.sas.rochester.edu/mth/sites/doug-ravenel/telescope.html
- The videos of the talks at *A Panorama of Homotopy - A Conference in Honour of Mike Hopkins* were Burklund, Hahn, Levy, and Schlank announced their result: https://www.maths.ox.ac.uk/groups/topology/panorama-homotopy-theory/links-recordings-lectures
- The videos of a series lecture on *Algebraic K-theory and Chromatic Homotopy Theory* that Dustin Clausen gave at IHES: https://indico.math.cnrs.fr/event/13651/

Acknowledgments. — I am grateful to Nicolas Bourbaki for organizing a seminar on Ravenel's Telescope Conjecture and its disproof, and for the opportunity to speak. I also thank his collaborators for very helpful support and comments on an earlier draft. In preparing this talk, I found the above reference and resources very helpful and inspiring.

1. The chromatic filtration of the Adams–Novikov spectral sequence

The examples of the α, β and γ periodic families in $\pi_*(\mathbb{S})$ led to the question of classifying such periodic families, and if any class in $\pi_*(\mathbb{S})$ belongs to such a family.

One approach to the question is provided by the work of Miller, Ravenel, and Wilson (1977) on the Adams–Novikov spectral sequence. This spectral sequence is the case of the following generalized Adams spectral sequence where $E_* = MU_*$, the complex cobordism theory, or in the (p)-local-case, where $E_* = BP_*$, the Brown–Peterson theory. In the sequel, we abbreviate $E_*(E)$ to E_*E.

Theorem 1.1 (Adams, 1974a). — *Given a multiplicative homology theory E_* such that E_*E is flat as an E_*-module, and a spectrum X, there is a spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(X)) \Rightarrow \pi_{t-s}(\hat{X}),$$

with differentials of the form $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$*, where* $\hat{X}$ *is the* E*-completion of* X*.*

Here E_*E is a Hopf-algebroïd over the ring E_*, and the Ext-functor above is taken in the category of left E_*E-comodules. Ravenel's "Green Book" (Ravenel, 2004) is a celebrated text-book reference for the study of $\pi_*(\mathbb{S})$ by generalized Adams spectral sequences; Appendix A1 of the Green Book presents the homological algebra of comodules over a Hopf-algebroïd needed to define and study the Ext-groups of the theorem as derived Hom-functors. In theory, to perform the computation of $\pi_*(X)$ using the above theorem, one needs to address the following steps.

1. Compute E_*, E_*E, $E_*(X)$ and the E_2-term $\operatorname{Ext}^{s,t}_{E_*E}(E_*, E_*(X))$;
2. Determine the d_2-differential and compute the E_3-term, which is the homology of differential bi-graded chain-complex (E_2, d_2). Repeat this with the successive pages (E_r, d_r), and determine the limiting term E_∞.
3. The graded group $\{E_\infty^{s,s+*}\}_s$ is the associated graded of the Adams–filtration on $\pi_*(\hat{X})$. Solve extensions to recover the filtration pieces and $\pi_*(\hat{X})$.
4. Determine how $\hat{X}$ relates to X.

Of course the feasibility of this program depends on the choices of E and X. We will concentrate here on the case where X is a finite spectrum, for example $X = \mathbb{S}$. Except in the cases where E_* is a $\mathbb{Q}$-algebra and the spectral sequence collapses, Adams spectral sequences for a non-trivial finite spectrum X have only been evaluated in a finite range, giving partial information.

When E_* and E_*E are know, the Ext-group of the E_2-page in point (1) is already very complicated, but is in practice computable by homological algebra tools, using small resolutions, and sometimes with the help of computers, see for example Bruner and Rognes (2021) and Bruner's Ext-calculator. Again, this is done in practice in a finite range only. Concerning the differentials in point (2), there is no known algorithmic method of computation for them, and their evaluation requires ad-hoc, often sophisticated arguments, for example using geometric input. Fortunately, in many examples of interest in a given total degree $t-s$, there are only finitely many filtration pieces: in the E_2-page or a further page, the groups displayed above a line of finite slope vanish, see for example Hopkins, Palmieri, and Smith (1999). This means that for computing $E_\infty^{s,*}$ there are non-zero differentials only in a known finite range that grows with s.

Let us be more explicit on the points (1) and (4) above in two specific examples of most interest. The classical Adams spectral corresponds to $E = H\mathbb{F}_p$, the mod p ordinary homology. In this case, $H\mathbb{F}_{p*} = H\mathbb{F}_{p*}(\mathbb{S}) = \mathbb{F}_p$ (concentrated in degree zero), and in general $H\mathbb{F}_{p*}(X)$ is easily computable. The Hopf algebra $\mathcal{A}_* = H\mathbb{F}_{p*}(H\mathbb{F}_p)$ is known as the *dual Steenrod algebra*; it was computed by Milnor.

Theorem 1.2 (Milnor, 1958). — *The Hopf algebra* $\mathcal{A}_*$ *is commutative and non-cocommutative; as an algebra it is free (in the graded-commutative sense) on infinitely* *many generators, more explicitly*

$$\mathcal{A}_* = \begin{cases} \mathbb{F}_2[\xi_1, \xi_2, \dots] & p = 2,\ |\xi_i| = 2^i - 1; \\ \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \dots) & p \text{ odd},\ |\xi_i| = 2p^i - 2,\ |\tau_i| = 2p^i - 1. \end{cases}$$

Definition 1.3. — *For any spectrum X, there is a natural spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_*, H\mathbb{F}_{p_*}(X)) \Rightarrow \pi_{t-s}(\widehat{X}),$$

which converges for X connective. It is called the (classical) mod (p) Adams spectral sequence *of X.*

Concerning point (4), in the case under consideration here, with X a finite spectrum, the $H\mathbb{F}_p$-completion of $\hat{X}$ is the p-completion (see Examples 3.8), and we have $\pi_*(\hat{X}) \cong \pi_*(X) \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers (this doesn't need to hold for non-finite spectra). In particular, the Adams spectral sequence associated to $H\mathbb{F}_p$, modulo $H\mathbb{F}_p$ extensions, determines the p-torsion subgroup of $\pi_*(X)$.

The second example is provided by the complex cobordism spectrum MU, which is a commutative ring spectrum introduced by Thom (1954); its homotopy groups are known as the complex cobordism ring, whose elements are equivalence classes of closed complex manifolds under the bordism relation. It has been computed by Milnor and Novikov independently.

Theorem 1.4 (Milnor, 1960; Novikov, 1960). — *The complex cobordism ring MU_* is a polynamial algebra in countably many generators,*

$$MU_* = \mathbb{Z}[x_1, x_2, \cdots], \quad |x_i| = 2i \text{ for all } i \geq 1.$$

This theory became one of the main actors in chromatic stable homotopy theory, as the universal complex oriented (co-)homology theory. The complex projective space $\mathbb{C}\mathrm{P}^\infty = \operatorname{colim}_n \mathbb{C}\mathrm{P}^n$ classifies complex line bundles on topological spaces, and is endowed with the structure of an abelian topological group, with a product classifying the tensor product of line bundles. We have the inclusion map $i\colon S^2 = \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$.

Definition 1.5. — *A ring spectrum E is called* complex orientable *if the map*

$$i^*\colon E^2(\mathbb{C}\mathrm{P}^\infty) \to E^2(S^2) \cong E^0 = \pi_0(E)$$

is surjective. This is equivalent to asking that there is a class $x \in E^2(\mathbb{C}P^\infty)$ that is mapped by i^* to $1 \in \pi_0(E)$; such a class is then called a complex orientation of E*. The ring spectrum E is called* complex oriented *if it is complex orientable, with a chosen complex orientation x_E.*

Lemma 1.6 (Adams, 1974b). — *Let (E, x_E) be a complex oriented ring spectrum. The group multiplication $m\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ induces a map of power series rings over E^**

$$E^*[[x_E]] = E^*(\mathbb{C}\mathrm{P}^\infty) \to E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) = E^*[[x, y]]\,,$$

and the power series in two variables $F_E(x,y) := m^*(x_E)$ *is a commutative (one dimensional) formal group law over the ring* E^**; in other words, the following identities hold:*

$$F(x,0)=x=F(0,x),\quad F(x,y)=F(y,x)\ \text{ and }\ F(x,F(y,z))=F(F(x,y),z).$$

Note that $E_n = E^{-n}$, so up to regrading we might also consider that the formal group F_E is one over the ring E_*.

Example 1.7. — Ordinary integral cohomology $H\mathbb{Z}$ is complex orientable; The associated formal group law is the additive one, with $F_{H\mathbb{Z}}(x,y)=x+y$.

— Complex K-theory KU, with $KU^*=\mathbb{Z}[u^{\pm 1}]$ where $|u|=-2$, is complex orientable; the associated formal group is the multiplicative one, with $F_{KU}(x,y)=x+y+uxy$.

— The ring spectra $\mathbb{S}$ and KO, real K-theory, are not complex orientable.

Given two formal group laws F, G over a ring R, a strict isomorphism $f\colon F\to G$ is a power series $f\in tR[[t]]$ with leading coefficient 1 (in particular it is invertible), such that $f(F(x,y))=G(f(x),f(y))$. Let us denote by FGL the functor that associate to a ring R the groupoid whose objects are the formal group laws over R, with morphisms the strict isomorphisms.

Following Lazard (1955), one can show that this functor is corepresented by a Hopf algebroid, which is a cogroupoid object (L, LB) in the category of commutative rings: more explicitly, we have a pair of commutative rings (L, LB), together with the following collection of ring homomorphism:

$$\begin{aligned} \eta_L\colon L &\to LB && \text{the left unit (source),}\\ \eta_R\colon L &\to LB && \text{the right unit (target),}\\ \Delta\colon LB &\to LB\otimes_L LB && \text{the coproduct (composition),}\\ \epsilon\colon LB &\to L && \text{the counit (identity), and}\\ c\colon LB &\to LB && \text{the conjugation (inverse).}\end{aligned}\tag{1}$$

These satisfy the obvious relations, dual to the defining properties of a groupoid, see Ravenel, 2004, Appendix A1. Note that there are two L-algebra structures on LB, given by η_L and η_R. This generalizes Hopf algebras, which are cogroups objects in the category of algebras (or in other words, for which $\eta_L=\eta_R$).

The Lazard ring L carries the universal formal group law $F^{\mathrm{u}}\in \mathrm{FGL}(L)$, and LB carries the universal strict isomorphism $h\colon \eta_L^*(F^{\mathrm{u}})\to\eta_R(F^{\mathrm{u}})$. Indeed, denoting $\mathrm{CAlg}_{\mathbb{Z}}$ the category of commutative rings, there are natural bijections for any commutative ring R

$$\begin{aligned} \mathrm{CAlg}_{\mathbb{Z}}(L,R) &\to \mathrm{Obj}(\mathrm{FGL}(R)), && f\mapsto f^*(F^{\mathrm{u}}),\\ \mathrm{CAlg}_{\mathbb{Z}}(LB,R) &\to \mathrm{Mor}(\mathrm{FGL}(R)), && g\mapsto g^*(h)\colon g^*\eta_L^*(F^{\mathrm{u}})\to g^*\eta_R^*(F^{\mathrm{u}}),\end{aligned}$$

and the structure maps of the groupoid $\mathrm{FGL}(R)$ are corepresented by the ring morphisms given in (1) above.

In particular, if E is a complex oriented ring spectrum, then there is a unique map $\theta_E \colon L \to E_*$ with $\theta_E{}^*(F^{\mathrm{u}}) = F_E$. Complex cobordism is complex orientable, with a unique complex orientation x_{MU}. Let us denote F_{MU} the associated formal group law.

Theorem 1.8 (Quillen, 1969). — *The ring homomorphism $\theta_{MU} \colon L \to MU_*$ is an isomorphism; in particular, F_{MU} is the universal formal group law.*

This recovers Milnor's Theorem 1.4 from Lazard's computation of L. But there is more: in fact, (M_*, MU_*MU) admits the structure of a Hopf algebroid; the structure homomorphisms are given as follows:

$$\begin{aligned}
\eta_L &\ \text{ is induced by } MU \cong MU \otimes \mathbb{S} \to MU \otimes MU\,,\\
\eta_R &\ \text{ is induced by } MU \cong \mathbb{S} \otimes MU \to MU \otimes MU\,,\\
\Delta &\ \text{ is induced by } MU \otimes MU \cong MU \otimes \mathbb{S} \otimes MU \to MU \otimes MU \otimes MU\,,\\
\epsilon &\ \text{ is induced by the multiplication } MU \otimes MU \to MU\,, \text{ and}\\
c &\ \text{ is induced by interchange } MU \otimes MU \to MU \otimes MU\,.
\end{aligned} \tag{2}$$

Adams constructed an isomorphism $\phi \colon \eta_L^*(F_{MU}) \to \eta_R^*(F_{MU})$.

Theorem 1.9 (Novikov, 1969; Landweber, 1967). — *The pair of homomorphisms*

$$(L, LB) \xrightarrow{(\theta_{MU},g)} (MU_*, MU_*MU)\,.$$

where $g \colon LB \to MU_*MU$ is caracterized by $g^*(h) = \phi$, is an isomorphism of Hopf algebroids.

This also recovers the computation of MU_*MU by standard methods, given as

$$MU_*MU \cong MU_*[b_1, b_2, \ldots] \ \text{ with } \ |b_i| = 2i\,,$$

but also provides formulas for the structure maps given in (2) of the Hopf algebroid (MU_*, MU_*MU), see Ravenel, 2004, A2.1.16.

These theorems classify ring homomorphisms out of MU_* and MU_*MU. Adams showed that one can also classify commutative ring spectrum maps out of MU.

Theorem 1.10 (Adams, 1974b, Lemma 2.2.4). — *Let E be a commutative ring spectrum. Then the correspondence*

$$(f \colon MU \to E) \to f_*(x_{MU}) \in \widetilde{E}^2(\mathbb{C}\mathrm{P}^\infty)$$

is a bijection between homotopy classes of ring spectrum maps $MU \to E$ and complex orientations of E.

Remark 1.11. — Jack Morava's program for studying the chromatic filtration of the stable homotopy category was to relate it to the height filtration of the moduli stack of (1-dimensional) formal groups, the chromatic height of a complex oriented cohomology theory corresponding to the height of the associated formal group (law). This gives the complex cobordism theory MU a central rôle in the chromatic perspective on stable homotopy, and the (co-)homology theories used to detect periodicity are all derived from it, as for example the Morava K-theories $K(n)$ introduced in Definition 2.1. Given a graded ring endowed with a formal group law, the Landweber Exactness Theorem gives a flatness condition for the construction of a complex oriented (co-)homology theory having this ring as coefficients, with the same formal group law. Using spectral algebraic geometry, Gregoric (2021) has recently proven that the category Sp is equivalent to a category of ind-coherent sheaves on a non-connective moduli stack of oriented formal groups, which has a filtration that recovers both the chromatic filtration of Sp and the height filtration of formal groups.

After this incursion in complex orientations, we return to the Adams spectral sequence, and consider a first, integral version of our second example, obtained from Theorem 1.1 by specializing to the case $E = MU$; it has the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*(X)) \Rightarrow \pi_{t-s}(\hat{X})\,. \tag{3}$$

Here, when X is connective spectrum, we have $X = \hat{X}$, so that the spectral sequence converges to $\pi_*(X)$. However, as usual in computations, it is more efficient to work one prime p at a time, by localizing away from p, replacing MU with $MU_{(p)}$, see Examples 3.8 in Section 3. The associated Hopf algebroid is just the localization away from of $(MU_*, MU_*(X))$, with

$$MU_{(p)_*} = \mathbb{Z}_{(p)}[x_1, x_2, \dots] \quad \text{and} \quad MU_{(p)_*}MU_{(p)} = \mathbb{Z}_{(p)}[x_1, x_2, \dots][b_1, b_2, \dots]\,.$$

The polynomial ring $MU_{(p)_*}$ can be rewritten as

$$MU_{(p)_*} = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \otimes \mathbb{Z}_{(p)}[x_i \,|\, i \neq p^k - 1 \;\forall k \geq 1]$$

with $|v_i| = 2p^i - 2$. Brown and Peterson (1966) defined a (p)-local ring spectrum denoted BP, and showed that there is a splitting of $MU_{(p)}$ into a sum of infinitely many shifted copies of BP. More precisely, we have

$$MU_{(p)} \cong \bigoplus_{x_I} \Sigma^{|x_I|} BP\,,$$

where the sum is indexed by the $\mathbb{Z}_{(p)}$-module monomial generators x_I of the second tensor factor above. The ring-spectrum BP is homotopy commutative, and the associated Hopf algebroid has coefficients

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \;\text{ and }\; BP_*BP = BP_*[t_1, t_2, \dots]\,. \;\; |t_i| = 2p^i - 2$$

See Ravenel, 2004, A2.1.27 for a description of the structure homomorphisms.

This splitting implies that the (co-)homology theories $MU_{(p)}$ and BP will capture the same information on the structure of the category of spectra, or in more precise terms that the $MU_{(p)}$-based and BP-based localization functors are equivalent.

We also have a universal property of the Hopf algebroïd (BP_*BP, BP_*), analogous to Theorem 1.9 and the discussion preceeding it.

PROPOSITION 1.12 (Ravenel, 2004, Theorem A2.1.27). — *The Hopf algebroid* (BP_*BP, BP_*) *corepresents the groupoid of* p*-typical formal group laws and their strict isomorphism over a* $\mathbb{Z}_{(p)}$*-algebra.*

See Ravenel (2004, A2.1.22) for the definition of p-typical formal group laws. Cartier (1967) proved that any formal group law over a $\mathbb{Z}_{(p)}$-algebra is canonically strictly isomorphic to a p-typical one. We now have a (p)-local version of the spectral sequence (3), first constructed by Novikov (1969).

DEFINITION 1.13. — *For any spectrum* X*, there is a natural spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X)) \Rightarrow \pi_{t-s}(X_{(p)}),$$

which converges for X *connective. It is called* the Adams–Novikov spectral sequence of X.

This spectral sequence is sparser than the classical Adams spectral sequence 1.3; in fact the Adams–Novikov E_2-term corresponds to the abutment of spectral sequence whose E_2-term is the classical Adams spectral sequence (Miller, 1981). Note that despite the nice structure of the algebras (MU_*, MU_*MU) and (BP_*, BP_*BP), the Ext-terms above are nevertheless tedious to compute. In fact, the structure maps η_R, Δ and c as in (2) are not given by explicit formulas; they are characterized by equations that require inductive computations on the algebra generators, see Ravenel, 2004, A2.1.16/27.

Miller, Ravenel, and Wilson (1977) introduced a first notion of chromatic filtration of the stable homotopy groups $\pi_*(\mathbb{S})$. The motivation, also inspired from ideas of Jack Morava, was to understand how periodic families in $\pi_*(\mathbb{S})$, such as the α, β and γ families mentioned in the introduction can be detected in the Adams–Novikov spectra sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi_{t-s}(\mathbb{S}_{(p)}),$$

In fact, these author show that the above E_2-term admits a filtration the layers of which consist of v_n-periodic classes. It is obtaining by splicing together inductively defined short exact sequences of BP_*BP-comodules

$$0 \longrightarrow N^n \to M^n \to N^{n+1} \to 0\,, \tag{4}$$

starting with $N^0 = BP_*$ and defining $M^n := N^n[v_n^{-1}]$. Here, as usual, $v_0 = p$. In particular, we have

$$M^0 = BP_*[v_0^{-1}] = BP_* \otimes \mathbb{Q} \quad \text{and} \quad N^1 = BP_* \otimes \mathbb{Q}/\mathbb{Z}_{(p)} = BP_*/p^\infty\,,$$

and by induction $M^n = BP_*/(v_0^\infty, \ldots, v_{n-1}^\infty)[v_n^{-1}]$. Abbreviating $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, -)$ to $\mathrm{Ext}^{s,t}(-)$, and applying this functor to the short exact sequence (4), we obtain a long exact sequence represented by a triangle in the diagram below; splicing this triangles we obtain an (unrolled) exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Ext}^{s-2,*}(N^2) \arrow[r] \arrow[d] & \operatorname{Ext}^{s-1,*}(N^1) \arrow[r] \arrow[d] & \operatorname{Ext}^{s,*}(N_0) \arrow[d] \\
 & \operatorname{Ext}^{s-2,*}(M^2) \arrow[ul] & \operatorname{Ext}^{s-1,*}(M^1) \arrow[ul] & \operatorname{Ext}^{s,*}(M^0) \arrow[ul]
\end{tikzcd} \tag{5}$$

Definition 1.14 (Miller, Ravenel, and Wilson, 1977). — The chromatic spectral sequence *is the spectral sequence associated to the exact couple (5). It is a tri-graded spectral sequence with signature*

$$E_1^{n,s,t} = \operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, M^n) \Rightarrow \operatorname{Ext}^{n+s,t}_{BP_*BP}(BP_*, BP_*),$$

Heuristically, the top line of the above unrolled exact couple gives a descending filtration of $\operatorname{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$, the n-th piece of which is the $(v_0, \ldots, v_{n-1})$-torsion subgroup. This converges to a filtration of $\pi_*(\mathbb{S}_{(p)})$ that is called by Ravenel *the algebraic chromatic filtration*. A more precise and general form of this filtration will be given below, see 4.8.

2. The Periodicity Theorem and Telescopes

To state the next theorem we need to introduce the (co-)homology theories known as Morava K-theories. They were introduce by Jack Morava (unpublished), see for example Johnson and Wilson, 1975 for a published account. Fix a prime p, and let the map $v_n \colon \Sigma^{|v_n|}\mathbb{S} \to BP$ represent the class $v_n \in BP_*$. For any BP-module, we also denote v_n the map obtained as

$$\Sigma^{|v_n|} M = \Sigma^{|v_n|}\mathbb{S} \otimes M \overset{v_n \otimes \mathrm{id}}{\longrightarrow} BP \otimes M \longrightarrow M\,,$$

and we denote M/v_n its cofiber. Using this, we can iteratively define $BP/(v_{i_0}, \ldots, v_{i_n})$ for chosen v_i's, and using a colimit, we can also define $BP/(v_{i_0}, v_{i_1}, \ldots)$ for a possibly infinite sequence of v_i's. Since the v_i act injectively on BP_*, we deduce that

$$\Big(BP/(v_{i_0}, v_{i_1}, \ldots)\Big)_* = BP_*/(v_{i_0}, v_{i_1}, \ldots)$$

Definition 2.1. — *Let p be a prime and $n \in \mathbb{N}$.* The 0-th Morava K-theory spectrum *is defined as $K(0) = H\mathbb{Q}$.* The n-th connective Morava K-theory *is defined as $k(n) = BP/(v_i \,|\, i \neq n)$.* The n-th Morava K-theory *is defined as the colimit*

$$K(n) = k(n)[v_n^{-1}] \coloneqq \operatorname{colim}(k(n) \xrightarrow{v_n} \Sigma^{-|v_n|}k(n) \xrightarrow{v_n} \Sigma^{-2|v_n|}k(n) \xrightarrow{v_n} \ldots)\,.$$

It is common to designate by $K(\infty)$ the mod (p) Eilenberg–Mac Lane spectrum $H\mathbb{F}_p$. We summarize the main properties of the Morava K-theories $K(n)$ in the next proposition.

Definition 2.2. — *A spectrum Y is called* finite *if it is a finite cell $\mathbb{S}$-module, or equivalently if it is a (de-)suspension of a spectrum of the form $\Sigma^\infty W$ for a finite pointed CW-complex W. A spectrum X is called* a finite p-local spectrum *if there is a finite spectrum Y and a map $\eta\colon Y \to X$ inducing the localization away from (p) in any generalized homology theory: $\eta_*\colon E_*(Y) \to E_*(X) \cong E_*(Y) \otimes \mathbb{Z}_{(p)}$. In other words, X is the (p)-localization of a finite spectrum in the sense of Example 3.8-1.*

Proposition 2.3. — *Let p be a prime and $n \in \mathbb{N}$.*

1. *By definition, $K(0)_* = \mathbb{Q}$ concentrated in degree 0, and for any spectrum X and any p, $K(0)_*(X)$ is the rational homology of X.*
2. *If p is odd and $n \geq 1$, $K(n)$ has a unique homotopy commutative MU-algebra structure; it has a non-unique MU-algebra structure at the prime 2. Its coefficient ring forms a (graded) field over $\mathbb{F}_p$:*

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}] \quad \text{with} \quad |v_n| = 2p^n - 2\,.$$

3. *For $n \geq 1$, $K(n)$ is complex-oriented, and its formal group law is the Honda formal group law with p-series $[p](x) = v_n x^{p^n}$.*
4. *There is a Künneth isomorphism $K(n)_*(X \otimes Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$.*
5. *Let X be a finite, non-trivial spectrum. Then $K(n)_*(X) \neq 0$ for n large enough. Moreover, for all $n \geq 1$,*

$$\big(K(n)_*(X) = 0\big) \Rightarrow \big(K(n-1)_*(X) = 0\big)\,.$$

The MU-algebra structure was constructed by Strickland (1999); Point 5 is proven in (Ravenel, 1984, Theorem 2.11). The other properties are established for example in (Johnson and Wilson, 1975) and (Würgler, 1991).

Definition 2.4. — *Let X be a finite (p)-local spectrum. We say the X* is of type n *if n is the smallest integer such that $K(n)_*(X) \neq 0$. If X is trivial, we say it has type ∞.*

Note that because of Proposition 2.3, Point 5, if X is non-trivial, then it is of type n for some $n \in \mathbb{N}$, and we have $K(m)_*(X) = 0$ for all $m < n$, while $K(m)_*(X) \neq 0$ for all $m \geq n$.

Periodicity Theorem 2.5 (Hopkins and Smith, 1998). — *Let X and Y be finite, (p)-local spectra of type n.*

1. *There exists $k \geq 1$ and self-map $f\colon \Sigma^{k|v_n|}X \to X$ such that $K(n)_*(f)$ is an isomorphism, and $K(m)_*(f) = 0$ for $m > n$. Such a map is called* a v_n-self-map of *X.*
2. *Suppose further given a v_n-self-map $g\colon \Sigma^{\ell|v_n|}Y \to Y$ of Y, and a map $h\colon X \to Y$. Then there are iterated compositions f^i of f and g^j of g, with $ik|v_n| = j\ell|v_n| =: m$,*

such that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\Sigma^m X \arrow[r, "\Sigma^m h"] \arrow[d, "f^i"] & \Sigma^m Y \arrow[d, "g^j"] \\
X \arrow[r, "h"] & Y
\end{tikzcd}$$

Note that above, and in the sequel, with f a v_n-self map of X, we also denote by f the map $\Sigma^{\ell k|v_n|} f$ for any $\ell \in \mathbb{Z}$. The iterates $f^i \colon \Sigma^{ik|v_n|} X \to X$ are non trivial for any $i \in \mathbb{N}$, since $K(n)_*(f^i)$ is non-trivial.

Definition 2.6. — *Let X be a finite, (p)-local spectrum of type n, and let f be a v_n-self map of X. The colimit*

$$\operatorname{colim}(X \xrightarrow{f} \Sigma^{-k|v_n|} X \xrightarrow{f} \Sigma^{-2k|v_n|} X \xrightarrow{f} \dots)$$

is called the telescope of X, *and is denoted by* $X[f^{-1}]$ *or* $\mathrm{Tel}(X)$.

Remark 2.7. — Note that the natural map $X \to X[f^{-1}]$ is non-trivial, since it is an isomorphism in $K(n)_*$-homology, while $K(m)_*(X[f^{-1}]) = 0$ for $m \neq n$. Also, note that by Point 2 of the Periodicity Theorem 2.5, taking h to be the identity of X, we see that up to iterations f is unique, so that $X[f^{-1}]$ does not depend on the choice of f. This justifies the notation $\mathrm{Tel}(X) \coloneqq X[f^{-1}]$.

Remark 2.8. — Let us comment on the name "telescope" for this colimit, which gave its name to Ravenel's conjecture. Depending on the setting used, it is necessary to ensure that the colimits as in the above definition are homotopically meaningful. Classically, this is done by requiring the maps in the direct system be cofibrations. Here is an example: a map of spaces $f \colon X \to Y$ can be replaced by a cofibration up to homotopy, using the cylinder construction: take

$$\mathrm{Cyl}(f) = (X \times [0,1] \coprod Y)/\sim$$

with the quotient topology, where the quotient is taken using the equivalence relation generated by $(x, 1) \sim f(x)$ for all $x \in X$. Then the inclusion $i \colon X \to \mathrm{Cyl}(f)$ given by $x \mapsto [(x, 0)$ is a cofibration, and f factors as $q \circ i$, where $q \colon \mathrm{Cyl}(f) \to Y$ is the homotopy equivalence that compresses the cylinder onto Y. Then sequential homotopy colimits can be realized by splicing such cylinders, which resembles a telescope in the astronomical sense.

Example 2.9. —
1. Let $\mathbb{S}_{(p)}$ be the (finite) (p)-local sphere spectrum. We have $K(0)_*(\mathbb{S}_{(p)}) = H\mathbb{Q}_*(\mathbb{S}_{(p)}) = \mathbb{Q} \neq 0$, so $\mathbb{S}_{(p)}$ is of type 0, and by the Periodicity Theorem it admits a non-trivial v_0-self-map $f_0 \colon \mathbb{S}_{(p)} \to \mathbb{S}_{(p)}$. An example of such is given by the multiplication by p (or any non-trivial power of p, up to a unit in $\mathbb{Z}_{(p)}$); it sits in a cofiber sequence

$$\mathbb{S}_{(p)} \xrightarrow{p} \mathbb{S}_{(p)} \longrightarrow \mathbb{S}/p \longrightarrow \Sigma\mathbb{S}_{(p)} \,,$$

and induces the multiplication by p on $K(m)_*(\mathbb{S}_{(p)})$ for all $m \geq 0$. In particular, it is an isomorphism for $m = 0$ and is zero for $m > 0$. This shows that it is a v_0-self map. By Serre's Theorem, we have $\mathrm{Tel}(\mathbb{S}_{(p)}) = \mathbb{S}_{(p)}[p^{-1}] = H\mathbb{Q}$.

2. Denoting $V(0)$ the (p)-local spectrum $\mathbb{S}/p$, we deduce from the cofiber sequence above that $K(0)_*(V(0)) = 0$, and if $m > 0$, $K(m)_*(V(0)) \cong K(m)_* \oplus K(m)\{\varepsilon\}$ for a class $\varepsilon \in K(m)_{-1}(V(0))$. This implies that $V(0)$ is of type 1, and by the Periodicity Theorem it admits a v_1-self-map $f_1 \colon \Sigma^{k|v_1|}V(0) \to V(0)$. Such a map with $k = 1$ was constructed by Adams at odd primes p, and Mahowald showed that $k = 4$ is the smallest value at $p = 2$ for which such a self-map exists. This v_1-self-map now sits in a cofiber sequence

$$\Sigma^{k|v_1|}V(0) \xrightarrow{f_1} V(0) \longrightarrow V(0)/f_1 \longrightarrow \Sigma^{k|v_1|+1}V(0)\,.$$

The homotopy groups of $\mathrm{Tel}(V(0))$ were computed by Miller for p odd, see Theorem 5.8.

3. Arguing as for $V(0)$, using the cofiber sequence just above, we deduce that $\mathbb{S}/(p, f_1) \coloneqq V(0)/f_1$ is of type 2, with $K(m)_*(V(0)/f_1))$ of rank 4 over $K(m)_*$, and thus admits a v_2-self-map. We can proceed by induction to construct, for all n, a finite (p)-local spectrum $\mathbb{S}/(f_0, \dots, f_{n-1})$ of type n. The associated telescope $\mathrm{Tel}(\mathbb{S}/(f_0, \dots, f_{n-1})$ remains very mysterious for $n \geq 2$, in particular its homotopy groups are not known.

Definition 2.10. — *Let p be a prime. We abuse notation and denote*

$$T(n) \coloneqq \mathrm{Tel}(\mathbb{S}/(f_0, \dots, f_{n-1})$$

where $\mathbb{S}/(f_0, \dots, f_{n-1})$ is a finite (p)-local spectrum of type n as given in Example 2.9

We now generalize the notion of a Moore spectrum $V(0)$ used above.

Definition 2.11. — *Let p be a prime and $n \in \mathbb{N}$.* A Smith–Toda complex $V(n)$ *is a finite, (p)-local spectrum of type n of the form $\mathbb{S}/(f_0, \dots, f_{n-1})$, where f_i induces multiplication by v_i in $K(i)_*$. Equivalently,*

$$BP_*(V(n)) = BP_*/(v_0, \dots, v_{n-1})\,.$$

The question of their existence is a difficult one; for example, $V(1)$ does not exist at the prime 2.

The Periodicity Theorem provides a notion of chromatic filtration of the homotopy groups of a (p)-local spectrum Y. Let $y \in \pi_k(Y)$ be a class represented by a map

$\Sigma^k\mathbb{S}_{(p)} \to Y$. Consider the diagram below.

(6)

$$\begin{tikzcd}
Y & \Sigma^k\mathbb{S}_{(p)} \arrow[l, "y"'] \arrow[d] & \Sigma^k\mathbb{S}_{(p)} \arrow[l, "f_0"'] \\
 & W(0) \arrow[ul, dotted, "y_1"] \arrow[d] & \Sigma^{k_1}W(0) \arrow[l, "f_1"'] \\
 & W(1) \arrow[uul, dotted, "y_2"] \arrow[d] & \Sigma^{k_2}W(1) \arrow[l, "f_2"'] \\
 & \vdots &
\end{tikzcd}$$

Here f_0 is some v_0-self-map of $\mathbb{S}_{(p)}$ and $W(0) = \mathbb{S}/f_0$; one continues so inductively, with $W(n) = \mathbb{S}/(f_0, \ldots, f_n)$, which is of type $n+1$, and f_{n+1} is a v_{n+1}-self-map of $W(n)$, as detailed in Example 2.9. Obviously, the class $y \in \pi_k(Y)$ is p-torsion if it is 0 when precomposed with some v_0-self-map f_0 of $\Sigma^k\mathbb{S}_{(p)}$, or alternatively, if it factors through $W(0)$; otherwise, it is non-p-torsion (or "v_0-periodic"). This is generalized to v_n-torsion in the following definition.

Definition 2.12. — *Let Y be a (p)-local spectrum, and $y \in \pi_k(Y)$. The class y is said to be a* $(v_0, \ldots, v_n)$-torsion class *if it factors through $W(n)$ in the diagram (6) for some choices of $f_0, \ldots, f_n$. If furthermore is does not extend over $W(n+1)$ for any choice of f_{n+1}, it is called* a v_{n+1}-periodic class. *We have a descending filtration by subgroups*

$$\cdots \subset F_2(Y) \subset F_1(Y) \subset F_0(Y) \subset \pi_*(Y)\,,$$

where $F_n(Y)$ is the subgroup consisting of $(v_0, \ldots, v_n)$-torsion classes. We call it the (geometric) chromatic filtration of $\pi_*(Y)$.

Here is a first form of the Telescope Conjecture, we will state it in more general form below (4.11)

Telescope Conjecture - First formulation 2.13. — *The* algebraic *and* geometric *chromatic filtrations of the homotopy groups of the (p)-local sphere spectrum (or of a finite (p)-local spectrum) agree.*

3. Bousfield Localization

Definition 3.1. — *Let E be a spectrum, and E_* the associated homology theory.*

1. *A spectrum W is called* E-acyclic *if $E_*(W) = 0$.*
2. *A map $f\colon A \to B$ is called an E-equivalence if $E_*(f)\colon E_*(A) \to E_*(B)$ is an isomorphism.*
3. *A spectrum X is called* E-local *if for any E-acyclic spectrum W we have $[W, X] = 0$.*

4. *A* Bousfield localization with respect to E *is a natural transformation* $\eta_E\colon \mathrm{id} \to L_E$ *of functors* $\mathrm{ho}(\mathrm{Sp}) \to \mathrm{ho}(\mathrm{Sp})$*, such that for any spectrum* X*,*
   (a) the spectrum $L_E(X)$ *is* E*-local, and*
   (b) the map $\eta_{E,X}\colon X \to L_E(X)$ *is an* E*-equivalence.*

It follows immediately from the definition that if such a localization $\eta_E\colon \mathrm{id} \to L_E$ exists, then for any spectrum X, $\eta_{E,X}$ is terminal among E-equivalences out of X, and initial among maps out of X to an E-local object. In particular, it is unique.

Theorem 3.2 (Bousfield, 1979, Thm 1.1). — *For any spectrum* E*, there exists a Bousfield localization functor* $\eta_E\colon \mathrm{id} \to L_E$*.*

For any spectrum X, we observe that X is E-acyclic if and only if $L_E(X) = 0$. We denote by $\ker(L_E)$ the full-subcategory of spectra X with $L_E(X) = 0$.

Note that in Definition 3.1, the notion of E-equivalence, and hence of E-local objects and E-localization, depend only on the class of E-acyclic spectra, not on the spectrum E itself; indeed, a map is an E-equivalence if and only if its cofiber is E-acyclic. For example, the spectra E and $E \oplus E$ have the same class of associated acyclic objects, thus the localization functors L_E and $L_{E\oplus E}$ are naturally equivalent. This motivates the following definition.

Definition 3.3. — *Two spectra are called* Bousfield equivalent *if they have the same class of acyclic objects. The Bousfield equivalence class of a spectrum* E *is denoted* $\langle E\,\rangle$*. We define a partial order relation on Bousfield classes as follows:* $\langle F\,\rangle \le \langle E\,\rangle$ *if and only if the class of* F*-acyclic spectra contains the class of* E*-acyclic spectra.*

Intuitively, the larger the Bousfield class of E is, the more information on X will be captured by the localization functor $X \to L_E X$. The following lemma states this more precisely.

Lemma 3.4. — *Suppose* $\langle F\,\rangle \le \langle E\,\rangle$*. Then any* F*-local spectrum is* E*-local, and for any spectrum* X*,* $\eta_{F,X}\colon X \to L_F X$ *factors uniquely as*

$$X \xrightarrow{\eta_{E,X}} L_E X \longrightarrow L_F X\,.$$

There are equivalences $L_F X \simeq L_F L_E X \simeq L_E L_F X$*.*

The following useful results shows how to reconstruct $L_{E\oplus F}$ from L_E and L_F, in good cases, see for example (Antolín-Camarena and Barthel, 2022, p. 3.3) for an account.

Proposition 3.5. — *Suppose* E *and* F *are spectra such that* $L_E L_F = 0$*. Then for any spectrum* X *there is a homotopy cartesian square*

$$\begin{tikzcd}
L_{E\oplus F}(X) \arrow[r] \arrow[d] & L_F(X) \arrow[d] \\
L_E(X) \arrow[r] & L_F(L_E(X))
\end{tikzcd}$$

If M is a $\mathbb{Z}$-module, and $R \subset \mathbb{Q}$ is a subring, the R-localization of M is just given by $M \mapsto M \otimes_{\mathbb{Z}} R$. Some localization functors on spectra have a similar property.

Proposition 3.6 (Ravenel, 1984, Proposition 1.27). — *Let E be a spectrum. The following conditions are equivalent.*

1. *$\langle E \rangle = \langle L_E \mathbb{S} \rangle$;*
2. *For any spectrum X, the natural map $L_E(\mathbb{S}) \otimes X \to L_E(X)$ is an equivalence. In particular,*
$$X \cong \mathbb{S} \otimes X \xrightarrow{\eta_{E,\mathbb{S}} \otimes \mathrm{id}} L_E(\mathbb{S}) \otimes X$$
is a Bousfield localization with respect to E.
3. *The functor L_E commutes with colimits.*

Definition 3.7. — *A spectrum E, or the associated E-localization functor L_E, is called* smashing *if it satisfies the equivalent conditions of Proposition 3.6.*

The following examples are taken from (Bousfield, 1979), which contains many more.

Example 3.8. —
1. Let p be a prime. *The (p)-localization functor* is the localization with respect to $\mathbb{S}_{(p)}$ (which is the telescope inverting all primes different than p). It is a smashing localization. It is usually denoted by $X_{(p)} \coloneqq L_{\mathbb{S}_{(p)}}(X)$, and we have
$$\pi_*(X_{(p)}) = \pi_*(X) \otimes \mathbb{Z}_{(p)}\,.$$
2. Let p be a prime. *The p-completion functor* is the localization with respect to $V(0) = \mathbb{S}/p$. It is usually denoted $X_p \coloneqq L_{V(0)}(X)$ or $X_p^\wedge$. It is not smashing. For a spectrum X, we have a splitable short exact sequence
$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_n(X)) \to \pi_n(X_p) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n-1}(X)) \to 0\,.$$
If the groups $\pi_*(X)$ are finitely generated, then
$$\pi_*(X_p) = \pi_*(X) \otimes \mathbb{Z}_p\,.$$
3. If E is a connective ring spectrum with $E_0 = \mathbb{Z}$ (or $E_0 = \mathbb{Z}_{(p)}$), and if X is a connective spectrum, then $L_E(X) = X$ (respectively $L_E(X) = X_{(p)}$).

Miller (1992) introduced a variant of Definition 3.1, where the condition for being E-local is tested only against finite spectra.

Definition 3.9. — *Let E be a spectrum, and E_* the associated homology theory.*

1. *A spectrum X is called* finitely E-local *if for any finite, E-acyclic spectrum W we have $[W, X] = 0$.*
2. *A spectrum Y is called* finitely E-acyclic *if $[X, Y] = 0$ for any finitely E-local spectrum X.*
3. *A map $f\colon A \to B$ is called a* finite E-equivalence *if its cofiber is finitely E-acyclic.*
4. *A* finite localization with respect to E *is a natural transformation $\eta_E\colon \mathrm{id} \to L_E^f$ of functors* $\mathrm{ho}(\mathrm{Sp}) \to \mathrm{ho}(\mathrm{Sp})$*, such that for any spectrum X,*
   - *(a) the spectrum $L_E^f(X)$ is finitely E-local, and*

(b) the map $\eta_{E,X}\colon X \to L_E^f(X)$ *is a finite* E*-equivalence.*

Theorem 3.10 (Miller, 1992, Thm 4, Prop 9). — *For any spectrum* E*, there exists a finite localization functor* $\eta_E\colon \mathrm{id} \to L_E^f$*. Moreover this functor is smashing.*

Note that the proposition corresponding to Proposition 3.5 for finite localizations also holds.

Since we have $\ker(L_E^f) \subset \ker(L_E)$ by definition, we deduce the existence of a unique natural transformation $L_E^f \to L_E$ such that for any spectrum X, we have a factorization of $\eta_{E,X}\colon X \to L_E(X)$ as

$$X \xrightarrow{\eta_{E,X}} L_E^f X \longrightarrow L_E X\,. \tag{7}$$

4. The Telescope Conjecture

Using these localizations, it is now possible to introduce spectrum-level chromatic filtrations that realize the chromatic filtrations of homotopy groups introduced above. Again, the main player is the Brown–Peterson spectrum BP. If we are interested in studying only the $v_0, \ldots, v_n$ related torsion and periodicity phenomena, an appropriate theory is the Johnson–Wilson spectrum $E(n)$, constructed from BP by the same procedure as for the Morava K-theories. It has coefficients

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}][v_n^{\pm 1}]\,.$$

Proposition 4.1 (Ravenel, 1984, Theorem 2.1). — *There are equivalences of Bousfield classes*

$$\langle BP[v_n^{-1}]\,\rangle = \langle E(n)\,\rangle = \langle K(0) \oplus K(1) \oplus \cdots \oplus K(n)\,\rangle\,.$$

Definition 4.2. — *The functor*

$$L_n := L_{E(n)} = L_{BP[v_n^{-1}]} = L_{K(0)\oplus\cdots\oplus K(n)}$$

is called the n-th chromatic localization functor*, and the functor*

$$L_n^f := L_{E(n)}^f = L_{BP[v_n^{-1}]}^f = L_{K(0)\oplus\cdots\oplus K(n)}^f$$

is called the n-th chromatic finite localization functor*.*

Remark 4.3. — Note that by Proposition 2.3, point 5, we have $L_n^f = L_{K(n)}^f$.

The localizations L_n^f and L_n are smashing (Definition 3.7). This is generic for finite localizations, but for L_n it is a more surprising, deep result: for example, the localizations $L_{K(n)}$ are not smashing for $n \geq 1$.

Proposition 4.4 (Miller, 1992, Proposition 9). — *Any finite localization functor is smashing.*

Smash Product Theorem 4.5 (Ravenel, 1992a, Prop 7.5.6)
For any $n \in \mathbb{N}$*, the functor* L_n *is smashing.*

The functors L_n^f are closely related to telescopes, as highlighted in the following result; for this reason they are sometime called *the telescopic localizations.* Recall the notation $T(n)$ introduced in 2.10.

Proposition 4.6 (Mahowald and Sadofsky, 1995; Miller, 1992)

Let $n \in \mathbb{N}$. For all m we have $\langle T(m) \rangle \geq \langle K(m) \rangle$, and hence $\langle T(0) \oplus \cdots \oplus T(n) \rangle \geq \langle K(0) \oplus \cdots \oplus K(n) \rangle$. The natural map $L_{T(0)\oplus\cdots\oplus T(n)} \to L_{K(0)\oplus\cdots\oplus K(n)} = L_n$ factors through an equivalence

$$L_{T(0)\oplus\cdots\oplus T(n)} \xrightarrow{\simeq} L_n^f$$

Remark 4.7. — It follows from the above result that $\langle T(0) \oplus \cdots \oplus T(n) \rangle$ depends only on n and not the choices of the telescopes $T(i)$, since it is equal to $\ker(L_n^f)$. In fact, Mahowald and Sadofsky (1995, Lemma 2.1) show that $\langle T(n) \rangle$ depends only on n.

We have therefore two competing notions of "chromatic filtrations" of a spectrum X, given by the towers

$$\begin{aligned} X \cdots \to L_2^f X \to L_1^f X \to L_0^f X \quad \text{and} \\ X \cdots \to L_2 X \to L_1 X \to L_0 X\,. \end{aligned} \tag{8}$$

These also induce filtrations of the homotopy groups of X,

$$\cdots \subseteq G_2^f(X) \subseteq G_1^f(X) \subseteq G_0^f(X) \subseteq \pi_*(X)$$

with $G_n^f(X) = \ker\left(\pi_*(X) \to \pi_*(L_n^f X)\right)$, and

$$\cdots \subseteq G_2(X) \subseteq G_1(X) \subseteq G_0(X) \subseteq \pi_*(X)$$

with $G_n(X) = \ker\left(\pi_*(X) \to \pi_*(L_n X)\right)$.

By the factorization (7)

$$X \to L_n^f X \to L_n X\,, \tag{9}$$

we have

$$G_n^f(X) \subseteq G_n(X) \quad \text{for all } n \in \mathbb{N}\,.$$

Definition 4.8. — *For a (p)-local spectrum X, the filtration $\{G_n(X)\}_n$ is called the* algebraic chromatic filtration, *and the filtration $\{G_n^f(X)\}_n$ is called the* geometric chromatic filtration.

Proposition 4.9 (Mahowald and Sadofsky, 1995, Proposition 4.1)

Let X be a spectrum. The filtrations $\{G_n^f(X)\}_n$ in 4.8 and $\{F_n(X)\}_n$ in 2.12 coincide.

This is related to the following result.

Proposition 4.10 (Miller, 1992, Proposition 14). — *Let X be a finite, (p)-spectrum of type n, and f be a v_n-self-map of X. Then the natural map $X \to L_n^f(X)$ factors through an equivalence $X[f^{-1}] \to L_n^f(X)$.*

Ravenel (1987) shows that in the construction of the chromatic spectral sequence 1.14, the resolution (4) can be realized inductively, using $N_0 = \mathbb{S}$ and $M_n := L_n(N_n)$, so that $BP_*(N_n) = N^n$ and $BP_*(M_n) = M^n$, and that the algebraic filtration of $\pi_*(\mathbb{S}_{(p)})$ defined by means of the chromatic spectral sequence as in Definition 1.14 agrees with the filtration $\{G_n(\mathbb{S}_{(p)})\}_n$ given above.

We can now state the Telescope Conjecture 2.13 at spectrum level.

RAVENEL'S TELESCOPE CONJECTURE 4.11. — *The natural map* $L_n^f \to L_n$ *is an equivalence for all* $n \in \mathbb{N}$ *.*

Remark 4.12. — Note that since these localizations are smashing, the Telescope Conjecture is equivalent to the statement that for all $n \in \mathbb{N}$, the map

$$L_n^f(\mathbb{S}) \to L_n(\mathbb{S})$$

is an equivalence. In that sense, the Conjectures 2.13 and 4.11 are equivalent.

The Telescope Conjecture holds for $n = 0$ and $n = 1$, as we will sketch in the next section. If the Conjecture holds for $n - 1$, it will hold for n if and only the following equivalent statements hold. Note that item 3 in the proposition below is Ravenel's original formulation as Ravenel, 1984, Conjecture 10.5.

PROPOSITION 4.13 (Mahowald, Ravenel, and Shick, 2001, Proposition 1.13)

For each prime p *and* $n \in \mathbb{N}$ *, let* X *be a chosen finite* (p)*-local spectrum of type* n*. The following statements are equivalent.*

1. *If* $L_{n-1}^f \to L_{n-1}$ *is an equivalence, so is* $L_n^f \to L_n$*;*
2. *The natural map* $\mathrm{Tel}(X) \to L_{K(n)}X$ *is an equivalence;*
3. $\langle T(n) \rangle = \langle K(n) \rangle$*, i.e. the natural map* $L_{T(n)} \to L_{K(n)}$ *is an equivalence;*
4. *The Adams–Novikov spectral sequence for* $\mathrm{Tel}(X)$ *converges to* $\pi_*(\mathrm{Tel}(X))$*.*

This Conjecture is appealing, since it equates the "geometric" and "algebraic" chromatic filtrations. The first one is the one we are most interested in, since it precisely captures the v_n-torsion and periodicity phenomena in homotopy; however, it remains mysterious and incomputable.

On the other hand, there are well established strategies to evaluate the second. Indeed, if X is a finite (p)-local spectrum of type n, the Adams–Novikov-spectral sequence 1.13 for $\mathrm{Tel}(X)$ converges to $\pi_*(L_nX)$, and thus, if the Telescope Conjecture holds, will also converge to $\pi_*(L_n^f(X)) = \pi_*(\mathrm{Tel}(X))$, see Ravenel, 1992b. Note that $\mathrm{Tel}(X)$ is not connective, so convergence of the Adams–Novikov spectral sequence need not hold (and indeed fails for $n \geq 2$, by failure of the Telescope Conjecture).

For the sphere spectrum $\mathbb{S}$, there is an iterative approach to compute $L_n(\mathbb{S})$ (and thus L_nX for any spectrum X, since L_n is smashing). It uses the following version of

Proposition 3.5: for $n \geq 1$, using $L_{K(n)}L_{n-1} = 0$, the homotopy cartesian square

$$\begin{tikzcd} L_n(X) \arrow[r] \arrow[d] & L_{n-1}(X) \arrow[d] \\ L_{K(n)}(X) \arrow[r] & L_{n-1}(L_{K(n)}(X)) \end{tikzcd} \tag{10}$$

is called *the n-th chromatic fracture square.* We also have a corresponding square relating L_n^f, L_{n-1}^f and $L_{T(n)}$. Assuming that $L_{n-1}(\mathbb{S})$ (or $L_{n-1}^f(\mathbb{S})$) is known, it remains to evaluate $L_{K(n)}(\mathbb{S})$ (resp. $L_{T(n)}(\mathbb{S})$) to determine $L_n(\mathbb{S})$ (resp. $L_n^f(\mathbb{S})$).

One approach that has been successfully applied for low values of n (see for example Goerss, Henn, Mahowald, and Rezk, 2005) relies on Morava's E_n theory (Morava, 1985), with coefficients

$$E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$$

with $|u_i| = 0$ and $|u| = 2$, and where $W(\mathbb{F}_{p^n})$ denotes the Witt-ring of $\mathbb{F}_{p^n}$. It is a commutative, complex oriented and $K(n)$-local ring spectrum. The ring $W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$ is the Lubin–Tate deformation ring (Lubin and Tate, 1966) of the p-typical, height n Honda formal group law H_n over $\mathbb{F}_{p^n}$. The *extended Morava stabilizer group* G_n is the automorphism group of H_n. It is a profinite group that acts in a continuous sense on E_n, and one can take continuous homotopy fixed-points.

Theorem 4.14 (Devinatz and Hopkins, 2004). — *There is an equivalence*

$$L_{K(n)}(\mathbb{S}) \simeq E_n^{hG_n} \, .$$

In particular, there is an associated continuous homotopy fixed-points spectral sequence

$$E_2^{s,t} = H_c^s(G_n, \pi_t(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}(\mathbb{S})) \, .$$

The following Theorem is another interesting feature of the L_n localization tower.

Theorem 4.15 (Ravenel, 1992a, Chromatic Convergence Theorem 7.5.7)
For a (p)-local finite spectrum X, the chromatic tower

$$X \cdots \to L_2(X) \to L_1(X) \to L_0(X)$$

converges in the sense that $X \simeq \lim_n(L_n(X))$.

It is unknown if the corresponding result for the tower

$$X \cdots \to L_2^f(X) \to L_1^f(X) \to L_0^f(X)$$

holds; however, we know from the factorizations (9) and the above Theorem that, if X a (p)-local finite spectrum, then X is a direct summand of $\lim_n(L_n^f(X))$.

5. The cases of height zero and one

In this section we outline why the Telescope Conjecture holds in the cases of height 0 and 1. Along the way we will also illustrate several of the results stated above.

The case of height 0 follows from classical results on the (unstable) homotopy groups of the spheres S^n.

The Hurewicz Theorem 5.1. — *The group* $\pi_m(S^n)$ *is trivial for* $m < n$, *and* $\pi_n(S^n) \cong \mathbb{Z}$, *generated by* $\mathrm{id}\colon S^n \to S^n$.

The Freudenthal Suspension Theorem 5.2. — *The suspension homomorphism*

$$\sigma\colon \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism if $n > k+1$, *and a surjection for* $n = k+1$.

The Serre Finiteness Theorem 5.3. — *For all* $k, \ell \in \mathbb{N}$, *the homotopy group* $\pi_k(S^\ell)$ *is a finite abelian group, except in the following cases:*

$$\pi_k(S^\ell) \cong \begin{cases} \mathbb{Z} & \text{if } k = \ell \text{, and} \\ \mathbb{Z} \oplus F_k & \text{if } \ell \text{ is even and } k = 2\ell - 1, \end{cases}$$

where F_k *denotes a finite abelian group.*

From Freudenthal's Suspension Theorem 5.2, we see that the k-th homotopy group $\pi_k(\mathbb{S})$ of the sphere spectrum can be computed by the isomorphism

$$\pi_{n+k}(S^n) \xrightarrow{\cong} \operatorname{colim}\Big(\pi_{n+k}(S^n) \xrightarrow{\sigma} \pi_{n+k+1}(S^{n+1}) \xrightarrow{\sigma} \cdots\Big) =: \pi_k(\mathbb{S})$$

whenever $n > k+1$, since the σ's in the colimit are all isomorphisms for these values. We then have the following corollary of Hurewicz' Theorem and Serre's Finiteness Theorem.

Corollary 5.4. — *The homotopy group* $\pi_k(\mathbb{S})$ *is trivial if* $k < 0$, *is isomorphic to* $\mathbb{Z}$ *if* $k = 0$, *and is a finite abelian group if* $k > 0$.

The (p)-local sphere spectrum $\mathbb{S}_{(p)}$ can be constructed as the telescope over all primes $q \neq p$; it is of type 0, see Example 2.9.1, and it admits the periodic v_0-self-map given by the multiplication by p. In particular, we deduce that $\mathrm{Tel}(\mathbb{S}_{(p)})$ is obtained from $\mathbb{S}$ as a telescope inverting all primes. Therefore

$$\mathrm{Tel}(\mathbb{S}_{(p)}) = \operatorname*{colim}_{\mathbb{N}^*}(\mathbb{S} \xrightarrow{1} \mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{3} \cdots)$$

and

$$\pi_*(\mathrm{Tel}(\mathbb{S}_{(p)})) = \operatorname*{colim}_{\mathbb{N}^*}(\pi_*(\mathbb{S}) \xrightarrow{1} \pi_*(\mathbb{S}) \xrightarrow{2} \pi_*(\mathbb{S}) \xrightarrow{3} \cdots) = \pi_*(\mathbb{S}) \otimes \mathbb{Q}\,.$$

Thus from Corollary 5.4 we deduce the following proposition. Since $K(0) = H\mathbb{Q}$ by definition, this settles the case $n = 0$ of the conjecture.

Corollary 5.5. — *The group* $\pi_k(\mathrm{Tel}(\mathbb{S}_{(p)}))$ *is* 0 *for* $k \neq 0$, *and isomorphic to* $\mathbb{Q}$ *for* $k = 0$. *In particular, the unit map* $\mathbb{S}_{(p)} \to H\mathbb{Q}$ *factors through an equivalence*

$$\mathrm{Tel}(\mathbb{S}_{(p)}) \xrightarrow{\simeq} H\mathbb{Q}\,.$$

In other terms, Ravenel's Telescope Conjecture holds for $n = 0$.

We now turn to the case $n = 1$. Since the Conjecture holds for $n = 0$, and since $\mathbb{S}/p$ is a finite (p)-local spectrum of type 1, by Proposition 4.13, it suffices to show that the natural map

$$\mathrm{Tel}(\mathbb{S}/p) \to L_{K(1)}(\mathbb{S}/p)$$

is an equivalence. We can first compute $L_{K(1)}(\mathbb{S})$ by applying Theorem 4.14 to the $n = 1$ case,

$$L_{K(1)}(\mathbb{S}) \simeq E_1^{hG_1}\,. \tag{11}$$

Here E_1 is the p-completed periodic complex K-theory spectrum KU_p, with coefficients $(KU_p)_* = \mathbb{Z}_p[u^{\pm 1}]$, $|u| = 2$ (note that $W(\mathbb{F}_p) = \mathbb{Z}_p$). The first Morava stabilizer group G_1 is isomorphic to the p-adic units $\mathbb{Z}_p^\times$, where $k \in \mathbb{Z}_p^\times$ acts on KU_p via the p-adic Adams operation

$$\psi^k \colon KU_p \to KU_p\,.$$

Here ψ^k is a map of ring spectra, that can be evaluated on coefficients by the formula $\psi^k(u) = ku$.

Such operations were first constructed by Adams for $k \in \mathbb{Z}$ on the complex K-theory group $KU^0(X)$ of a space, by using the definition of the group $KU^0(X)$, for a suitably finite space X (a finite CW-complex), as the group completion of the monoid of isomorphism classes of finite dimensional complex vector bundles on X, with the direct sum. The Adams operations were then extended to the stable (spectrum level) p-adic Adams operations above, see for example (Atiyah and Tall, 1969).

In the sequel, we concentrate on the case where p is an odd prime, since there are some differences with the case $p = 2$. The group $\mathbb{Z}_p^\times$ is topologically generated by any element $k \in \mathbb{Z}_p^\times$ that reduces to a generator of $(\mathbb{Z}/p^2)^\times$. For such k, the homotopy fixed points spectrum $KU_p^{h\mathbb{Z}_p^\times}$ is equivalent to $KU_p^{h\langle k\rangle}$, where $\langle k\rangle \cong \mathbb{Z}$ is the (discrete) subgroup generated by k. It follows that $KU_p^{h\mathbb{Z}_p^\times}$ is the homotopy equalizer of the maps ψ^k and id from KU_p to itself; equivalently, it fits in a cofiber sequence

$$KU_p^{h\mathbb{Z}_p^\times} \longrightarrow KU_p \xrightarrow{\psi^k - \mathrm{id}} KU_p\,. \tag{12}$$

Remark 5.6. — The fiber of $KU_p \xrightarrow{\psi^k - \mathrm{id}} KU_p$ is usually called the p-complete J-spectrum, since it is closely related to the image of the stable j-homomorphism $j \colon \pi_*(O) \to \pi_*(\mathbb{S})$ introduced by G. W. Whitehead, whose image was determined by Adams and Quillen.

It is then easy to evaluate the long exact sequence in homotopy associated to the fiber sequence (12), using the formula $(\psi^k - \mathrm{id})_*(u^n) = (k^n - 1)u^n$ and the formula

$$v_p(k^n - 1) = \begin{cases} v_p(n) + 1 & \text{if } (p-1) \text{ divides } n, \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

(for k a topological generator of $\mathbb{Z}_p^\times$), where $v_p \colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ is the p-adic valuation. We obtain the following computation:

$$\pi_n(L_{K(1)}(\mathbb{S})) \cong \pi_n(KU_p^{h\mathbb{Z}_p^\times}) \cong \begin{cases} \mathbb{Z}_p & \text{if } n = 0, -1, \\ \mathbb{Z}/p^{v_p(m)+1} & \text{if } n = 2(p-1)m - 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the p-torsion groups above precisely correspond to those given by the $\{\alpha_k\}_k$-family in Theorem 0.2, which is detected by $\mathbb{S} \to L_{K(1)}(\mathbb{S})$.

Remark 5.7. — Notice that we deduce from the above formula that

$$\pi_n(L_{K(0)}L_{K(1)}(\mathbb{S})) \cong \pi_n(L_{K(1)}(\mathbb{S})) \cong \begin{cases} \mathbb{Q}_p & \text{if } n = 0, -1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have an equivalence $L_{K(0)}L_{K(1)}(\mathbb{S}) \simeq H\mathbb{Q}_p \oplus \Sigma^{-1}H\mathbb{Q}_p$. On the other hand, from Corollary 5.4, we know that $L_0(\mathbb{S}) \simeq H\mathbb{Q}$. Using the first chromatic fracture square (10), we can also evaluate $L_1(\mathbb{S})$:

$$\pi_n(L_1(\mathbb{S})) \cong \begin{cases} \mathbb{Z}_{(p)} & \text{if } n = 0, \\ 0 & \text{if } n = -1, \\ \mathbb{Z}/p^\infty & \text{if } n = -2, \\ \pi_n(L_{K(1)}(\mathbb{S})) & \text{otherwise.} \end{cases}$$

This appears in Ravenel, 1984, Theorem 8.10. Notice that the isomorphism $\pi_0(L_1(\mathbb{S})) \cong \mathbb{Z}_{(p)}$ corresponds to the following pull-back square of ordinary rings

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\ \mathbb{Z}_p \arrow[r] & \mathbb{Q}_p , \end{tikzcd}$$

which is the (p)-local version of the classical arithmetic square.

From the cofiber sequence (12), we deduce a cofiber sequence

$$L_{K(1)}(\mathbb{S}/p) \longrightarrow KU/p \xrightarrow{\psi^k - \mathrm{id}} KU/p ,$$

and using the formulas given in (13), we see that we have an isomorphism of graded rings

$$\pi_*(L_{K(1)}(\mathbb{S}/p)) \cong \mathbb{F}_p[v_1^{\pm 1}] \otimes E(\varepsilon) ,$$

where $|\varepsilon| = -1$, $E(\varepsilon)$ denotes the exterior (or square-zero) $\mathbb{F}_p$-algebra on the generator ε, and where we have used the relation $u^{p-1} = v_1$ (see 5.11).

In his work on the image of the j-homomorphism, long before the proof of the Periodicity Theorem2.5, Adams (1966) showed the existence of a periodic v_1 self-map

$$\Sigma^{|v_1|}\mathbb{S}/p \xrightarrow{v_1} \mathbb{S}/p$$

at an odd-prime, using detection in complex K-theory. Before the Telescope Conjecture was formulated, Miller (1981, Theorem 4.11) computed the homotopy groups of $\mathrm{Tel}(\mathbb{S}/p) = \mathbb{S}/p[v_1^{-1}]$ using a localized Adams spectral sequence, and proving its convergence. Combined with the evaluation $\pi_*(L_{K(1)}(\mathbb{S}/p))$, this implies the following result.

Theorem 5.8 (Miller). — *Let p be an odd prime. The natural map $\mathbb{S}/p \to L_{K(1)}(\mathbb{S}/p)$ induces an equivalence*

$$\mathbb{S}/p[v_1^{-1}] \simeq L_{K(1)}(\mathbb{S}/p)\,.$$

In particular, the Telescope Conjecture holds at height 1 for odd primes.

In contrast, at the prime 2, there is a v_1-periodic self-map

$$\Sigma^8\mathbb{S}/2 \xrightarrow{v_1^4} \mathbb{S}/2,$$

but no lower powers of v_1 can be realized. Mahowald (1981, Theorem 6.3) computed $\pi_*(\mathbb{S}/2)[v_1^{-4}]$; with a computation of $L_{K(1)}(\mathbb{S}/2)$ by methods closely related to the odd-primary case above, this implies the following result.

Theorem 5.9 (Mahowald). — *The Telescope Conjecture at height 1 holds for the prime* 2*.*

In the above discussion, we could have worked also with a "subring" of KU_p known as the Adams summand L. Note that the mod p reduction map of $\mathbb{Z}_p^\times$ fits in a short exact sequence

$$1 \to (1 + p\mathbb{Z}_p) \to \mathbb{Z}_p^\times \to \mathbb{F}_p^\times \to 1$$

which splits by including $\mathbb{F}_p^\times$ as the subgroup Δ of $(p-1)$th roots of unity.

Definition 5.10. — *Let p be an odd prime. The commutative ring spectrum*

$$L = KU_p^{h\Delta}$$

is called the (p-complete, periodic) Adams summand of KU_p.

Proposition 5.11 (Adams, 1969, Corollary 4.8). — *The coefficients of L are given by $L_* = \mathbb{Z}_p[v_1^{\pm 1}]$, and the ring spectrum map given by the inclusion of fixed-points $L \to KU_p$ induces the homomorphism*

$$L_* \to KU_p \quad v_1 \mapsto u^{p-1}\,.$$

The specrum KU_p spilts as an L-module into a sum

$$KU_p = \bigoplus_{i=0}^{p-2} \Sigma^{2i} L\,.$$

Remark 5.12. — From the above result, we deduce that $KU_p^{h\mathbb{Z}_p^\times} \simeq (KU_p^{h\Delta})^{h(\mathbb{Z}_p^\times/\Delta)} \simeq L^{h(1+p\mathbb{Z}_p)}$. As above, if k is a topological generator of $1+p\mathbb{Z}_p$, for example $k=1+p$, and if $\mathbb{Z} \cong \langle k\rangle \subset (1+p\mathbb{Z}_p)$ acts on L by the Adams operation ψ^k, we have the reformulation of (11) given by

$$L_{K(1)}(\mathbb{S}) \simeq L^{h\mathbb{Z}}\,.$$

Definition 5.13. — The connective Adams summand *is the* 0*-connective cover of* L*, denoted* ℓ*, with coefficients* $\ell_* = \mathbb{Z}_p[v_1]$*. The action of* $\mathbb{Z}$ *on* L *discussed above restricts to an action on* ℓ*, and the homotopy fixed points spectrum*

$$\ell^{h\mathbb{Z}}$$

is the the (-1)*-connective cover of* $L^{h\mathbb{Z}} \simeq L_{K(1)}(\mathbb{S})$*.*

6. The counter-example at height two

At any prime, the Telescope Conjecture 4.11 fails at all heights larger or equal to 2. The next Theorem offers the first disproof by exhibiting an explicit counter-example. Hereafter the article Burklund, Hahn, Levy, and Schlank (BHLS) will be mentioned as BHLS.

Theorem 6.1 (BHLS, Theorem A). — *Let* p *be any prime and* $n \geq 1$*. For any* $k \geq 0$*, the spectrum*

$$L_{T(n+1)}(K(BP\langle n\rangle^{hp^k\mathbb{Z}}))$$

is not $K(n+1)$*-local. In particular,*

$$\langle T(n+1)\rangle \gneqq \langle K(n+1)\rangle\,,$$

and the Telescope Conjecture fails at height $n+1$*.*

Here $BP\langle n\rangle = BP/(v_{n+1}, v_{n+2}, \dots)$ is a BP-algebra with coefficients

$$BP\langle n\rangle_* = \mathbb{Z}_p[v_1, \dots, v_n]\,,$$

with a specific action of $\mathbb{Z}$ by an analog of Adams operations, and $K(X)$ denotes the algebraic K-theory of a ring spectrum X. Below, we will give a sketch of the proof of this theorem in the simplest case, when $n = 1$ and $p \geq 7$. Before that, let us briefly introduce the algebraic K-theory functor, and indicate some of the features that made it a successful source of counter-examples for the Telescope Conjecture.

Algebraic K-theory is a functor

$$K\colon \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Sp}$$

from structured (or A_∞) ring spectra to spectra; if the input is a structured commutative (or E_∞) ring-spectrum, so is the output, and algebraic K-theory thus specializes to a functor

$$K\colon \mathrm{CAlg}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp})\,.$$

When applied to the Eilenberg–Mac Lane spectrum of an ordinary ring R, $\pi_0(K(R))$ coincides with Grothendieck's group $K_0(R)$, and its first and second homotopy groups correspond to classical algebraically defined invariants. Quillen (1973) gave the first definition of *higher algebraic K-theory*, where the n-th K-theory group $K_n(R)$ is defined as the homotopy group of a connective spectrum $K(R)$. Grothendieck's group $K_0(R)$ it obtained from group-completing the monoid of (isomorphism classes) of finitely generated projective R-modules, with the direct sum. In an informal sense, higher algebraic K-theory corresponds to a higher-categorical group-completion of the same underlying monoid, but taking into account the isomorphisms (i.e. group-completing a groupoid). Quillen's $+$-construction and Q-construction were first realizations of this idea; this was later generalized by the $S_\bullet$-construction of Waldhausen (1985). Barwick (2016) extended the $S_\bullet$-construction to suitable ∞-categories, providing the modern framework for algebraic K-theory.

If A is a ring spectrum, $\mathrm{Perf}(A)$ denotes the full subcategory of the category of left A-modules spanned by A under finite colimits, desuspensions and retracts. For an ordinary ring R, objects of $\mathrm{Perf}(R)$ coincide with bounded chain complexes of finitely generated projective R-modules. The K-theory of A is then defined as

$$K(A) \coloneqq K(\mathrm{Perf}(A))\,.$$

This functor is difficult to compute; Quillen (1972), in a tour de force, computed $K_*(\mathbb{F}_p)$, while $K_*(\mathbb{Z})$ has been only partially determined because of number theoretical issues. The functor K has many fascinating properties; the ones that interest us most here concern its interactions with chromatic homotopy theory.

The K-theory spectrum $K(\mathbb{S})$ is the receptacle of an invariant used in the study of the geometric topology of manifolds, see Waldhausen, Jahren, and Rognes, 2013. Waldhausen (1984) proposed to study $K(\mathbb{S})$ by climbing the chromatic tower

$$K(\mathbb{S}) \cdots \to K(L_2^f(\mathbb{S})) \to K(L_1^f(\mathbb{S})) \to K(L_0^f(\mathbb{S})) = K(\mathbb{Q})\,,$$

which motivated the study of $K(A)$ when A is for example a $T(n)$-local spectrum for small values of n, and also motivated the development of chromatic homotopy theory.

Note that when A is a non-zero commutative ring spectrum, $K(A)$ is never a finite spectrum; in fact its chromatic behavior sharply contrasts with the chromatic behavior of finite spectra as stated in the Periodicity Theorem 2.5. A first example is the following theorem. Recall that $K(n)$ denotes the n-th Morava K-theory at a prime p. If R is an ordinary ring, then $K(1)_*(R) = 0$, but for example we have $K(1)_*(K(\mathbb{Z})) \neq 0$.

Theorem 6.2 (Mitchell, 1990, Theorem A). — *For any (ordinary) ring R, any prime p and any $n \geq 2$, the n-th Morava K-theory of $K(R)$ vanishes:*

$$K(n)_*(K(R)) = 0\,.$$

This provides examples where algebraic K-theory increase the chromatic complexity of its input from 0 to 1 (but no more). Another example, to which we will return below, is given by the Adams summand ℓ introduced in Definition 5.13. Note that $L_{T(1)}(\ell) = L \neq 0$ while $L_{T(2)}(\ell) = 0$. In (Ausoni and Rognes, 2002), we showed that if $p \geq 5$, then $L_{T(2)}(K(\ell)) \neq 0$ while $L_{T(3)}(K(\ell)) = 0$.

Such examples led John Rognes to state the *Redshift Conjectures* (Ausoni and Rognes, 2008; Rognes, 2014) on the chromatic and arithmetic behavior of the algebraic K-theory of ring spectra. Here *Redshift* refers to the idea that K-theory promotes the v_n-periodicity of its input to v_{n+1}-periodicity of its output, thus "increasing" the wavelength (shifting towards red). This has recently been proven in full generality for commutative ring spectra.

Hahn (2016) proved than if $A \in \mathrm{CAlg}(\mathrm{Sp})$ and if $n \in \mathbb{N}$, then $T(n)_*(A) = 0$ implies $T(n+1)_*(A) = 0$. This motivates the next definition. Note that by (Land, Mathew, Meier, and Tamme, 2024, Lemma 2.3), when A is a ring spectrum, $K(n)_*(A) = 0$ if and only if $T(n)_*(A) = 0$, so in the next definition one could replace $T(n)_*$ by $K(n)_*$.

Definition 6.3. — *Let* $A \in \mathrm{CAlg}(\mathrm{Sp})$ *be non-zero.* The height of A *is defined as*

$$\mathrm{ht}(A) = \max\{n \in \mathbb{N} \mid T(n)_*(A) \neq 0\}$$

Theorem 6.4. — *Let* $A \in \mathrm{CAlg}(\mathrm{Sp})$ *be non-zero, with* $\mathrm{ht}(A) \geq 0$*. Then*

$$\mathrm{ht}(K(A)) = \mathrm{ht}(A) + 1\,.$$

This result was proven in a series of amazing papers: the inequality $\mathrm{ht}(K(A)) \leq \mathrm{ht}(A) + 1$ is proven in (Land, Mathew, Meier, and Tamme, 2024; Clausen, Mathew, Naumann, and Noel, 2024) while the inequality $\mathrm{ht}(K(A)) \geq \mathrm{ht}(A)+1$ is proven in (Yuan, 2021; Burklund, Schlank, and Yuan, 2022).

The next result shows that for $n \geq 1$, $L_{T(n)}K(A)$ depends only on $L_{T(n-1)\oplus T(n)}(A)$.

Purity Theorem 6.5 (Land, Mathew, Meier, and Tamme, 2024)

Let A be a ring spectrum. For $n \geq 1$, the canonical map $A \to L_{T(n-1)\oplus T(n)}(A)$ induces an equivalence

$$L_{T(n)}(K(A)) \to L_{T(n)}(K(L_{T(n-1)\oplus T(n)}(A)))\,.$$

Examples of computations have shown that when A is a ring-spectrum of height n, $L_{T(n+1)}(K(A))$ is accessible but still has particularly interesting features, while $L_{T(m)}(K(A))$ will be increasingly difficult to compute when the $0 < m \leq n$ gets smaller. This contrasts with the case of finite spectra, where v_m-periodicity gets more difficult to study when m increases.

Topological cyclic homology (denoted TC), which builds on topological Hochschild homology (denoted THH), has been an extremely powerful tool for performing computations of the algebraic K-theory of ring spectra. THH was invented by Bökstedt (unpublished), and TC first appeared in Bökstedt, Hsiang, and Madsen (1993). Nikolaus and Scholze (2018) offered new foundations for the theory. For $A \in \mathrm{Alg}(\mathrm{Sp})$, topological Hochschild homology of A, as a spectrum, is given by

$$THH(A) = A \otimes_{A \otimes A^{\mathrm{op}}} A\,,$$

which is the tensor product of A with itself in the category of bi-modules. It can be realized as a cyclic-bar construction, which shows that $THH(A)$ has an action by the circle $\mathbb{T} = S^1$. Nikolaus and Scholze promote $THH(A)$ to a *cyclotomic spectrum*: they endow it with a $\mathbb{T}/C_p$-equivariant map

$$\varphi\colon THH(A) \to THH(A)^{tC_p}\,,$$

where $C_p \subset \mathbb{T}$ is the cyclic subgroup of order p, and where $THH(A)^{tC_p}$ denotes the Tate construction that sits in a cofiber sequence with the norm map N,

$$THH(A)_{hC_p} \xrightarrow{N} THH(A)^{hC_p} \longrightarrow THH(A)^{tC_p}\,.$$

This is analogous to the construction of the Tate cohomology of a group G by splicing its homology and its cohomology. They then define topological cyclic homology of A (at a prime p) to fit in a cofiber sequence

$$TC(A) \longrightarrow THH(A)^{h\mathbb{T}} \xrightarrow{\varphi^{h\mathbb{T}}-\mathrm{can}} (THH(A)^{tC_p})^{h\mathbb{T}/C_p}\,.$$

There is a natural transformation for connective A ring spectra, called the cyclotomic trace $\mathrm{trc}\colon K(A) \to TC(A)$. The following theorem is the fundamental tool for computing K-theory from topological cyclic homology.

Theorem 6.6 (Dundas, Goodwillie, and McCarthy, 2013, Theorem 7.2.2.1)
Let $f\colon A \to B$ be a map of connective ring spectra, such that $f_*\colon \pi_0(A) \to \pi_0(B)$ is surjective with nilpotent kernel. Then the following square is homotopy cartesian after p-completion:

$$\begin{tikzcd}
K(A) \arrow[r, "\mathrm{trc}"] \arrow[d, "f_*"'] & TC(A) \arrow[d, "f_*"] \\
K(B) \arrow[r, "\mathrm{trc}"] & TC(B)
\end{tikzcd}$$

In particular, the difference between $K(A)$ and $TC(A)$ (in other word the fiber F of the cyclotomic trace $K(A) \to TC(A)$) is the same as the difference between $K(B)$ and $TC(B)$. Assuming that F is known (for example via a computation of the bottom map of the square), to compute $K(A)$, it essentially remains to compute $TC(A)$. This is very useful because $TC(A)$ is entirely defined by homological constructions, starting with $THH(A)$, fixed-points and the Tate construction, all of which can be in principle evaluated by homological algebra methods, such as spectral sequences. These homological algebra methods permit explicit computations in good cases, with suitably finite coefficients, namely $V_*(TC(A))$ where $\mathrm{ht}(A) = n$ and V is a finite (p)-local spectrum of type at least n. On the other side, $K(A)$ is defined via categorification and group-completion, which is a much more mysterious construction.

Let us now return to disproof of the Telescope Conjecture, and sketch the proof in these easiest case where $n = 1$ and $p \geq 7$, treated in BHLS, Theorem D. This differs a little from the corresponding case in Theorem 6.1. The authors take $p \geq 7$ to ensure the existence of the Smith–Toda complex $V(2)$ (see Definition 2.11) as a homotopy commutative and associative ring spectrum. They consider the Adams summand L, see Definition 5.10, with the action of $\mathbb{Z}$ given in Remark 5.12.

Note that the $\mathbb{Z}$-action on L restricts to an action of any subgroup of $\mathbb{Z}$, and in particular $p^k\mathbb{Z} \subset \mathbb{Z}$. The following theorem is based on the work of Ben-Moshe, Carmeli, Schlank, and Yanovski (2025), see also the discussion in (BHLS, Section 6).

Theorem 6.7. — *For any $k \geq 0$ and odd p, the coassembly map induces an equivalence*

$$L_{K(2)}(K(L^{hp^k\mathbb{Z}})) \to L_{K(2)}(K(L)^{hp^k\mathbb{Z}}) .$$

This contrasts with the following result.

Theorem 6.8 (BHLS). — *Let $p \geq 7$. For k large enough, the coassembly map*

$$L_{T(2)}K(L^{hp^k\mathbb{Z}}) \to L_{T(2)}K(L)^{hp^k\mathbb{Z}} .$$

is not an equivalence. Therefore, the Telescope Conjecture fails at height 2 and $p \geq 7$.

In fact, the above theorem is a consequence of the stronger Theorem D of the same paper, where the authors explicitly compute

$$T(2)_*(K(L^{hp^k\mathbb{Z}})) \to T(2)_*(K(L)^{hp^k\mathbb{Z}}) \tag{14}$$

for k large enough, and show that this is not an isomorphism. We sketch below the several reductions allowing the authors to achieve this.

We start with the right hand side of (14). Blumberg and Mandell (2008, Localization Theorem) have proven the existence of a localization sequence

$$K(\mathbb{Z}_p) \to K(\ell) \to K(L) , \tag{15}$$

where $\mathbb{Z}_p = \ell/v_1$ and $L = \ell[v_1^{-1}]$.

Bökstedt and Madsen (1994) computed $K(\mathbb{Z}_p)$ and $TC(\mathbb{Z}_p)$, and it follows that these vanish $T(2)$-locally, so that $K(\ell) \to K(L)$ is a $T(2)$-equivalence. Note that this also follows from the Purity Theorem 6.5.

Applying Theorem 6.6 to the map of rings $\ell \to \pi_0(\ell) = \mathbb{Z}_p$, we obtain a homotopy cartesian square after p-completion

$$\begin{tikzcd}
K(\ell) \arrow[r] \arrow[d] & TC(\ell) \arrow[d] \\
K(\mathbb{Z}_p) \arrow[r] & TC(\mathbb{Z}_p).
\end{tikzcd}$$

Since $K(\mathbb{Z}_p)$ and $TC(\mathbb{Z}_p)$ are $T(2)$-acyclic, we deduce that we have a diagram of $\mathbb{Z}$-equivariant $T(2)$-equivalences

$$K(L) \leftarrow K(\ell) \to TC(\ell)\,.$$

In (Ausoni and Rognes, 2002, Theorem 0.3), we computed $V(1)_*(TC(\ell))$ (for $p \geq 5$) and showed that it is a finitely generated free $\mathbb{F}_p[v_2]$-module on $4(p+1)$ explicit generators. This computation has been simplified (and extended to the $p = 3$ case) in (Hahn, Raksit, and Wilson, 2025, Theorem 1.4.1) using the even filtration. In particular, we have

$$T(2)_*(TC(\ell)) \cong \mathbb{F}_p[v_2^{\pm 1}] \otimes_{\mathbb{F}_p(v_2)} V(1)_*(TC(\ell))\,,$$

so that $T(2)_*(TC(\ell)$ is also finite in any degree. Just as in the case (12) considered above, we have a cofiber sequence of the form

$$TC(\ell)^{hp^k\mathbb{Z}} \to TC(\ell) \to TC(\ell)\,,$$

which implies the following result.

Lemma 6.9. — *For any $k \geq 0$, the $\mathbb{F}_p$-module $T(2)_*(TC(\ell)^{hp^k\mathbb{Z}})$ is finite in any degree.*

Note that if $p^k\mathbb{Z}$ acted trivially on $V(1)_*(TC(\ell))$, we would have

$$V(1)_*(TC(\ell)^{hp^k\mathbb{Z}}) \cong V(1)_*(TC(\ell)) \otimes E(\zeta_k) \quad \text{with } |\zeta_k| = -1,$$

and the same would hold if we replaced $V(1)_*$ by $T(2)_*$.

Let us now consider the left hand side of (14). The spectrum $\ell^{hk\mathbb{Z}}$ is (-1)-connective, and therefore the existence of a localization sequence for $\ell^{hp^k\mathbb{Z}} \to L^{hp^k\mathbb{Z}}$ analogous to (15) doesn't follow from (Blumberg and Mandell, 2008). In addition, the connectivity hypothesis of Theorem 6.6 is not satisfied by $\ell^{hp^k\mathbb{Z}}$. However, variants of these results where established by Levy (2022, Theorems A and B) for $\ell^{h\mathbb{Z}}$. These results are generalized in BHLS, Corollary 6.3, using the Purity Theorem 6.5. We formulate this result in the case of ℓ, using $L_{T(1)}(\ell) = L$.

Proposition 6.10. — *Suppose ℓ equipped with an action of $\mathbb{Z}$ by ring-maps. There is a commuting diagram were the horizontal maps are equivalences, and the vertical maps are the coassembly maps:*

$$\begin{tikzcd}
L_{T(2)}(K(L^{h\mathbb{Z}})) \arrow[d] & L_{T(2)}(K(\ell^{h\mathbb{Z}})) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & L_{T(2)}(TC(\ell^{h\mathbb{Z}})) \arrow[d] \\
L_{T(2)}(K(L)^{h\mathbb{Z}}) & L_{T(2)}(K(\ell)^{h\mathbb{Z}}) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & L_{T(2)}(TC(\ell)^{h\mathbb{Z}})\,.
\end{tikzcd}$$

It remains to compute $T(2)_*(TC(\ell^{hp^k\mathbb{Z}}))$. McClure and Staffeldt (1993) computed $V(0)_*(THH(\ell))$, and we deduce the formula

$$V(1)_*THH(\ell) \cong \mathbb{F}_p[\mu] \otimes E(\lambda_1, \lambda_2)\,,$$

where $|\lambda_1| = 2p-1$, and $|\mu| = |\lambda_2| + 1 = 2p^2$. Lee and Levy (2023, Theorem 1.1) evaluated $V(1)_*(THH_*(\ell^{h\mathbb{Z}}))$, and their computation adapts directly to the case $\ell^{hp^k\mathbb{Z}}$.

Theorem 6.11 (Lee–Levy). — *For $k \geq 0$, there is an isomorphsm of $\mathbb{F}_p$-algebras*

$$V(1)_*(THH(\ell^{hp^k\mathbb{Z}})) \cong V(1)_*(THH(\ell)) \otimes HH_*(\mathbb{F}_p^{hp^k\mathbb{Z}}/\mathbb{F}_p)\,.$$

Moreover, we have an isomorphism of $\mathbb{F}_p$-algebras $HH_*(\mathbb{F}_p^{hp^k\mathbb{Z}}/\mathbb{F}_p) \cong C^0(p^k\mathbb{Z}_p, \mathbb{F}_p) \otimes E(\zeta_k)$, where $C^0(p^k\mathbb{Z}_p, \mathbb{F}_p)$ denotes the ring of continuous functions $f\colon p^k\mathbb{Z}_p \to \mathbb{F}_p$ (concentrated in degree 0), and $|\zeta_k| = -1$.

This formula would be the same if the action of $\mathbb{Z}$ on ℓ were the trivial one, showing that $V(1)_*(THH(-))$ doesn't see the difference. We have the following crucial result as a special case of (BHLS, Theorem 4.30): the same holds for $V(1)_*TC(-)$ when k is large enough.

Proposition 6.12. — *For k large enough, the coassembly map for the above $p^k\mathbb{Z}$-action (by Adams operations)*

$$TC(\ell^{hp^k\mathbb{Z}}) \to TC(\ell)^{hp^k\mathbb{Z}}$$

and the coassembly map for the trivial $\mathbb{Z}$-action

$$TC(\ell^{B\mathbb{Z}}) \to TC(\ell)^{B\mathbb{Z}}$$

induce the same homomorphism in $V(1)$-homotopy $V(1)_*(-)$.

The next step is to evaluate

$$V(1)_*TC(\ell^{B\mathbb{Z}}) \to V(1)_*TC(\ell)^{B\mathbb{Z}}\,, \tag{16}$$

by computing TC from THH using a standard approach. The input is the following result by Lee and Levy, 2023 (recall Theorem 6.11): the coassembly map

$$V(1)_*(THH(\ell^{B\mathbb{Z}})) \to V(1)_*(THH(\ell)^{B\mathbb{Z}}) \tag{17}$$

is given by tensoring $V(1)_*(THH(\ell))$ with the $\mathbb{F}_p$-module map

$$C^0(\mathbb{Z}_p, \mathbb{F}_p) \otimes E(\zeta) \to E(\zeta)$$

that sends ζ to itself, and sends $f \in C^0(\mathbb{Z}_p, F_p)$ to $f(0)$. In particular, the kernel of the homomorphism (17) is infinite dimensional over $\mathbb{F}_p$ in any degree. From there, (BHLS, Theorem 7.1) evaluates the coassembly homomorphism (16), determining how this phenomenon persists through the identification of the equalizer defining $TC(\ell^{B\mathbb{Z}})$. In particular, $V(1)_*(TC(\ell^{B\mathbb{Z}}))$ is a free $\mathbb{F}_p[v_2]$-module, and for infinitely many values of m, the group $V(1)_m(TC(\ell^{B\mathbb{Z}}))$ is infinite. We deduce that

$$T(2)_*(TC(\ell^{B\mathbb{Z}})) \cong \mathbb{F}_p[v_2^{\pm 1}] \otimes_{\mathbb{F}_p(v_2)} V(1)_*(TC(\ell^{B\mathbb{Z}}))\,,$$

and hence $T(2)_*(TC(\ell^{B\mathbb{Z}}))$ is also infinite in infinitely many degrees. By Proposition 6.12, the same holds for $T(2)_*(TC(\ell^{hp^k\mathbb{Z}}))$ when k is large enough. Comparison with Lemma 6.9 finishes the proof of Theorem 6.8.

Remark 6.13. — In BHLS, Section 1.5, the authors present their intention to explore, in forthcoming work with Shachar Carmeli and Lior Yanovski, additional consequences of their disproof; for example, for any $n \geq 1$, they expect that there exists $k \in \mathbb{Z}$ such that $\pi_k(T(n+1))$ is not finitely generated as an abelian group.

Abbreviations

BHLS Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank (2023). *K-theoretic counterexamples to Ravenel's telescope conjecture.* arXiv: `2310.17459 [math.AT]` (heretofore cited as: BHLS).

References

John F. Adams (1966). "On the groups $J(X)$. IV", *Topology* **5**, pp. 21–71.

——— ed. (1969). *Stable homotopy theory. Lectures delivered at the University of California at Berkeley, 1961. Notes by A.T. Vasquez.* Vol. 3. Lect. Notes Math. Springer, Cham.

——— (1974a). *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. Chicago - London: The University of Chicago Press. X, 373 p. £ 3.00 (1974).

——— (1974b). *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. Chicago - London: The University of Chicago Press. X, 373 p. £ 3.00 (1974).

Omar Antolín-Camarena and Tobias Barthel (2022). "Chromatic fracture cubes", in: *Equivariant topology and derived algebra. Based on the conference, Trondheim, Norway, 2019. In honour of Professor J. P. C. Greenlees' 60th birthday.* Cambridge: Cambridge University Press, pp. 100–118.

Michael F. Atiyah and David O. Tall (1969). "Group representations, λ-rings and the J-homomorphism", *Topology* **8**, pp. 253–297.

Christian Ausoni and John Rognes (2002). "Algebraic K-theory of topological K-theory", *Acta Math.* **188** (1), pp. 1–39.

——— (2008). "The chromatic red-shift in algebraic K-theory", *Enseign. Math. (2)* **54** (1-2), pp. 9–11.

Clark Barwick (2016). "On the algebraic K-theory of higher categories", *J. Topol.* **9** (1), pp. 245–347.

Shay Ben-Moshe, Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski (2025). "Descent and cyclotomic redshift for chromatically localized algebraic K-theory", *J. Am. Math. Soc.* **38** (2), pp. 521–583.

Andrew J. Blumberg and Michael A. Mandell (2008). “The localization sequence for the algebraic K-theory of topological K-theory”, *Acta Math.* **200** (2), pp. 155–179.

Marcel Bökstedt, Wu-Chung Hsiang, and Ib Madsen (1993). “The cyclotomic trace and algebraic K-theory of spaces”, *Invent. Math.* **111** (3), pp. 465–539.

Marcel Bökstedt and Ib Madsen (1994). “Topological cyclic homology of the integers”, in: *K-theory. Contributions of the international colloquium, Strasbourg, France, June 29-July 3, 1992.* Paris: Société Mathématique de France, pp. 57–143.

Aldridge K. Bousfield (1979). “The localization of spectra with respect to homology”, *Topology* **18**, pp. 257–281.

Edgar H. jun. Brown and Franklin P. Peterson (1966). “A spectrum whose $\mathbb{Z}/p$ cohomology is the algebra of reduced p-th powers”, *Topology* **5**, pp. 149–154.

Robert R. Bruner and John Rognes (2021). *The cohomology of the mod 2 Steenrod algebra.* Preprint, arXiv:2109.13117 [math.AT] (2021).

Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank (2023). *K-theoretic counterexamples to Ravenel’s telescope conjecture.* arXiv: `2310.17459 [math.AT]` (heretofore cited as: BHLS).

Robert Burklund, Tomer M. Schlank, and Allen Yuan (2022). *The Chromatic Nullstellensatz.* Preprint, arXiv:2207.09929 [math.AT] (2022).

Pierre Cartier (1967). “Modules associées à un groupe formel commutatif. Courbes typiques”, *C. R. Acad. Sci., Paris, Sér. A* **265**, pp. 129–132.

Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel (2024). “Descent and vanishing in chromatic algebraic K-theory via group actions”, *Ann. Sci. Éc. Norm. Supér. (4)* **57** (4), pp. 1135–1190.

Ethan S. Devinatz and Michael J. Hopkins (2004). “Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups.” *Topology* **43** (1), pp. 1–47.

Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith (1988). “Nilpotence and stable homotopy theory. I”, *Ann. Math. (2)* **128** (2), pp. 207–241.

Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy (2013). *The local structure of algebraic K-theory.* Vol. 18. Algebr. Appl. London: Springer.

Paul G. Goerss, Hans-Werner Henn, Mark Mahowald, and Charles Rezk (2005). “A resolution of the (K(2)-local sphere at the prime 3”, *Ann. Math. (2)* **162** (2), pp. 777–822.

Rok Gregoric (2021). *Moduli stack of oriented formal groups and the chromatic filtration.* Preprint, arXiv:2111.15202 [math.AT] (2021).

Jeremy Hahn (2016). *On the Bousfield classes of H_∞-ring spectra.* Preprint, arXiv:1612.04386 [math.AT] (2016).

Jeremy Hahn, Arpon Raksit, and Dylan Wilson (2025). *A motivic filtration on the topological cyclic homology of commutative ring spectra.* Preprint, arXiv:2206.11208 [math.KT] (2025).

Michael J. Hopkins, John H. Palmieri, and Jeffrey H. Smith (1999). “Vanishing lines in generalized Adams spectral sequences are generic”, *Geom. Topol.* **3**, pp. 155–165.

Michael J. Hopkins and Jeffrey H. Smith (1998). “Nilpotence and stable homotopy theory. II”, *Ann. Math. (2)* **148** (1), pp. 1–49.

David Copeland Johnson and W. Stephen Wilson (1975). “BP operations and Morava’s extraordinary K-theories”, *Math. Z.* **144**, pp. 55–75.

Markus Land, Akhil Mathew, Lennart Meier, and Georg Tamme (2024). “Purity in chromatically localized algebraic K-theory”, *J. Am. Math. Soc.* **37** (4), pp. 1011–1040.

Peter S. Landweber (1967). “Cobordism operations and Hopf algebras”, *Trans. Am. Math. Soc.* **129**, pp. 94–110.

Michel Lazard (1955). “Sur les groupes de Lie formels à un paramètre”, *Bull. Soc. Math. Fr.* **83**, pp. 251–274.

David Jongwon Lee and Ishan Levy (2023). *Topological Hochschild homology of the image of j.* Preprint, arXiv:2307.04248 [math.AT] (2023).

Ishan Levy (2022). *The algebraic K-theory of the K(1)-local sphere via TC.* Preprint, arXiv:2209.05314 [math.AT] (2022).

Jonathan Lubin and John Tate (1966). “Formal moduli for one-parameter formal Lie groups”, *Bull. Soc. Math. Fr.* **94**, pp. 49–59.

Jacob Lurie (2006). *Stable Infinity Categories.* Preprint, arXiv:math/0608228 [math.CT] (2006).

——— (2017). *Higher Algebra.* Preprint.

Mark Mahowald (1981). “*bo*-resolutions”, *Pac. J. Math.* **92**, pp. 365–383.

Mark Mahowald, Douglas C. Ravenel, and Paul Shick (2001). “The triple loop space approach to the telescope conjecture”, in: *Homotopy methods in algebraic topology. Proceedings of an AMS-IMS-SIAM joint summer research conference, University of Colorado, Boulder, CO, USA, June 20–24, 1999.* Providence, RI: American Mathematical Society (AMS), pp. 217–284.

Mark Mahowald and Hal Sadofsky (1995). “v_n telescopes and the Adams spectral sequence”, *Duke Math. J.* **78** (1), pp. 101–129.

James E. McClure and R. E. Staffeldt (1993). “On the topological Hochschild homology of *bu*. I”, *Am. J. Math.* **115** (1), pp. 1–45.

Haynes R. Miller (1981). “On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space”, *J. Pure Appl. Algebra* **20**, pp. 287–312.

——— (1992). “Finite localizations”, *Bol. Soc. Mat. Mex., II. Ser.* **37** (1-2), pp. 383–389.

Haynes R. Miller, Douglas C. Ravenel, and Stephen W. Wilson (1977). “Periodic phenomena in the Adams–Novikov spectral sequence”, *Ann. Math. (2)* **106**, pp. 469–516.

John W. Milnor (1958). “The Steenrod algebra and its dual”, *Ann. Math. (2)* **67**, pp. 150–171.

John W. Milnor (1960). “On the cobordism ring Ω^* and a complex analogue. I”, *Am. J. Math.* **82**, pp. 505–521.

Stephen A. Mitchell (1990). “The Morava K-theory of algebraic K-theory spectra”, *K-Theory* **3** (6), pp. 607–626.

Jack Morava (1985). “Noetherian localisations of categories of cobordism comodules”, *Ann. Math. (2)* **121**, pp. 1–39.

Thomas Nikolaus and Peter Scholze (2018). “On topological cyclic homology”, *Acta Math.* **221** (2), pp. 203–409.

Sergei P. Novikov (1960). “Some problems in the topology of manifolds connected with the theory of Thom spaces”, *Dokl. Akad. Nauk. SSSR.* **132**, pp. 1031–1034.

——— (1969). “The methods of algebraic topology from the viewpoint of cobordism theory”, *Math. USSR, Izv.* **1**, pp. 827–913.

Daniel Quillen (1969). “On the formal group laws of unoriented and complex cobordism theory”, *Bull. Am. Math. Soc.* **75**, pp. 1293–1298.

——— (1972). “On the cohomology and K-theory of the general linear groups over a finite field”, *Ann. Math. (2)* **96**, pp. 552–586.

——— (1973). *Higher algebraic K-theory. I.* Algebr. K-Theory I, Proc. Conf. Battelle Inst. 1972, Lect. Notes Math. 341, 85-147 (1973).

Douglas C. Ravenel (1984). “Localization with respect to certain periodic homology theories”, *Am. J. Math.* **106**, pp. 351–414.

——— (1987). *The geometric realization of the chromatic resolution.* Algebraic topology and algebraic K-theory, Proc. Conf., Princeton, NJ (USA), Ann. Math. Stud. 113, 168-179 (1987).

——— (1992a). *Nilpotence and periodicity in stable homotopy theory.* Vol. 128. Ann. Math. Stud. Princeton, NJ: Princeton University Press.

——— (1992b). “Progress report on the telescope conjecture”, in: *Adams memorial symposium on algebraic topology. Volume 2. Proceedings of a symposium, held in Manchester, UK, in July 1990.* Cambridge: Cambridge University Press, pp. 1–21.

——— (2004). *Complex cobordism and stable homotopy groups of spheres.* 2nd ed. Providence, RI: AMS Chelsea Publishing.

John Rognes (2014). *Chromatic redshift.* Preprint, arXiv:1403.4838 [math.AT] (2014).

Larry Smith (1977). “On realizing complex bordism modules. IV: Applications to the stable homotopy groups of spheres”, *Am. J. Math.* **99**, pp. 418–436.

Neil P. Strickland (1999). “Products on MU-modules”, *Trans. Am. Math. Soc.* **351** (7), pp. 2569–2606.

René Thom (1954). “Quelques propriétés globales des variétés différentiables”, *Comment. Math. Helv.* **28**, pp. 17–86.

Hirosi Toda (1971). “On spectra realizing exterior parts of the Steenrod algebra”, *Topology* **10**, pp. 53–65.

Friedhelm Waldhausen (1984). *Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy.* Algebraic topology, Proc. Conf., Aarhus 1982, Lect. Notes Math. 1051, 173-195 (1984).

——— (1985). *Algebraic K-theory of spaces.* Algebraic and geometric topology, Proc. Conf., New Brunswick/USA 1983, Lect. Notes Math. 1126, 318-419 (1985).

Friedhelm Waldhausen, Bjørn Jahren, and John Rognes (2013). *Spaces of PL manifolds and categories of simple maps.* Vol. 186. Ann. Math. Stud. Princeton, NJ: Princeton University Press.

Urs Würgler (1991). *Morava K-theories: A survey.* Algebraic topology, Proc. Conf., Poznań/Pol. 1989, Lect. Notes Math. 1474, 111-138 (1991).

Allen Yuan (2021). *Examples of chromatic redshift in algebraic K-theory.* Preprint, arXiv:2111.10837 [math.KT] (2021).

Christian Ausoni

LAGA – CNRS (UMR 7539)
Institut Galilée
Université Sorbonne Paris Nord
99 avenue Jean-Baptiste Clément
93430 Villetaneuse
E-mail : `ausoni@math.univ-paris13.fr`