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MINIMAL SURFACES
VIA EQUIVARIANT EIGENVALUE OPTIMIZATION
[after Karpukhin, Kusner, McGrath, and Stern]

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Introduction

Spectral geometry concerns the interplay between the geometry of a manifold and suitable eigenvalue problems defined on it. One classical renowned example is the Faber–Krahn inequality (Faber, 1923; Krahn, 1925), which states that the first Dirichlet eigenvalue of the Laplace operator on Euclidean domains with fixed area is minimized by a round disk. In other words, of all drums with the same area, the round one has the lowest pitch.

Here, we are interested in other classical shape optimization problems that have surprising connections with the theory of minimal surfaces. In particular, we examine the groundbreaking work of Karpukhin, Kusner, McGrath, and Stern (2024), who exploit this connection to construct many new examples of minimal surfaces in the sphere and free boundary minimal surfaces in the ball. Minimal surfaces are critical points of the area functional, one of the oldest and most classical objects in differential geometry, and they play a central role in current research. Beyond their relevance in physical applications, they are beautiful and rich mathematical objects, exhibiting both a great diversity of examples and strong rigidity properties.

We will now briefly introduce the objects and results we are going to discuss, before delving into the details in the next sections.

Laplace eigenvalue optimization and minimal surfaces in spheres

The first problem discussed in this exposé regards the optimization of the eigenvalues of the Laplace operator on a Riemannian surface with fixed area. Hersch (1970) proved that among all Riemannian two-spheres with fixed area, the first eigenvalue of the Laplacian is maximized by the round sphere. It is then natural to pose the same question when the Riemannian surface has more complicated topology. Surprisingly, Nadirashvili (1996) discovered that, if a Riemannian surface maximizes the first eigenvalue of the Laplacian among all Riemannian surfaces with the same topology and the same area, then it can be isometrically immersed as a *minimal surface* in the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ for some $n \geq 3$. We recall that basic examples of minimal surfaces in $\mathbb{S}^3$ are the equatorial two-sphere (whose metric indeed coincides with the round metric in Hersch's result) and the Clifford torus (see Example 1.1).

This connection between Laplacian eigenvalue optimization and minimal surfaces in round spheres spurred advances in the study of shape optimization, as well as in the study of minimal surfaces in spheres. The work of Karpukhin, Kusner, McGrath, and Stern (KKMS, from now on) is a striking example of this. Indeed, the authors use *equivariant* Laplace eigenvalue optimization to construct many new examples of embedded minimal surfaces in $\mathbb{S}^3$. This is the first time that eigenvalue optimization has been successfully employed to systematically construct new families of embedded minimal surfaces in a three-dimensional manifold.

THEOREM A (KKMS, Theorem 1.1). — *For every $\gamma \in \mathbb{N}$, there exist at least $\lfloor \frac{\gamma-1}{4} \rfloor + 1$ nonisometric, embedded, orientable, minimal surfaces in $\mathbb{S}^3$ with genus γ and area less than 8π.*

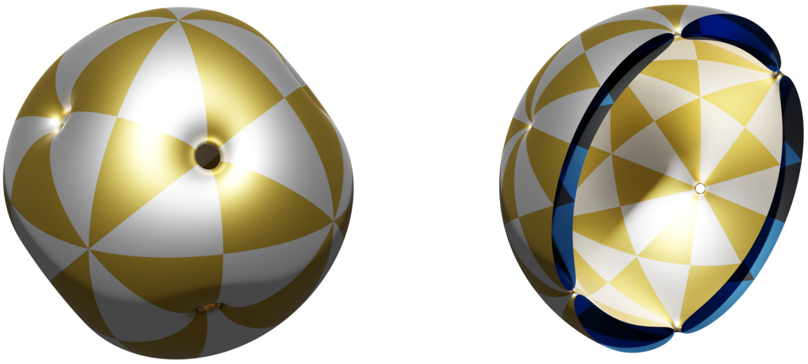

FIGURE 1. Numerical simulations from Schulz (2025) of two minimal surfaces obtained in Theorem A, with genus 7 and 11 respectively. Both surfaces are represented after stereographic projection of $\mathbb{S}^3$ into $\mathbb{R}^3$. The right image represents only half of the surface.

Note that, prior to this result, Ketover (2022) proved that the number of embedded minimal surfaces in $\mathbb{S}^3$ with genus γ goes to infinity as $\gamma \to \infty$ (with rate at least $\gamma / \log\log\gamma$). All the surfaces constructed in (Ketover, 2022) have area less than $4\pi^2$ and they are constructed via Simon–Smith min-max theory. It is an interesting open question to understand the asymptotic number of nonisometric minimal surfaces as the genus γ goes to infinity. However, though expected, so far we do not even know if there are finitely many nonisometric minimal surfaces for every given genus. Only compactness, in the smooth topology, of the space of embedded minimal surfaces in $\mathbb{S}^3$ with fixed genus is known by a result of Choi and Schoen (1985).

Theorem A is proven by finding a Riemannian surface M maximizing the first Laplace eigenvalue among all Riemannian metrics on M which are symmetric with respect to a reflection τ such that $M/\langle\tau\rangle$, the quotient of M by the two-element group generated by τ, has genus zero. By work of Nadirashvili (1996), the surface M can be isometrically immersed as a minimal surface in a sphere $\mathbb{S}^n$. However, in general, the immersion is not an embedding, and the target sphere cannot be expected to have low dimension. One of the core ideas of KKMS, which we will see in detail in Section 4, is that imposing the symmetry with respect to the reflection τ ensures that the immersion is actually an embedding in $\mathbb{S}^3$ with area control, as stated in the theorem.

Besides the reflection τ, it is possible to further assume symmetry with respect to any larger discrete group $\Gamma = \langle \tau \rangle \times G$ acting on M and generated by reflections. By choosing different groups Γ and different actions on a surface with fixed genus γ, it is possible to prove the existence of many nonisometric examples as stated in the theorem. Indeed, the number $\lfloor \frac{\gamma-1}{4} \rfloor + 1$ in Theorem A is obtained by performing equivariant eigenvalue optimization with respect to different actions of the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. The number is not optimized and can be improved by taking different groups Γ.

Theorem A fits into a long history of existence results for minimal surfaces in $\mathbb{S}^3$, beginning with the result of Lawson (1970), who proved the existence of embedded minimal surfaces in $\mathbb{S}^3$ with arbitrary prescribed genus. Active research on existence questions has continued ever since (see e.g. Karcher, Pinkall, and Sterling, 1988; Kapouleas and Yang, 2010; Ketover, 2022; Kapouleas and McGrath, 2023). Observe that proving existence of minimal surfaces is in general hard, as these objects display instability and rigidity properties, as witnessed for example by uniqueness results for low topologies (see e.g. Brendle, 2013a; Marques and Neves, 2014; and related Bourbaki seminars by Carron, 2016; Rivière, 2015) or with suitable symmetries (see Choe and Soret, 2009; Kapouleas and Wiygul, 2020, 2022 for properties of the Lawson surfaces, including uniqueness results). Despite these advances, many fundamental questions concerning the number and uniqueness of minimal surfaces remain open (see e.g. KKMS, Section 1.6, for open questions related to Theorem A).

Remark 0.1. — All the closed (i.e., compact without boundary) surfaces obtained in KKMS are doublings of an equatorial sphere $\mathbb{S}^2 \subset \mathbb{S}^3$. Heuristically, this means that they resemble two copies of an equatorial sphere attached by some necks (cf. Figure 1). We refer to (Kapouleas and McGrath, 2023, Definition 1.1) for a precise definition.

Since their introduction by Kapouleas and Yang (2010), doublings of the sphere and more generally of any minimal surface have been extensively studied. We mention for example recent papers via gluing methods by Kapouleas and McGrath (2024) and Chu and Stern (2025). So far, all the known examples of embedded minimal surfaces with area less than 8π are the equatorial sphere itself or a doubling of it. It is an interesting open question to understand if this is always the case or whether there are examples of embedded minimal surfaces in $\mathbb{S}^3$ with area less than 8π which are not a doubling of the equatorial sphere (see e.g. Kapouleas and McGrath, 2019).

Remark 0.2. — The closed minimal surfaces constructed in KKMS have area less than 8π, and the area converges to 8π as the genus goes to infinity. This asymptotic behavior has been studied by Karpukhin, McGrath, and Stern (2025), who prove that the convergence is exponential in the genus γ for most of the surfaces and, in general, at least as fast as the area growth for the Lawson surfaces $\xi_{\gamma,1}$, which are conjectured to have the least area among minimal surfaces with genus γ (cf. Kusner, 1989, p. 343).

Steklov eigenvalue optimization and free boundary minimal surfaces in balls

Minimal surfaces in spheres have a natural counterpart in Euclidean unit balls $\mathbb{B}^n$ in $\mathbb{R}^n$, namely free boundary minimal surfaces. As the name suggests, these objects are surfaces with boundary and they are critical points of the area functional with respect to variations that constrain the boundary of the surface to the boundary $\partial\mathbb{B}^n$ of the ball. Two examples are the equatorial disk and the critical catenoid in $\mathbb{B}^3$ (see Example 2.1).

One might hope that there is a connection with an eigenvalue problem on surfaces with boundary, in analogy with the closed case. It turns out that this is indeed the case and the right eigenvalue problem is the Steklov problem (or Dirichlet-to-Neumann problem), thanks to work of Fraser and Schoen (2011, 2013, 2016). After Fraser–Schoen's results, the study of free boundary minimal surfaces in $\mathbb{B}^3$ has seen a lot of activity over the last 10 years, with many existence results (see e.g. Folha, Pacard, and Zolotareva, 2017; Kapouleas and Li, 2021; Fernández, Hauswirth, and Mira, 2023; Franz and Schulz, 2023) and works on the properties of these objects (see e.g. Kusner and McGrath, 2024). However, the analogue of Lawson's problem, posed by Fraser and Li (2014, Question 1; see also the survey by Li, 2020, Open Question 1), about the existence of free boundary minimal surfaces in $\mathbb{B}^3$ of every topological type, was still open until Karpukhin, Kusner, McGrath, and Stern (2024) affirmatively answered this question.

Theorem B (KKMS, Theorem 1.2). — *For every $\gamma, \beta \in \mathbb{N}$, $\beta \geq 1$, there exist at least $\lfloor \frac{\gamma-2}{4} \rfloor + 1$ nonisometric, embedded, orientable, free boundary minimal surfaces in $\mathbb{B}^3$ with genus γ and β boundary components. Moreover, these embeddings have area less than 2π and are doublings of an equatorial disk $\mathbb{B}^2 \subset \mathbb{B}^3$.*

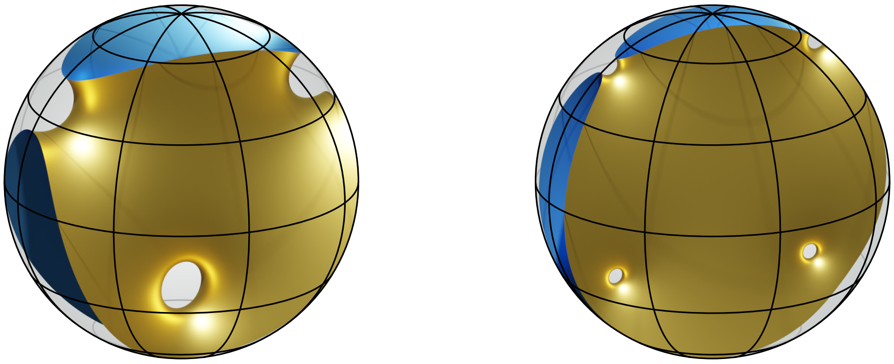

Figure 2. Simulations from Schulz (2025) of two free boundary minimal surfaces obtained in Theorem B. The left one has genus one and two boundary components, while the right one has genus two and two boundary components.

The surfaces in the previous theorem are obtained as embeddings by first eigenfunctions (i.e., the coordinate functions are first eigenfunctions) of the Steklov operator on a compact orientable surface M with genus γ and β boundary components, with respect to a metric maximizing the first Steklov eigenvalue with fixed area, among metrics symmetric with respect to a suitable action of the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. As in the closed case, many different groups and many different actions can be considered, even if, for technical reasons, the possibilities are not as general as in the closed case (see KKMS, Theorem 9.15 and Remark 6.2 below). Similarly to the closed case, many questions remain open, concerning for example the existence, uniqueness and number of examples for each topology.

Existence of a maximizing metric

As mentioned above, the proofs of Theorems A and B are based on proving the existence of a metric maximizing the first Laplace eigenvalue (in the closed case) or Steklov eigenvalue (in the case with boundary) among metrics with suitable symmetries and fixed area. Then, one shows that the associated first eigenfunctions give a (free boundary) minimal embedding in $\mathbb{S}^3$ (resp. $\mathbb{B}^3$). One of the novelties of the paper KKMS is a new technique for proving the existence of such a maximizer. This problem is extremely delicate, whether one assumes symmetry or not, and has seen many partial results and attempts (see KKMS, Section 1.3 for a more detailed discussion). We will explain the new method introduced in KKMS in more detail in Sections 6. Besides proving the existence of a maximizing metric under suitable symmetry assumptions, this new technique has already led to important advances in the existence of maximizing metrics for the eigenvalues of the Laplacian.

THEOREM C (Petrides, 2024, Karpukhin, Petrides, and Stern, 2025)
Every closed surface M admits a metric (smooth away from finitely many conical singularities) maximizing the first Laplace eigenvalue among metrics with fixed area.

The result was first proved by Petrides (2024) for M orientable and then extended to the nonorientable case by Karpukhin, Petrides, and Stern (2025), who also provide a different proof for the orientable case. We refer to Section 6.3 for more details.

Note that the analogous question for the first Steklov eigenvalue on general surfaces with boundary is still open. However, we know that a maximizer exists for some topologies of M with boundary (e.g. for all surfaces with genus zero and one, thanks to KKMS, Corollary 1.11), and it is expected that the methods developed by Karpukhin, Kusner, McGrath, and Stern (KKMS), Petrides (2024) and Karpukhin, Petrides, and Stern (2025) will help in this direction.

Plan of the article

The goal of this exposé is to give an introduction to the study of (free boundary) minimal surfaces via eigenvalue optimization and present the main ideas in the proofs of KKMS, after developing the most significant ingredients. We introduce both the closed case and the case with boundary, but we will mainly focus on the *closed orientable* case for what concerns the proofs, in order to simplify the exposition, while still presenting the key ideas. Below is an outline of the content of the article.

Preliminaries

1. The closed setting
– Minimal surfaces in the sphere.
– Laplace eigenvalue problem on closed surfaces.

2. The setting with boundary
– Free boundary minimal surfaces in the ball.
– Steklov eigenvalue problem on surfaces with boundary.

3. Li–Yau conformal volume
A milestone in spectral geometry, the paper by Li and Yau (1982) serves to prove some results in Section 4 and as a motivation for Section 5.

Results of KKMS

4. Eigenvalue optimization on basic reflection surfaces
– Problem of maximizing the first eigenvalue on surfaces with suitable symmetries.
– Proof in KKMS that if a maximizing metric exists, then Theorem A follows.

5. Existence of a conformal eigenvalue maximizer
Proof of the existence of a maximizing metric in each conformal class via min-max characterization of conformal eigenvalues by Karpukhin and Stern (2024c).

6. Existence of an eigenvalue maximizer
Proof of the existence of a maximizing metric of the first eigenvalue on surfaces with symmetries considered in Section 4, with new technique introduced in KKMS.

1. The closed setting

1.1. Minimal surfaces in spheres

Let (M, g) be a closed Riemannian surface and let $u\colon (M, g) \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a smooth isometric(1) immersion in the sphere $\mathbb{S}^n$. We say that the immersion is *minimal* if it is a critical point of the area functional. Namely, for every one-parameter family of immersions $u_t\colon M \to \mathbb{S}^n$ for $t \in (-\varepsilon, \varepsilon)$ such that $u_0 = u$, we have

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(u_t(M)) = 0.$$

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(1) By isometric immersion, we mean that $g = u^* g_{\mathbb{S}^n} = u^* g_{\mathbb{R}^{n+1}}$, in other words, the metric g is induced by the Euclidean metric on $\mathbb{R}^{n+1}$ via the map u. In this case, we denote by $\operatorname{area}(u(M)) = \operatorname{area}(M, u^* g_{\mathbb{S}^n}) = \operatorname{area}(M, g)$ the area of M with respect to the metric g.

By computing the first variation of area, it turns out that an immersion is minimal if and only if its mean curvature is equal to zero. We refer to (Colding and Minicozzi, 2011; White, 2016) for classical references in the study of minimal surfaces.

Example 1.1. — The equatorial sphere $\mathbb{S}^2 \subset \mathbb{S}^3$ is a minimal surface. Another classical example of minimal surface in $\mathbb{S}^3$ is the Clifford torus, given by

$$\frac{1}{\sqrt{2}}\mathbb{S}^1 \times \frac{1}{\sqrt{2}}\mathbb{S}^1 = \left\{ \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) \in \mathbb{R}^4 \ : \ \theta, \varphi \in [0, 2\pi) \right\} \subset \mathbb{S}^3.$$

By results of Almgren (1966) and Brendle (2013a), these are the only *embedded* minimal surfaces (up to ambient isometry) in $\mathbb{S}^3$ of genus zero and one respectively. However, note that there are infinitely many nonisometric immersed minimal tori in $\mathbb{S}^3$ by a result of Lawson (1969). We refer also to the survey (Brendle, 2013b).

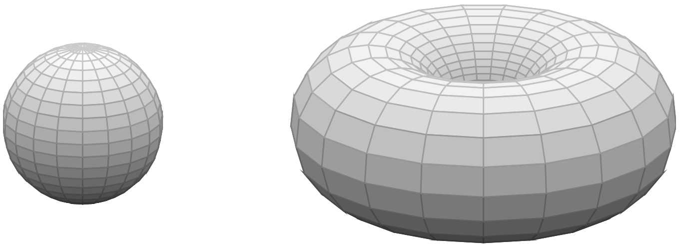

FIGURE 3. Image of the equatorial sphere $\mathbb{S}^2 \subset \mathbb{S}^3$ and the Clifford torus $\frac{1}{\sqrt{2}}\mathbb{S}^1 \times \frac{1}{\sqrt{2}}\mathbb{S}^1 \subset \mathbb{S}^3$ after stereographic projection of $\mathbb{S}^3$ into $\mathbb{R}^3$.

Remark 1.2. — In the study of minimal surfaces, including some of the results discussed in this paper, minimal immersions with *branch points* may appear. A map $u\colon M \to \mathbb{R}^{n+1}$ is said to be a branched immersion if it is an immersion away from finitely many points, called branch points, where the map locally behaves like the complex map $\mathbb{C} \ni z \mapsto z^k$ for some $k > 1$. We refer to (Gulliver, Osserman, and Royden, 1973, Definition 1.2) for a precise definition. See also (Colding and Minicozzi, 2011, Chapter 4, §4) for a discussion of branch points in the context of minimal surfaces. If $u\colon (M, g) \to \mathbb{R}^{n+1}$ is an isometry, then g is smooth away(2) from finitely many *conical singularities* at the branch points, namely it degenerates to zero there.

In this exposé, we do not insist on branch points and conical singularities (also because they do not appear in the main theorems A and B). The reader not interested in the technical details can ignore these elements.

It was first observed by Takahashi (1966, Theorem 3) that an isometric immersion $u\colon (M, g) \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is minimal if and only if the coordinate functions $u_j = x_j \circ u$

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(2) Note that, even if the expression "smooth away from" is used, the metric g is actually smooth everywhere, but degenerates to zero at the conical singularities.

for $j = 1, \dots, n+1$ are eigenfunctions of the Laplacian Δ_g with eigenvalue 2. Indeed, for every isometric immersion $u\colon (M, g) \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$, we have

$$-\Delta_g u_j = -\Delta_g(\langle x, e_j\rangle \circ u) = -\langle H^{M \subset \mathbb{R}^{n+1}}, e_j\rangle = -\langle H^{M \subset \mathbb{S}^n}, e_j\rangle + 2u_j, \tag{1}$$

where $\{e_j\}_{j=1}^{n+1}$ is the canonical basis of $\mathbb{R}^{n+1}$, and where $H^{M \subset \mathbb{R}^{n+1}}$ and $H^{M \subset \mathbb{S}^n}$ are the mean curvatures of M as a submanifold of $\mathbb{R}^{n+1}$ and $\mathbb{S}^n$, respectively. Therefore, the immersion u is minimal, namely $H^{M \subset \mathbb{S}^n} = 0$, if and only if u_j is a Laplace eigenfunction with eigenvalue 2 for every $i = 1, \dots, n+1$.

Remark 1.3. — In 1982, Yau conjectured that 2 is the *first* eigenvalue of the Laplacian for every *embedded* minimal surface in $\mathbb{S}^3$. This question is still open.

Remark 1.4. — If we do not require the immersion $u\colon (M, g) \to \mathbb{S}^n$ to be an isometry, we have that u is immersed by eigenfunctions (with respect to the same eigenvalue) of the Laplacian Δ_g if and only if it is harmonic (and not necessarily minimal). We recall that a map $u\colon (M, g) \to \mathbb{S}^n$ is harmonic if it is a critical point of the *Dirichlet energy*

$$E(u) \coloneqq \frac{1}{2}\int_M |du|_g^2 \, dv_g, \tag{2}$$

where dv_g is the volume form on (M, g). When M has dimension two, the Dirichlet energy is a *conformal invariant*, namely it does not change if computed with respect to any other metric in the conformal class(3) $[g]$ of g.

Let us now show that u is harmonic if and only if it is immersed by λ-eigenfunctions of the Laplacian (possibly(4) with respect to a metric conformal to g). If $u\colon (M, g) \to \mathbb{S}^n$ is immersed by eigenfunctions with eigenvalue λ, then $-\Delta_g u_i = \lambda u_i$ for all $i = 1, \dots, n+1$. Multiplying the equation by u_i and summing over $i = 1, \dots, n+1$, we then get

$$\lambda = \lambda \sum_{i=1}^{n+1} u_i^2 = -\sum_{i=1}^{n+1} u_i \Delta_g u_i = -\sum_{i=1}^{n+1}\left[\frac{1}{2}\Delta_g(u_i^2) - |du_i|_g^2\right] = |du|_g^2.$$

This implies that $-\Delta_g u = |du|_g^2 u$, and this equation precisely says that u is harmonic (see Struwe, 2008, Section 6).

Conversely, if $u\colon (M, g) \to \mathbb{S}^n$ is harmonic, then $-\Delta_g u = |du|_g^2 u$. Therefore, by taking $g_u = \frac{1}{2}|du|_g^2 g$, we have $-\Delta_{g_u} u = 2u$, in other words, the components of u are eigenfunctions with eigenvalue 2 with respect to the metric g_u.

Note that in general the pull-back metric $u^* g_{\mathbb{S}^n}$ may not belong to the conformal class $[g]$ of g, but when it does the immersion u is minimal. Indeed, we have that the map u is minimal if and only if it is conformal and harmonic.

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(3) Given a Riemannian surface (M, g), the *conformal class* of g is defined as the set of all metrics that differ from g by a positive multiplicative function, i.e., $[g] \coloneqq \{fg \,:\, f \in C^\infty(M),\ f > 0\}$.

(4) Recall that being harmonic is a conformal invariant, while being immersed by eigenfunctions is not.

1.2. Laplace eigenvalue problem on closed surfaces

Let M be a smooth connected closed surface and let g be a smooth metric on M. Then, the Laplace operator Δ_g on M admits a discrete spectrum consisting of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$. Namely, there exists an $L^2(M)$-orthonormal basis $\{u_k\}_{k\in\mathbb{N}}$ of smooth functions such that

$$-\Delta_g u_k = \lambda_k u_k.$$

We recall that the eigenvalues λ_k admit a variational characterization. In particular, for the first eigenvalue λ_1 we have(5)

$$\lambda_1 = \inf_{\substack{u\in C^\infty(M)\setminus\{0\}\\ \int_M u=0}} \frac{\int_M |du|^2\, dv_g}{\int_M u^2\, dv_g} = \inf_{\substack{u\in \mathrm{Lip}(M)\setminus\{0\}\\ \int_M u=0}} \frac{\int_M |du|^2\, dv_g}{\int_M u^2\, dv_g}. \tag{3}$$

Note that the condition $\int_M u\, dv_g = 0$ is necessary to exclude the constant functions, which are eigenfunctions with respect to eigenvalue $\lambda_0 = 0$.

Remark 1.5. — Recall that, if $u \in \mathrm{Lip}(M)$, that is to say $u\colon M \to \mathbb{R}$ is Lipschitz continuous, then u is differentiable almost everywhere by Rademacher's theorem. Therefore, the integral $\int_M |du|^2\, dv_g$ is well-defined.

We refer to (Chavel, 1984, Chapter 1, §5) for further background concerning the spectral theory of the Laplacian on manifolds, including the following theorem. We recall that the *nodal set* of an eigenfunction u is defined as its zero set $u^{-1}(0) \subset M$, denote by $\mathcal{N}_u$, while its *nodal domains* are the connected components of $M \setminus \mathcal{N}_u$.

THEOREM 1.6 (Courant's nodal domain theorem). — *Let $u = u_k$ be the kth eigenfunction of the Laplacian Δ_g on M. Then, its nodal set consists of a finite number of C^2-immersed circles that meet at a finite number of points, and the number of its nodal domains is at most $k+1$.*

Remark 1.7. — A direct consequence of Courant's nodal domain theorem is that a first eigenfunction of the Laplacian has exactly two nodal domains. Indeed, it cannot have one nodal domain (i.e., be nowhere zero) because it is orthogonal to the constant functions, which are eigenfunctions with respect to eigenvalue $\lambda_0 = 0$.

Remark 1.8. — A variant of Courant's nodal domain theorem works also for Laplace eigenvalues with Dirichlet or Neumann boundary conditions on a surface with boundary, and for the Steklov eigenvalues considered in Section 2. A corollary of this (cf. Chavel, 1984, Corollary 2 in Chapter 1, §5) is that the first Dirichlet eigenvalue on a surface with boundary is simple.(6)

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(5) From now on, we omit the subscript g, i.e., write $|du|^2 = |du|_g^2$, when it is clear with respect to which metric the norm is computed.

(6) Note that the zero eigenvalue of the Laplacian on a connected closed surface is also simple (the eigenfunctions are the constant functions). For the Dirichlet eigenvalue on a surface with boundary the first eigenvalue is already positive and thus we start enumerating the eigenvalues from one.

Since we consider the eigenvalue problem as the metric on M changes, we explicitly write the dependence of the eigenvalues on the surface and the metric as $\lambda_k(M,g)$. Moreover, let us introduce a renormalized version of the eigenvalues as follows

$$\overline{\lambda}_k(M,g) = \lambda_k(M,g)\,\mathrm{area}(M,g).$$

Observe that $\overline{\lambda}_k(M,g)$ is scale invariant and is uniformly bounded from above on the space of all metrics, by (Yang and Yau, 1980; Korevaar, 1993). Therefore, the maximum of $\overline{\lambda}_k(M,g)$ over all smooth metrics on g, denote by

$$\Lambda_k(M) \coloneqq \sup_{\substack{g \text{ smooth} \\ \text{metric}}} \overline{\lambda}_k(M,g),$$

is finite, and we can ask whether there exists a metric g_0 maximizing the kth renormalized eigenvalue over all metrics, that is to say $\overline{\lambda}_k(M,g_0) = \Lambda_k(M)$. Note that this is equivalent to maximizing $\lambda_k(M,g)$ among metrics of fixed area, as stated in the introduction. Nadirashvili realized that if we find such a maximizing metric g_0 then, surprisingly, we get an associated minimal immersion into a sphere.

Theorem 1.9 (Nadirashvili, 1996, Theorems 1 and 5). — *If g_0 is a metric on M maximizing the kth renormalized eigenvalue over all metrics, then there exist independent kth eigenfunctions $u_1,\dots,u_{n+1}$ for some $n \geq 2$ such that $u \coloneqq (u_1,\dots,u_{n+1})\colon M \to \mathbb{R}^{n+1}$ is a (branched) minimal immersion in $\mathbb{S}^n$. Moreover, the map u can be made isometric by rescaling g_0 using a positive constant.*

Remark 1.10. — In the previous theorem, we allow the metric g_0 to have finitely many conical singularities, which correspond to the branch points of u. This is because in general we expect the maximizing metric to have this regularity (see Kokarev, 2014, Regularity theorem). In fact, conical singularities arise for example for the maximizer in case M has genus 2 (cf. Nayatani and Shoda, 2019, see also Example 1.16 below). Note that the Laplace eigenvalues for metrics with conical singularities still make sense (see e.g. Kokarev, 2014, Example 1.1).

Remark 1.11. — The theorem above holds more generally for metrics g_0 that are critical points of the renormalized eigenvalue $\overline{\lambda}_k(M,g)$, giving a correspondence between critical points of renormalized eigenvalues and (branched) minimal immersions into spheres. We stated it for maximizers only, because $\overline{\lambda}_k(M,g)$ is Lipschitz continuous with respect to the metric g, in the smooth topology, but not differentiable in general. Therefore, one needs a definition of critical points for the Lipschitz function $\overline{\lambda}_k(M,g)$. This is given in (Nadirashvili, 1996, Section 3). The result also admits generalizations to manifolds M of higher dimension (see El Soufi and Ilias, 2000; Karpukhin and Métras, 2022).

Remark 1.12. — For every $n \geq 2$ for which Theorem 1.9 holds, we have that $n+1$ is no larger than the multiplicity of the eigenvalue $\lambda_k(M,g_0)$, but can be strictly smaller. However, we know that $n \geq 3$ unless M is a two-sphere.

Idea of the proof of Theorem 1.9. — We refer to (Fraser and Schoen, 2013, Proposition 2.1) for a complete proof of the theorem, giving here only an idea of it. The result follows from the first variation formula for $\lambda_k(M, g)$ at a critical point. The delicate point is that eigenvalues depend only Lipschitz continuously on the metric, therefore we cannot just take classical derivatives.

Let g_0 be a metric maximizing $\overline{\lambda}_k(M, g)$. By rescaling the metric, without loss of generality we can assume that $\lambda_k(M, g_0) = 2$. It turns out that, for every symmetric $(0, 2)$-tensor ω on M such that $\int_M \langle \omega, g_0 \rangle \, dv_{g_0} = 0$, there exists a kth eigenfunction u with $\|u\|_{L^2(M,g_0)} = 1$ and

$$Q_\omega(u) \coloneqq -\int_M \langle \omega, du \otimes du - \frac{1}{2}|du|^2 g_0 + u^2 g_0 \rangle \, dv_{g_0} = 0.$$

Note that ω represents the first derivative of a variation of g_0 with fixed area (which is assured by the condition $\int_M \langle \omega, g_0 \rangle \, dv_{g_0} = 0$).

Remark 1.13. — When the eigenvalue $\lambda_k(M, g_0)$ is simple, it is known that $g \mapsto \lambda_k(M, g)$ is differentiable at g_0, and we can compute the derivative of λ_k in the direction ω, obtaining $Q_\omega(u)$ (which is then equal to zero by criticality of g_0). The previous one is the general statement when $\lambda_k(M, g_0)$ is not simple (and not differentiable).

Thanks to an application of the Hahn–Banach theorem, we then obtain that g_0 belongs to the convex hull (in the space of symmetric $(0, 2)$-tensor fields on (M, g_0)) of the set

$$\left\{ du \otimes du - \frac{1}{2}|du|^2 g_0 + u^2 g_0 \ : \ u \text{ is a } k\text{th eigenfunction} \right\}.$$

In other words, there exist linearly independent kth eigenfunctions $u_1, \dots, u_{n+1}$ such that

$$g_0 = \sum_{j=1}^{n+1} \left(du_j \otimes du_j - \frac{1}{2}|du_j|^2 g_0 + u_j^2 g_0 \right). \tag{4}$$

This easily implies that $\sum_{j=1}^{n+1} u_j^2 = 1$ (by tracing the equation) and, applying the Laplace operator to this equation, we get

$$\sum_{j=1}^{n+1} |du_j|^2 = -\sum_{j=1}^{n+1} u_j \Delta_{g_0} u_j = \lambda_k \sum_{j=1}^{n+1} {u_j}^2 = 2, \tag{5}$$

where we used that the u_j's are kth eigenfunctions and $\lambda_k = \lambda_k(M, g_0) = 2$. Together with (4), this implies that $\sum_{j=1}^{n+1} du_j \otimes du_j = g_0$.

In other words, the map $u \coloneqq (u_1, \dots, u_{n+1}) \colon M \to \mathbb{R}^{n+1}$ takes values in $\mathbb{S}^n$ and it is an isometric immersion. Moreover, each u_j is an eigenfunction of Δ_{g_0} with eigenvalue 2. This implies that u is minimal by (Takahashi, 1966, Theorem 3), as discussed above. □

Recall that smooth closed surfaces are characterized by their topology, more precisely by specifying their genus and whether they are orientable or not. The simplest case is when M is topologically a two-dimensional sphere. This case was studied prior to Nadirashvili's result in seminal work of Hersch. Let us briefly present the result here.

PROPOSITION 1.14 (Hersch, 1970). — *The round metric on* $\mathbb{S}^2$ *uniquely (up to rescaling) maximizes the first renormalized Laplace eigenvalue on* $\mathbb{S}^2$ *among all smooth metrics. In particular, we get* $\Lambda_1(\mathbb{S}^2) = \bar{\lambda}_1(\mathbb{S}^2, g_{\mathbb{S}^2}) = 8\pi$.

Proof. — Let g be any metric on $\mathbb{S}^2$. By the uniformization theorem, there exists a conformal diffeomorphism $f\colon (\mathbb{S}^2, g) \to (\mathbb{S}^2, g_{\mathbb{S}^2})$. Up to composing with a conformal transformation of the round sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$, we can assume that $\int_{\mathbb{S}^2} f\, dv_g = 0$: this follows from the explicit characterization of the conformal transformations of $\mathbb{S}^2$ (see Section 3), which gives the freedom[7] to choose the value of $\int_{\mathbb{S}^2} f\, dv_g$. Therefore, by the variational characterization (3) of λ_1 applied to the components f_1, f_2, f_3 of f, we have that

$$(6) \quad \bar{\lambda}_1(\mathbb{S}^2, g) = \sum_{j=1}^{3} \lambda_1(\mathbb{S}^2, g) \int_{\mathbb{S}^2} f_j^2\, dv_g \le \sum_{j=1}^{3} \int_{\mathbb{S}^2} |df_j|_g^2\, dv_g = \sum_{j=1}^{3} \int_{\mathbb{S}^2} |dx_j|_{g_{\mathbb{S}^2}}^2\, dv_{g_{\mathbb{S}^2}} = 8\pi,$$

where we used that the Dirichlet energy is conformally invariant.

Now, let g be a metric for which equality holds. Up to a rescaling, we can assume that $\lambda_1(\mathbb{S}^2, g) = 2$. By equality in (6), the coordinates f_1, f_2, f_3 are first Laplace eigenfunctions on (M, g). Then, by applying the Laplace operator to the equation $\sum_{j=1}^3 f_j^2 = 1$ (similarly to (5)), we obtain that $\sum_{j=1}^3 |df_j|_g^2 = 2$. Since f is a conformal map, this implies that f is actually an isometry, concluding the proof. ☐

Remark 1.15. — Let M be a genus zero surface with boundary and let $\lambda_1^N(M, g)$ be the first Laplace eigenvalue on (M, g) with Neumann boundary conditions. Then, the same proof as above shows that $\lambda_1^N(M, g)\mathrm{area}(M, g) < 8\pi$. Indeed, if M has genus zero, there exists a conformal embedding $f\colon (M, g) \to (\mathbb{S}^2, g_{\mathbb{S}^2})$ and then the proof proceeds as above, using the variational characterization for the first Neumann eigenvalue:

$$(7) \qquad \lambda_1^N(M, g) = \inf_{\substack{u \in C^\infty(M)\setminus\{0\} \\ \int_M u = 0}} \frac{\int_M |du|^2\, dv_g}{\int_M u^2\, dv_g}.$$

Example 1.16. — The maximum of the first renormalized Laplace eigenvalue is known in the following cases. In most of these cases the maximizing metric is unique, meaning that it is unique up to rescaling.

- For the sphere, $\Lambda_1(\mathbb{S}^2) = 8\pi$ and the unique maximizer is the round metric on $\mathbb{S}^2$, thanks to the previous proposition.
- For the projective plane, $\Lambda_1(\mathbb{RP}^2) = 12\pi$ and the unique maximizer is the Veronese surface, minimally embedded in $\mathbb{S}^4$ (see Li and Yau, 1982, Corollary 5).

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[7] The idea of composing with a conformal transformation of a sphere $\mathbb{S}^n$ to prescribe a "center of mass", i.e., an integral quantity on the sphere, comes from (Szegö, 1954) and it is referred to as the *Hersch trick.*

- For the torus, $\Lambda_1(\mathbb{T}^2) = 8\pi^2/\sqrt{3}$ and the unique maximizer is the flat equilater torus (i.e., the torus $\mathbb{R}^2/L$, where L is the lattice generated by $(1,0)$ and $(1/2, \sqrt{3}/2)$), which can be isometrically minimally embedded in $\mathbb{S}^5$ by first eigenfunctions by (Nadirashvili, 1996, Theorem 1). By (El Soufi and Ilias, 2000, Theorem 2.1), the only other critical metric for $\overline{\lambda}_1$ on $\mathbb{T}^2$ is the flat square metric, which is the metric on the Clifford torus, minimally embedded in $\mathbb{S}^3$.
- For the Klein bottle, $\Lambda_1(\mathbb{K}^2) = 12\pi E(\sqrt{8/9})$, where $E(\sqrt{8/9})$ is the complete elliptic integral of the second kind evaluated at $8/9$, by (El Soufi, Giacomini, and Jazar, 2006, Theorem 1.1). The unique maximizing metric is a metric of revolution, with which $\mathbb{K}^2$ can be isometrically minimally embedded in $\mathbb{S}^4$.
- For the orientable genus 2 surface, $\Lambda_1(\mathbb{T}^2\#\mathbb{T}^2) = 16\pi$ and a maximizing metric is the metric on the Bolza surface, which is a branched double cover of the round $\mathbb{S}^2$, by (Nayatani and Shoda, 2019).

2. The setting with boundary

2.1. Free boundary minimal surfaces in balls

Let (M, g) be a compact Riemannian surface with boundary and let $u\colon (M, g) \to \mathbb{B}^n$ be a smooth isometric immersion into the ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $u(\partial M) \subset \partial\mathbb{B}^n$. Analogously to the closed case, we say that the immersion is *free boundary minimal* if it is a critical point of the area functional; more precisely, if for every smooth one-parameter family of immersions $u_t\colon M \to \mathbb{B}^n$, with $t \in (-\varepsilon, \varepsilon)$, such that $u_t(\partial M) \subset \partial\mathbb{B}^n$ and $u_0 = u$, we have

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(u_t(M)) = 0.$$

Note that we allow the boundary of the immersion to move inside the boundary of the ball, which is where the name “free boundary” comes from. Computing the first variation of area, we get that being a free boundary minimal surface is equivalent to having zero mean curvature and intersecting the boundary of $\mathbb{B}^n$ orthogonally.

Example 2.1. — The easiest examples of free boundary minimal surfaces in $\mathbb{B}^3$ are the equatorial disk $\mathbb{B}^2 \subset \mathbb{B}^3$ and the critical catenoid, explicitly parametrized by

$$[-aT, aT] \times [0, 2\pi] \ni (t, \theta) \mapsto \Big(a\cosh(a^{-1}t)\cos\theta, a\cosh(a^{-1}t)\sin\theta, t\Big) \in \mathbb{B}^3,$$

where $T > 0$ is the unique positive solution of $T\tanh T = 1$, and $a = (T\cosh T)^{-1}$.

The equatorial disk is the only free boundary minimal disk in $\mathbb{B}^3$ by a result of Nitsche (1985) but we still do not know if the critical catenoid is the only embedded free boundary minimal annulus in $\mathbb{B}^3$. Note that there are infinitely many examples of *immersed* free boundary minimal annuli in $\mathbb{B}^3$ by a result of Fernández, Hauswirth, and Mira (2023). See also the construction by Kapouleas and McGrath (2022) with a different method.

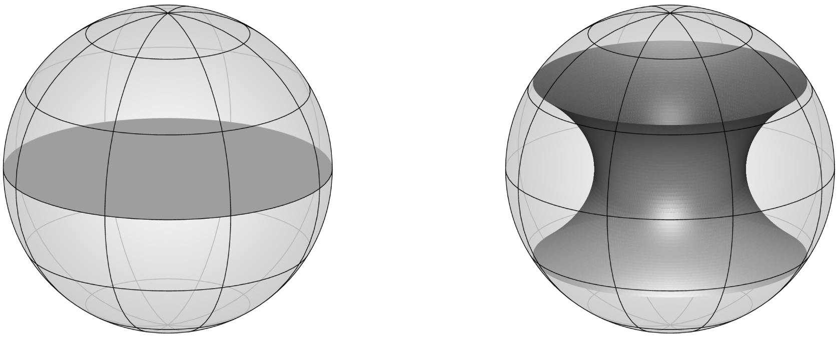

FIGURE 4. The equatorial disk and the critical catenoid in $\mathbb{B}^3$.

In analogy with (Takahashi, 1966) in the closed case, Fraser and Schoen (2011, Lemma 2.2) observed that an isometric immersion $u\colon (M,g) \to \mathbb{B}^n$ is free boundary minimal if and only if the coordinate functions $u_j = x_j \circ u$ for $j = 1, \ldots, n$ satisfy

$$\begin{cases} \Delta_g u_j = 0 & \text{on } M \\ \frac{\partial u_j}{\partial \eta} = u_j & \text{on } \partial M, \end{cases}$$

where η is the unit outward conormal to ∂M. By definition, this equation means that they are Steklov eigenfunctions with eigenvalue 1. We introduce the Steklov operator in more detail in the next section.

2.2. Steklov eigenvalue problem on surfaces with boundary

Let M be a smooth connected compact surface with boundary and let g be a smooth metric on M. The Steklov operator is the Dirichlet-to-Neumann map given by

$$u \in C^\infty(\partial M) \mapsto \frac{\partial \hat{u}}{\partial \eta} \in C^\infty(\partial M),$$

where $\hat{u} \in C^\infty(M)$ is the harmonic extension of u in the interior of M. The Steklov operator admits a discrete spectrum consisting of eigenvalues $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq \cdots \to \infty$. Namely, there exist smooth functions $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(M)$, which form an orthonormal basis of $L^2(\partial M)$, such that

$$\begin{cases} \Delta_g u_k = 0 & \text{on } M \\ \frac{\partial u_k}{\partial \eta} = \sigma_k u_k & \text{on } \partial M. \end{cases}$$

We often write $\sigma_k(M,g)$ to materialize the dependence of the Steklov eigenvalues on the Riemannian manifold (M,g). We consider the renormalized Steklov eigenvalue

$$\overline{\sigma}_k(M,g) = \sigma_k(M,g)\,\mathrm{length}(\partial M, g),$$

which is invariant with respect to rescalings of the metric. Then, we define

$$\Sigma_k(M) := \sup_{\substack{g \text{ smooth} \\ \text{metric}}} \overline{\sigma}_k(M, g),$$

and wonder whether there is metric g_0 maximizing the kth renormalized Steklov eigenvalue $\overline{\sigma}_k(M, g)$ over all metrics, that is to say $\overline{\sigma}_k(M, g_0) = \Sigma_k(M)$.

We refer to (Girouard and Polterovich, 2017; Colbois, Girouard, Gordon, and Sher, 2024) for surveys on the study of the Steklov operator. Here, we are interested in the following connection between maximizing metrics for Steklov eigenvalues and free boundary minimal surfaces, analogous to Theorem 1.9.

THEOREM 2.2 (Fraser and Schoen, 2013, Proposition 2.4). — *If g_0 is a metric on M maximizing the kth renormalized Steklov eigenvalue over all metrics, then there exist independent kth eigenfunctions $u_1, \dots, u_n$ for some $n \geq 2$ such that $u := (u_1, \dots, u_n) \colon M \to \mathbb{R}^n$ is a (branched) free boundary minimal immersion in $\mathbb{B}^n$. Moreover, up to rescaling g_0, the map u is an isometry when restricted to ∂M.*

Example 2.3. — The maximal value $\Sigma_1(M)$ is known explicitly in the following cases:

- On the disk $\mathbb{B}^2$, we have $\Sigma_1(\mathbb{B}^2) = \pi$ and the unique maximizer (up to σ-homothety, cf. Fraser and Schoen, 2016, Definition 2.1) is the flat disk by (Weinstock, 1954).
- On the annulus, $\overline{\sigma}_1$ is uniquely realized (up to σ-homothety) by the metric on the critical catenoid by (Fraser and Schoen, 2016, Theorem 1.3).
- On the Möbius band, $\overline{\sigma}_1$ is uniquely realized (up to σ-homothety) by the metric on the critical Möbius band, which is a free boundary minimal Möbius band embedded in $\mathbb{B}^4$, by (Fraser and Schoen, 2016, Theorem 1.5).

Remark 2.4. — Besides the obvious analogy between the closed case and the case with boundary, there is another very interesting connection between Theorems 1.9 and 2.2. Fix an arbitrary $\gamma \in \mathbb{N}$. Let M be the (differentiable) closed orientable surface with genus γ, and, for every $\beta \in \mathbb{N}$, let M_β be the (differentiable) orientable surface with genus γ and β boundary components. Then, Girouard and Lagacé (2021, Theorem 1.3) and Karpukhin and Stern (2024c, Theorem 1.6) proved that $\lim_{\beta\to\infty} \Sigma_1(M_\beta) = \Lambda_1(M)$. Moreover, by (Karpukhin and Stern, 2024b, Theorem 1.1), if maximizing metrics on M_β and M exist,(8) then the associated free boundary minimal immersions $M_\beta \to \mathbb{B}^{n+1}$ from Theorem 2.2 converge (in some suitable sense) to the minimal immersion $M \to \mathbb{S}^n$ from Theorem 1.9, as $\beta \to \infty$.

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(8) By Theorem C, we know that a maximizing metric for the first Laplace eigenvalue on closed surfaces exists, but we do not currently know that a maximizing metric for the first Steklov eigenvalue exists on all surfaces with boundary (see the discussion after Theorem C).

3. Li–Yau conformal volume

In this section, we recall the notion of conformal volume introduced by Li and Yau (1982), both because it is useful to prove results related to the first Laplace eigenvalue that will be important in what follows, and because it is one of the inspirations of the min-max characterization of conformal eigenvalues by Karpukhin and Stern (2024c), discussed in Section 5.

Unless otherwise stated, let us assume that (M,g) is a closed(9) Riemannian surface. If there exists a conformal immersion $\phi\colon M\to\mathbb{S}^n$, then we define

$$V_c(n,\phi) := \sup_{h\in\mathrm{Conf}(\mathbb{S}^n)} \mathrm{area}(M,(h\circ\phi)^*g_{\mathbb{S}^n}),$$

where $\mathrm{Conf}(\mathbb{S}^n)$ is the space of conformal automorphisms of $\mathbb{S}^n$. Since $h\circ\phi$ is a conformal map, $\mathrm{area}(M,(h\circ\phi)^*g_{\mathbb{S}^n})$ is equal to the Dirichlet energy $E(h\circ\phi)$ defined in (2) (see e.g. White, 2016, Section 4); in other words

$$\mathrm{area}(M,(h\circ\phi)^*g_{\mathbb{S}^n}) = E(h\circ\phi) := \frac{1}{2}\int_M |d(h\circ\phi)|^2\,dv_g. \tag{8}$$

Recall that the set of conformal automorphisms $\mathrm{Conf}(\mathbb{S}^n)$ of $\mathbb{S}^n$ has the structure of a finite-dimensional Lie group, called *Möbius group*. Its quotient(10) modulo the isometry group of $\mathbb{S}^n$ is homeomorphic to the open ball $\mathbb{B}^{n+1}$. In particular, we have that

$$\mathrm{Conf}(\mathbb{S}^n)\big/\mathrm{Isom}(\mathbb{S}^n) \cong \left\{ G_a(x) := \frac{1-|a|^2}{|x+a|^2}(x+a)+a \;:\; a\in\mathbb{B}^{n+1}\right\}. \tag{9}$$

Therefore, we can rewrite $V_c(n,\phi)$ as

$$V_c(n,\phi) = \sup_{a\in\mathbb{B}^{n+1}} \mathrm{area}(M,(G_a\circ\phi)^*g_{\mathbb{S}^n}) = \sup_{a\in\mathbb{B}^{n+1}} E(G_a\circ\phi). \tag{10}$$

Recall that the Dirichlet energy is invariant under conformal reparameterization of the domain, as mentioned in Remark 1.4. Thus, $V_c(n,\phi)$ depends only on the conformal class $[g]$.

Lemma 3.1 (Li and Yau, 1982, Proposition 1). — *If $\phi\colon (M,g)\to\mathbb{S}^n$ is an isometric minimal immersion, then $V_c(n,\phi)=\mathrm{area}(M,g)$.*

Proof. — Let $H^\phi = H^{\phi(M)\subset\mathbb{S}^n}$ be the mean curvature of the immersion ϕ, and let dv_ϕ be the volume measure on $(M,\phi^*g_{\mathbb{S}^n})$. By standard Riemannian geometry computations, one can check that the quantity $\int_M(|H^\phi|^2+1)\,dv_\phi$ is invariant under composition of ϕ with a conformal change of $\mathbb{S}^n$, that is to say

$$\int_M (|H^{G_a\circ\phi}|^2+1)\,dv_{G_a\circ\phi} = \int_M (|H^\phi|^2+1)\,dv_\phi$$

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(9) It is also possible to define the conformal volume for surfaces with boundary in the same way.

(10) The isometry group of $\mathbb{S}^n$ is a closed subgroup of $\mathrm{Conf}(\mathbb{S}^n)$. Therefore, the quotient is a well-defined topological space with the induced topology.

for every $a \in \mathbb{B}^{n+1}$. In particular, for every $a \in \mathbb{B}^{n+1}$, we get

$$\begin{aligned}\operatorname{area}(M,g) &= \int_M 1\,dv_g = \int_M (|H^\phi|^2+1)\,dv_\phi \\ &= \int_M (|H^{G_a\circ\phi}|^2+1)\,dv_{G_a\circ\phi} \geq \operatorname{area}(M,(G_a\circ\phi)^*g_{\mathbb{S}^n}),\end{aligned}$$

where we used that $H^\phi \equiv 0$, since ϕ is minimal, and $g = \phi^* g_{\mathbb{S}^n}$. By taking the supremum over $a \in \mathbb{B}^{n+1}$ and using that $\operatorname{area}(M,g) = \operatorname{area}(M,\phi^*g_{\mathbb{S}^n}) \leq V_c(n,\phi)$, we get the desired equality. □

In order to prove Theorem A, we will first obtain minimal immersions with area bounded by 8π. Then, their embeddedness will be a consequence of the following result.

Proposition 3.2 (Li and Yau, 1982, Corollary 10). — *If $\phi\colon (M,g) \to \mathbb{S}^n$ is an isometric (branched) minimal immersion and* $\operatorname{area}(M,g) < 8\pi$*, then ϕ is an embedding.*

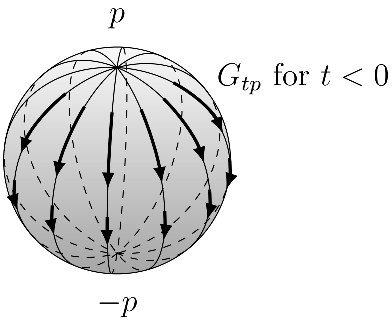
Proof. — Assume by contradiction that ϕ is not an embedding, then there exists $p \in \mathbb{S}^n$ such that the density of ϕ at p (cf. White, 2016, Section 1) is greater than two, namely

$$\Theta(\phi,p) \coloneqq \lim_{r\to 0} \frac{\operatorname{area}(\phi^{-1}(B_r(p)),\phi^*g_{\mathbb{S}^n})}{\pi r^2} \geq 2,$$

where $B_r(p)$ is the geodesic ball in $\mathbb{S}^n$ of center p and radius r. Observe that this takes into account both the cases where the preimage $\phi^{-1}(p)$ contains more than one point and where p is a branch point for ϕ. We now claim that $V_c(n,\phi) \geq \operatorname{area}(\mathbb{S}^2)\Theta(\phi,p) = 4\pi\Theta(\phi,p) \geq 8\pi$. This would contradict our assumption $\operatorname{area}(M,g) < 8\pi$ since $V_c(n,\phi) = \operatorname{area}(M,g)$ by the previous lemma.

Let us give a heuristic idea of the proof of the inequality $V_c(n,\phi) \geq \operatorname{area}(\mathbb{S}^2)\Theta(\phi,p)$. For $t \in (-1,1)$, consider the conformal transformation

$$G_{tp}(x) = \frac{1-t^2}{|x+tp|^2}(x+tp)+tp.$$



Note that the map G_{tp} fixes p and $-p$ and it converges to the constant map $-p$ on $\mathbb{S}^2 \setminus \{p\}$ as $t \to -1$. In other words, for t close to -1, G_{tp} maps a small neighborhood of p onto almost the entire sphere $\mathbb{S}^n$ and the rest of the sphere to a small neighborhood of $-p$. As a result, as t approaches -1, under the map $G_{tp}\circ\phi\colon M \to \mathbb{S}^n$ a small neighborhood of each point in $\phi^{-1}(p)$ approaches a collection of equatorial two-spheres (passing through p and $-p$) in $\mathbb{S}^n$ ($\Theta(\phi,p)$ of them, counted with multiplicity). The rest of the surface M is mapped to a small neighborhood of $-p$ and has small area with respect to the induced metric $(G_{tp}\circ\phi)^*g_{\mathbb{S}^n}$. In particular, we get that $\operatorname{area}(M,(G_{tp}\circ\phi)^*g_{\mathbb{S}^n}) \to \operatorname{area}(\mathbb{S}^2)\Theta(\phi,p) = 4\pi\Theta(\phi,p)$, as desired. □

Now, we define the *n-conformal volume of* $(M,[g])$, where $[g]$ is the conformal class of g (recall that $V_c(n,\phi)$ depends only on $[g]$), as

$$V_c(n,M,[g]) \coloneqq \inf_{\substack{\phi\colon M\to\mathbb{S}^n\\ \text{conformal}}} V_c(n,\phi). \tag{11}$$

If there are no conformal maps $\phi\colon M\to\mathbb{S}^n$, we set $V_c(n,M,[g])=\infty$.

Note that $V_c(n,M,[g])<\infty$ for every sufficiently large n by Nash's embedding theorem, which gives the existence of a conformal map $\phi\colon M\to\mathbb{S}^n$ for all n sufficiently large. Moreover, $V_c(n,M,[g])\geq V_c(n+1,M,[g])$, since a conformal map $\phi\colon M\to\mathbb{S}^n$ can be seen as a conformal map into $\mathbb{S}^{n+1}\supset\mathbb{S}^n$.

Thanks to a Hersch-type trick (cf. Proposition 1.14), it is possible to bound the first renormalized Laplace eigenvalue from above by the conformal volume.

Theorem 3.3 (Li and Yau, 1982, Theorem 1). — *For every closed Riemannian surface (M,g) and $n\in\mathbb{N}$, we have*

$$\overline{\lambda}_1(M,g)\leq 2V_c(n,M,[g]).$$

Moreover, equality holds if and only if (M,g) can be minimally (branched) immersed isometrically (up to a rescaling of the metric) in $\mathbb{S}^n$ by first eigenfunctions.

Remark 3.4. — The theorem holds also for surfaces (M,g) with boundary (cf. Remark 1.15), by considering the first Laplace eigenvalue $\lambda_1^N(M,g)$ with Neumann boundary conditions, using the variational characterization (7) of the first Neumann eigenvalue.

Proof. — If $V_c(n,M,[g])=\infty$, the result is obvious. Therefore, let us assume $V_c(n,M,[g])<\infty$. For every conformal map $\phi\colon M\to\mathbb{S}^n$, by the Hersch trick, there exists a conformal automorphism $h\colon\mathbb{S}^n\to\mathbb{S}^n$ such that $\int_M(h\circ\phi)\,dv_g=0\in\mathbb{R}^{n+1}$. Therefore, without loss of generality, we can assume that $\int_M\phi\,dv_g=0$. By the variational characterization (3) of the first eigenvalue of the Laplacian, we have

$$\lambda_1(M,g)\mathrm{area}(M,g)=\lambda_1(M,g)\int_M|\phi|^2\,dv_g\leq\int_M|d\phi|^2\,dv_g=2E(\phi)\leq 2V_c(n,\phi), \tag{12}$$

where we used (10) in the last inequality. Since this holds for every conformal map ϕ, we get the desired inequality. □

Remark 3.5. — If we take the supremum over the metrics in the conformal class $[g]$ in Theorem 3.3, we obtain that

$$\Lambda_1(M,[g])\coloneqq\sup_{\tilde g\in[g]}\overline{\lambda}_1(M,\tilde g)\leq 2V_c(n,M,[g]).$$

Observe that the equality case in the theorem tells us that every minimal immersion in $\mathbb{S}^n$ by first eigenfunctions realizes the maximum of $\overline{\lambda}_1$ in its conformal class. In particular, if $\phi\colon(M,g)\to\mathbb{S}^n$ is an isometric minimal immersion by first eigenfunctions, then $V_c(n,\phi)=V_c(n,M,[g])=\mathrm{area}(M,g)$.

4. Eigenvalue optimization on basic reflection surfaces

We now delve into eigenvalue optimization on a surface M for metrics satisfying suitable symmetry assumptions. This is one of the key aspects of the paper KKMS.

We mention that equivariant techniques have been fruitfully used to construct examples of (free boundary) minimal surfaces via other methods too, such as Simon–Smith min-max (see e.g. Ketover, 2016) and gluing (see e.g. Kapouleas, 2017). Moreover, Petrides (2023a) used equivariant eigenvalue optimization to construct an embedded, nonplanar minimal surface in every three-dimensional ellipsoid with suitable properties (see Petrides, 2023b for the free boundary analogue).

4.1. Basic reflection surfaces

Unless otherwise stated, let M be a smooth, connected, closed, orientable Riemannian surface. Given an isometry τ of M, we denote by $M^\tau := \{p \in M \,:\, \tau(p) = p\}$ its set of fixed points.

Definition 4.1 (KKMS, Definition 2.1, Lemma 5.3). — *We say that an isometry τ of M is a* reflection *if $M \setminus M^\tau$ is not connected. Equivalently, $d\tau_p \colon T_pM \to T_pM$ is a Euclidean reflection (with respect to the metric on M) for all $p \in M^\tau$.*

Remark 4.2. — By (KKMS, Lemma 2.3), a reflection τ is an involution (i.e., $\tau^2 = \mathrm{id}_M$), M^τ is a codimension one submanifold (possibly disconnected), and $M \setminus M^\tau$ has exactly two connected components.

One of the key ideas in KKMS is to perform *equivariant* eigenvalue optimization on so-called *basic reflection surfaces* (see Figure 5).

Definition 4.3 (KKMS, Definition 5.4). — *A pair (M, τ) is a* basic reflection surface *if τ is a reflection on M such that $M/\langle\tau\rangle$ has genus zero.*

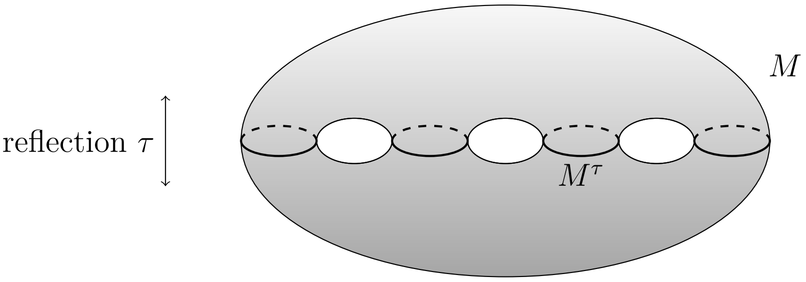


Figure 5. Example of a closed basic reflection surface (M, τ) of genus 3. Note that $M/\langle\tau\rangle$ is diffeomorphic to a sphere with 4 disks removed, in particular it has genus zero.

The key property of a basic reflection surface is that the quotient $M/\langle\tau\rangle$ has genus zero. Then, heuristically, equivariant eigenvalue optimization on such surfaces might be expected to behave well, in line with the nice properties enjoyed for example by

Neumann eigenvalue optimization on genus zero surfaces (see e.g. Remark 1.15) or Steklov eigenvalue optimization on genus zero surfaces. Note for example that the multiplicity of the first Steklov eigenvalue on genus zero surfaces is at most 3 by (Fraser and Schoen, 2016, Theorem 2.3). It is possible to recover an analogous multiplicity bound for the first Laplace eigenvalue on basic reflection surfaces.

Proposition 4.4 (KKMS, Proposition 5.12 and 5.15). — *Let (M, τ) be a closed basic reflection surface and let g be a Riemannian metric on M which is invariant with respect to the reflection τ. Then, the multiplicity of the first eigenvalue $\lambda_1(M, g)$ of the Laplacian on (M, g) is at most 4.*

Proof. — Let us denote by $\mathcal{E}$ the first eigenspace of the Laplacian on (M, g). We want to prove that $\dim(\mathcal{E}) \leq 4$. Note that $\mathcal{E}$ is invariant with respect to the action of τ (i.e., if u is a first eigenfunction, then $u \circ \tau$ is also a first eigenfunction). Therefore, we can decompose $\mathcal{E}$ as the orthogonal direct sum of its odd part $\mathcal{E}^- := \{u \in \mathcal{E} \,:\, u \circ \tau = -u\}$ and its even part $\mathcal{E}^+ := \{u \in \mathcal{E} \,:\, u \circ \tau = u\}$ with respect to the reflection τ.

Recall from Definition 4.1 (see Remark 4.2) that $M \setminus M^\tau$ consists of exactly two connected components M_1 and M_2. Then, we have the following bounds on the dimensions of $\mathcal{E}^-$ and $\mathcal{E}^+$, respectively.

First bound : $\dim(\mathcal{E}^-) \leq 1$. If $u \in \mathcal{E}^-$ then by definition $u \circ \tau = -u$ and hence $u \equiv 0$ on M^τ. Therefore, u is a Dirichlet eigenfunction on M_1 (and M_2). By Courant's nodal domain Theorem 1.6 (see Remark 1.7), u has exactly two nodal domains, which must coincide with M_1 and M_2. Therefore, $u|_{M_1}$ is a Dirichlet eigenfunction which is never zero on M_1. This implies that $u|_{M_1}$ is a *first* Dirichlet eigenfunction on M_1. Since the first Dirichlet eigenspace is simple (see Remark 1.8), we get that $\dim(\mathcal{E}^-) \leq 1$.

Second bound : $\dim(\mathcal{E}^+) \leq 3$. Let us consider any $u \in \mathcal{E}^+$, so that $u \circ \tau = u$. In particular, u is a Neumann eigenfunction on M_1. We claim that $u|_{M_1}$ has two nodal domains. Let U_+, U_- be the two nodal domains of u in M, i.e., $M \setminus \mathcal{N}_u = U_+ \sqcup U_-$, where $\mathcal{N}_u = \{u = 0\}$, $U_+ = \{u > 0\}$, $U_- = \{u < 0\}$. Observe that $U_+|_{M_1}$ and $U_-|_{M_1}$ are both connected subsets of M_1. Indeed, it is easily observed that, if a subset V of M_1 is not connected, then $V \cup \tau(V)$ is not connected in M (see also KKMS, Lemma 5.14). As a result, the Neumann eigenfunction $u|_{M_1}$ has exactly two nodal domains. By Theorem 1.6, $\mathcal{N}_u$ is a union of C^2-immersed circles. Since M_1 has genus zero and $\mathcal{N}_u$ divides M_1 into two connected components both touching ∂M_1, the nodal set $\mathcal{N}_u$ has to be a single C^2 circle without self-intersections. Therefore, we can apply the Hopf lemma (see e.g. Gilbarg and Trudinger, 2001, Lemma 3.4) to obtain that, for all $p \in \mathcal{N}_u$, $du_p \neq 0$. As a result, if we fix any point $p \in M_1$, then the linear map

$$\begin{aligned} T \colon \mathcal{E}^+ &\to \mathbb{R} \times T_p^* M \\ u &\mapsto (u(p), du_p) \end{aligned}$$

is injective. Therefore, $\dim(\mathcal{E}^+) \leq \dim(\mathbb{R} \times T_p^* M) = 3$. □

Recall that the first renormalized Neumann eigenvalue on a genus zero surface is less than 8π (cf. Remark 1.15). The same holds for the first renormalized Steklov eigenvalue on genus zero surfaces (Kokarev, 2014, Theorem A_1, see also Example 1.3 therein). We have a similar upper bound for the first renormalized Laplace eigenvalue on basic reflection surfaces.

Lemma 4.5 (KKMS, Lemma 5.19). — *Let (M, τ) be a closed basic reflection surface and let g be a metric on M which is invariant with respect to τ. We then have*

$$\bar{\lambda}_1(M, g) < 16\pi.$$

Proof. — Let $M_1 \subset M$ be one of the two connected components of $M \setminus M^\tau$. By the definition of a basic reflection surface, M_1 is a genus zero surface with boundary M^τ. Therefore, by Remark 1.15, we have

$$\lambda_1^N(M_1, g)\mathrm{area}(M_1, g) < 8\pi,$$

where $\lambda_1^N(M_1, g)$ is the first Neumann eigenvalue on (M_1, g). Now note that $\mathrm{area}(M, g) = 2\mathrm{area}(M_1, g)$, since g is invariant with respect to the reflection τ. Moreover, by the variational characterization (3) for eigenvalues, we have

$$\begin{aligned} \lambda_1^N(M_1, g) &= \inf\left\{\frac{\int_{M_1}|du|^2}{\int_{M_1}|u|^2} \ :\ u \text{ Lipschitz on } M_1, \int_{M_1} u = 0\right\} \\ &= \inf\left\{\frac{\int_{M}|du|^2}{\int_{M}|u|^2} \ :\ u\ \tau\text{-even Lipschitz on } M, \int_{M} u = 0\right\} \\ &\geq \inf\left\{\frac{\int_{M}|du|^2}{\int_{M}|u|^2} \ :\ u \text{ Lipschitz on } M, \int_{M} u = 0\right\} = \lambda_1(M, g). \end{aligned}$$

As a result, we get the desired inequality

$$\lambda_1(M, g)\mathrm{area}(M, g) \leq 2\lambda_1^N(M_1, g)\mathrm{area}(M_1, g) < 16\pi. \qquad \square$$

4.2. Equivariant basic reflection surfaces

Performing equivariant eigenvalue optimization on basic reflection surfaces is already sufficient to obtain embedded minimal surfaces in $\mathbb{S}^3$ with area less than 8π. However, KKMS imposes further symmetries with respect to a larger group Γ. This lets them obtain multiple examples of nonisometric minimal surfaces with the same topology.

Definition 4.6. — *A pair (M, τ) is a* Γ-equivariant basic reflection surface *if (M, τ) is a basic reflection surface, Γ is a finite group generated by reflections of M, and τ is contained in Γ and commutes with each of its elements. In other words, $\Gamma = \langle\tau\rangle \times G$ for some finite group G generated by reflections.*

Remark 4.7. — To be precise we should distinguish between the group Γ and its action $\Gamma \times M \to M$ on M, as is done in (KKMS, see for example Section 2.3 therein). However, to simplify our notation, we identify the group and its action. The reader should be aware that the same group could correspond to different actions.

Sections 5.3, 5.7, 5.8 and 5.9 of KKMS contain many properties and a precise classification of basic reflection surfaces and group actions as above. Here, we just give an example in the case when $G = \mathbb{Z}_2$, namely when $\Gamma = \langle\tau\rangle \times \langle\rho\rangle$ for reflections τ, ρ of M.

The quotient $M/\langle\tau\rangle$ can be represented as a sphere with $\gamma + 1$ disks removed, where γ is the genus of M. If we impose an extra symmetry with respect to a reflection ρ, the surface $M/\langle\tau\rangle$ can be represented as a sphere with $\gamma + 1$ disks removed, such that the disks are symmetric with respect to the reflection across an equator (representing the reflection ρ). In particular, let a be the number of disks centered along the equator and b be the number of disks in each hemisphere bounded by the equator, in such a way that $a + 2b = \gamma + 1$. See Figure 6.

Fixing $\gamma \in \mathbb{N}$ and performing equivariant eigenvalue optimization on surfaces of this type, as the number of disks a and b varies, we shall see that it is possible to obtain $\lfloor\frac{\gamma-1}{4}\rfloor + 1$ nonisometric minimal embeddings with genus γ in $\mathbb{S}^3$, as stated in Theorem A (see also Section 6).

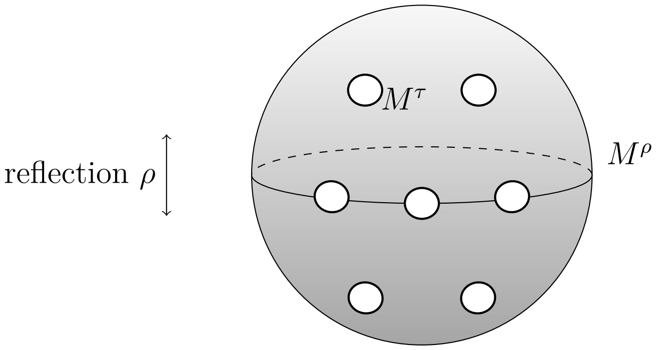


FIGURE 6. Representation of $M/\langle\tau\rangle$, half of a Γ-equivariant basic reflection surface (M, τ), where $\Gamma = \langle\tau\rangle \times \langle\rho\rangle$. The surface $M/\langle\tau\rangle$ is diffeomorphic to a sphere with $a = 3$ disks removed along an equator (which corresponds to M^ρ) and $b = 2$ disks removed in each hemisphere. In this case, M has genus $\gamma = 6$.

4.3. Equivariant eigenvalue optimization

Given a Γ-equivariant basic reflection surface (M, τ) as in Definition 4.6, we denote by $\mathrm{Met}_\Gamma(M)$ the space of Riemannian metrics on M that are invariant with respect to the action of Γ. Then, we can define

$$\Lambda_k^\Gamma(M) := \sup_{g \in \mathrm{Met}_\Gamma(M)} \overline{\lambda}_k(M, g),$$

and look for a metric g_0 that maximizes the kth renormalized Laplace eigenvalue among Γ-symmetric metrics on M, that is to say $\overline{\lambda}_k(M, g_0) = \Lambda_k^\Gamma(M)$. By the principle of symmetric criticality of Palais (1979), one expects that critical points of some eigenvalue in the space of symmetric metrics $\mathrm{Met}_\Gamma(M)$ are also unconstrained critical points. Since the eigenvalues depend only Lipschitz continuously on g, this needs to be proved, but goes through without problems, see (KKMS, Theorems 2.12 and 2.15). Moreover, as described above, since (M,τ) is a basic reflection surface, we expect the equivariant maximizer to have nice properties. In fact, KKMS gets the following result for equivariant maximizers of the *first* renormalized eigenvalue.

Theorem 4.8 (cf. KKMS, Theorem 2.12 and Lemma 5.35)
Let (M,τ) be a closed Γ-equivariant basic reflection surface with $\operatorname{genus}(M)\neq 0$. *If $g_0 \in \mathrm{Met}_\Gamma(M)$ is a metric maximizing the first renormalized Laplace eigenvalue among the Γ-symmetric metrics on M, then there exist independent first eigenfunctions u_1,u_2,u_3,u_4 such that $u := (u_1,u_2,u_3,u_4)\colon (M,g_0) \to \mathbb{R}^4$ is an isometric minimal embedding into $\mathbb{S}^3$. Moreover, the embedding has area less than 8π and it is a doubling of an equatorial sphere.*

Example 4.9 (cf. KKMS, Corollary 2.6). — When the genus of M is equal to 0, in other words $M=\mathbb{S}^2$ is a sphere, it is known that the round metric $g_{\mathbb{S}^2}$ on $\mathbb{S}^2$ is Γ-invariant for every finite group Γ acting on $\mathbb{S}^2$. Therefore, using Hersch's result (Proposition 1.14), we get

$$8\pi = \overline{\lambda}_1(\mathbb{S}^2, g_{\mathbb{S}^2}) \leq \Lambda_1^\Gamma(\mathbb{S}^2) \leq \Lambda_1(\mathbb{S}^2) = 8\pi,$$

which implies that $\Lambda_1^\Gamma(\mathbb{S}^2)=8\pi$ for every finite group Γ.

We will see in Sections 5 and 6 that a Γ-equivariant maximizing metric as in Theorem 4.8 indeed exists. For now, let us sketch the proof of Theorem 4.8.

Sketch of the proof of Theorem 4.8. — Let $g_0 \in \mathrm{Met}_\Gamma(M)$ be a metric maximizing $\overline{\lambda}_1$ among the Γ-equivariant metrics on M. Up to rescaling g_0, we can assume that $\lambda_1(M,g_0)=2$. We can apply Theorem 1.9 to our equivariant setting thanks to the principle of symmetric criticality of Palais (1979), as described at the beginning of this section. We obtain that there exist independent first eigenfunctions $u_1,\dots,u_{n+1}$ for some $n\geq 3$ such that $u := (u_1,\dots,u_{n+1})\colon (M,g_0)\to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is an isometric (branched) minimal immersion. Note that $n=3$ thanks to Proposition 4.4. Moreover, by Lemma 4.5, we have

$$\operatorname{area}(M,g_0) = \frac{1}{2}\overline{\lambda}_1(M,g_0) < 8\pi.$$

In particular, by Proposition 3.2, the map u is an embedding.

We are left with showing that $u(M)$ is a doubling of an equatorial sphere. This relies on the fact that the reflection τ on $u(M)$ is induced by that across an equatorial sphere in $\mathbb{S}^3$, and the coordinate functions of $\mathbb{R}^4$ span the first eigenspace of the Laplacian on (M,g_0), for which we know the properties described in the proof of Proposition 4.4. We refer to (KKMS, Lemma 3.35) for the details. □

We conclude this section with a sort of rigidity property of any minimal embedding of the type produced by Theorem 4.8, which says that any other map by first eigenfunctions to $\mathbb{S}^n$ must also be an embedding. The usefulness may be unclear now, but it will be helpful to prove the existence of a maximizing metric in Section 6. Indeed, this result is what makes it somewhat easier to prove the existence of a maximizing metric in this equivariant setting, compared with the general case.

PROPOSITION 4.10 (cf. KKMS, Theorem 8.5). — *Let* (M, g) *be an orientable Riemannian surface with* $\operatorname{genus}(M) \neq 1$. *Assume that the multiplicity of* $\lambda_1(M, g)$ *is* 4, *and that there exists a minimal embedding* $u\colon (M, g) \to \mathbb{S}^3$ *by first eigenfunctions. Then, any other map* $v\colon M \to \mathbb{S}^n$ *by first eigenfunctions of* Δ_g *satisfies* $v(p) \neq v(q)$ *for all* $p \neq q \in M$.

Proof. — When M has genus zero, the only minimal embedding $(M, g) \to \mathbb{S}^3$ is the one giving the equatorial sphere and the statement follows. Therefore, let us now assume that $\operatorname{genus}(M) > 1$. Since u is an embedding, the components of u span the first eigenspace. Therefore, we can write $v = Au$, where A is a matrix with $n + 1$ rows and 4 columns. Using that $1 = |v|^2 = \langle Au, Au\rangle = u^T A^T Au$, we have that $u(M)$ is contained in the intersection of $\mathbb{S}^3$ with the level set $\{Q(x) = 1\}$ of the quadratic form Q on $\mathbb{R}^4$ associated to $A^T A$ (i.e., $Q(x) = x^T A^T Ax$). If A has rank 4 then $v(p) - v(q) = A(u(p) - u(q)) \neq 0$ for all $p \neq q \in M$, since u is injective and so is the linear map $\mathbb{R}^4 \to \mathbb{R}^{n+1}$ attached to A. If on the other hand A does not have rank 4, (KKMS, p. 101) shows that M must have genus one, which is excluded by assumption. As a result, A has rank 4 and v is injective. □

Remark 4.11. — We stated the proposition for surfaces with genus greater than one because it actually fails when $\operatorname{genus}(M) = 1$. In that case, the only minimal embedding $u\colon (M, g) \to \mathbb{S}^3$ by first eigenfunctions is the one giving the Clifford torus, by (El Soufi and Ilias, 2000, Theorem 2.1).(11) In particular, (M, g) is (up to homothety) the flat square torus $(T^2 = \mathbb{R}^2/\mathbb{Z}^2, g_{\text{flat}})$ and the first eigenspace is generated by the functions $\cos(2\pi x), \sin(2\pi x), \cos(2\pi y), \sin(2\pi y)$ (see the proof of Theorem 2.1 in El Soufi and Ilias, 2000). In particular, the map $v\colon M \to \mathbb{S}^1$ given by $v(x, y) = (\cos(2\pi x), \sin(2\pi x))$ contradicts Proposition 4.10.

4.4. Basic reflection surfaces with boundary

The definition of a basic reflection surface (M, τ), Definition 4.3, carries over to the case where M has boundary, but we additionally require that the double $\widetilde{M}$ of M (obtained by gluing two copies of M along its boundary) is a closed basic reflection surface with respect to the natural extension of τ to $\widetilde{M}$. For an equivalent definition see (KKMS, Definition 5.4).

The definition of a Γ-equivariant basic reflection surface also carries over to the case where M has boundary. Note that, if (M, τ) is a Γ-equivariant basic reflection surface, then its double $(\widetilde{M}, \tau)$ is a $\widetilde{\Gamma}$-equivariant basic reflection surface, where $\widetilde{\Gamma} = \langle\iota\rangle \times \Gamma$ and ι is the natural reflection interchanging the two copies of M in $\widetilde{M}$.

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(11) Recall that by (Brendle, 2013a) (cf. Example 1.1), we could remove the assumption that u is by first eigenfunctions.

In this setting, one can define

$$\Sigma_1^\Gamma(M) := \sup_{g\in \mathrm{Met}_\Gamma(M)} \overline{\sigma}_1(M,g),$$

and say that a metric $g_0 \in \mathrm{Met}_\Gamma(M)$ maximizes the first renormalized Steklov eigenvalue if $\overline{\sigma}_1(M,g_0) = \Sigma_1^\Gamma(M)$. Then, we get the analogue of Theorem 4.8 for surfaces with boundary.

Theorem 4.12 (cf. KKMS, Theorem 2.15, Lemma 5.37). — *Let (M,τ) be a Γ-equivariant basic reflection surface with boundary. If $g_0 \in \mathrm{Met}_\Gamma(M)$ is a metric on M maximizing the first renormalized Steklov eigenvalue among the Γ-symmetric metrics on M, then there exist independent first eigenfunctions u_1, u_2, u_3 so that $u := (u_1,u_2,u_3)\colon M \to \mathbb{R}^3$ is a free boundary minimal embedding in $\mathbb{B}^3$ with area less than 2π. Moreover, the embedding is a doubling of an equatorial disk.*

We do not discuss the proof of this theorem, which presents some further technical difficulties compared with the closed case, but the main ideas are similar. In Sections 5 and 6, we will discuss the existence of an equivariant maximizing metric. However, again, we will not go into the details of the case with boundary. For technical reasons, KKMS does not get existence of a maximizer for every equivariant basic reflection surface with boundary (see also Remark 6.2). One case in which they do get existence is when (M,τ) is a Γ-equivariant basic reflection surface with $\Gamma = \langle\tau\rangle \times \langle\rho\rangle$ for two reflections τ, ρ. Let us discuss this setting in more detail.

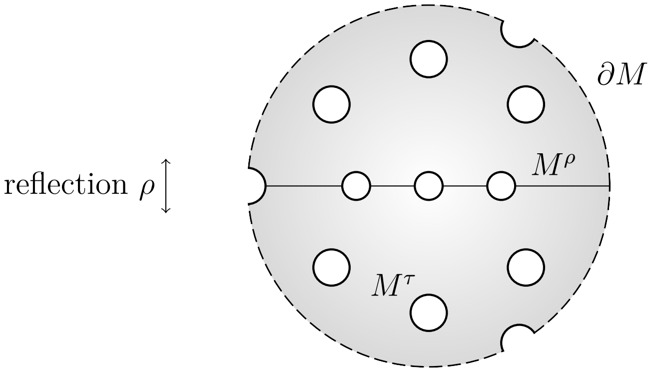

Figure 7. Schematic representation of $M/\langle\tau\rangle$, half of a Γ-equivariant basic reflection surface (M,τ), where $\Gamma = \langle\tau\rangle \times \langle\rho\rangle$. The surface M is obtained by gluing two copies of $M/\langle\tau\rangle$ along M^τ. In this case, M has genus 9 and 3 boundary components.

Half of the surface $M/\langle\tau\rangle$ can be represented as a disk with γ disks removed in the interior and β half-disks removed along the boundary. Here, γ and β correspond to the genus and the number of boundary components of M. The extra symmetry ρ can be represented as the reflection across a diameter of the disk. In this way, all the removed disks are symmetric with respect to this reflection. In particular, we have a of these disks centered along the diameter and b of them centered in each half-disk bounded by the diameter, in such a way that $a + 2b = \gamma$. See Figure 7.

One can perform equivariant eigenvalue optimization on $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-equivariant basic reflection surfaces as the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on M varies in order to have different values of a and b. In this way, one can obtain $\lfloor \frac{\gamma-2}{4} \rfloor + 1$ nonisometric free boundary minimal embeddings with genus γ and β boundary components in $\mathbb{B}^3$, as stated in Theorem B.

5. Existence of a conformal eigenvalue maximizer

Thanks to Theorem 4.8, in order to prove Theorem A, we are left with the proof of the existence of a maximizing metric $g_0 \in \mathrm{Met}_\Gamma(M)$. The first step, discussed in this section, is to prove the existence of a maximizer among the Γ-equivariant metrics in every given conformal class $[g]$. More precisely, defining

$$\Lambda_1^\Gamma(M,[g]) \coloneqq \sup\{\overline{\lambda}_1(M,\tilde{g}) \ : \ \tilde{g} \in \mathrm{Met}_\Gamma(M) \cap [g]\},$$

the following theorem states that, under suitable assumptions, there is a metric $g_0 \in \mathrm{Met}_\Gamma(M) \cap [g]$ maximizing the first renormalized Laplace eigenvalue among Γ-equivariant metrics in $[g]$, that is to say $\overline{\lambda}_1(M,g_0) = \Lambda_1^\Gamma(M,[g])$.

Theorem 5.1 (Petrides, 2014; KKMS, Theorem 3.12). — *Let (M,τ) be a Γ-equivariant basic reflection surface. Consider $g \in \mathrm{Met}_\Gamma(M)$ such that $\Lambda_1^\Gamma(M,[g]) > 8\pi$. Then there exists a metric $g_0 \in \mathrm{Met}_\Gamma(M) \cap [g]$, smooth away from finitely many conical singularities, maximizing the first renormalized Laplace eigenvalue among Γ-equivariant metrics in $[g]$. Moreover, there exists a harmonic map $u\colon (M,g_0) \to \mathbb{S}^n$ by first eigenfunctions, for some $n \in \mathbb{N}$.*

Remark 5.2. — Observe that, by maximizing in a conformal class, we only get a *harmonic* map and not a minimal immersion. This is because we cannot guarantee in general that the immersion is an isometry (cf. Remark 1.4).

Remark 5.3. — Note that, thanks to Theorem 3.3, if $u\colon (M,g) \to \mathbb{S}^n$ is an isometric immersion by first eigenfunctions, then g maximizes the first renormalized eigenvalue $\overline{\lambda}_1$ in its conformal class $[g]$. Moreover, by (Montiel and Ros, 1986, Theorem 1), there exists at most one such metric in each conformal class. However, this uniqueness regards the *intrinsic* metric on M. We do not know whether the same metric on M can give rise to minimal immersions that are not related by an ambient isometry.

Remark 5.4. — When M is not a sphere, for every conformal class $[g]$, the strict inequality $\Lambda_1(M,[g]) > 8\pi$ was proven by Petrides (2014, Theorem 1) in the case without symmetry and, in Section 6, we will see that it also holds in our setting with symmetry.

Theorem 5.1 in the case without symmetry was first proved by Petrides (2014), using a method which was later refined in (Petrides, 2025). The proof is based on Ekeland's variational principle, which allows one to find a maximizing sequence of metrics(12) in $[g]$ and associated maps by first eigenfunctions, which are almost harmonic. Careful analysis shows that this sequence of metrics converges to a maximizer for $\overline{\lambda}_1$ in $[g]$. See also (Petrides, 2026) for more details.

Subsequently, other proofs have been obtained (see e.g. Vinokurov, 2025). Here, we describe in more detail the approach taken by Karpukhin, Kusner, McGrath, and Stern (KKMS, Sections 3 and 4), since it is based on a min-max characterization of $\Lambda_1^\Gamma(M)$ developed in (Karpukhin and Stern, 2024c) which is of independent interest.

5.1. Definition of min-max energy and first results

The idea is to define a min-max energy $\mathcal{E}_n(M,[g])$, inspired by the Li–Yau conformal volume $V_c(n,M,[g])$ presented in Section 3, such that not only $\Lambda_1(M,[g]) \leq 2\mathcal{E}_n(M,[g])$ (as in Theorem 3.3 for the Li–Yau conformal volume), but also equality holds for every sufficiently large n. Indeed, as explained in (Karpukhin and Stern, 2024c, Section 2.4), the min-max energy $\mathcal{E}_n(M,[g])$ can be thought as a relaxation of the conformal volume $V_c(n,M,[g])$ such that Theorem 3.3 and its proof still hold. The energies $\mathcal{E}_n(M,[g])$ are defined on suitable spaces of maps and, by running a min-max scheme for these energies, the authors produce a special harmonic map which induces the desired maximizing metric for $\overline{\lambda}_1$ in $[g]$.

Given the subset $\{G_a\}_{a\in\mathbb{B}^{n+1}}$ of conformal transformations of $\mathbb{S}^n$ defined in (9), we recall from (11) that

$$V_c(n,M,[g]) := \inf_{\substack{\phi\colon M\to\mathbb{S}^n\\ \text{conformal}}} \sup_{a\in\mathbb{B}^{n+1}} E(G_a\circ\phi) = \inf_{\substack{\phi\colon M\to\mathbb{S}^n\\ \text{conformal}}} \sup_{a\in\mathbb{B}^{n+1}} \int_M \frac{1}{2}|d(G_a\circ\phi)|^2\,dv_g.$$

Now, let $C^0(\overline{\mathbb{B}}^{n+1}, W^{1,2}(M,\mathbb{S}^n))$ be the space of continuous functions from the closed ball $\overline{\mathbb{B}}^{n+1}$ to the Sobolev space $W^{1,2}(M,\mathbb{S}^n)$. We then define its subset $\widetilde{\mathcal{B}}_n$ as the set of functions $F=(a\mapsto F_a)\in C^0(\overline{\mathbb{B}}^{n+1}, W^{1,2}(M,\mathbb{S}^n))$ such that $F_a\equiv a$ for all $a\in\mathbb{S}^n$, meaning that F_a is equal to the constant function equal to a for all $a\in\mathbb{S}^n$. The first idea of Karpukhin and Stern (2024c) is to relax the definition of conformal volume to(13)

$$\widetilde{\mathcal{E}}_n(M,[g]) := \inf_{F\in\widetilde{\mathcal{B}}_n}\sup_{a\in\mathbb{B}^{n+1}} E(F_a) = \inf_{F\in\widetilde{\mathcal{B}}_n}\sup_{a\in\mathbb{B}^{n+1}} \int_M \frac{1}{2}|dF_a|^2\,dv_g,$$

The key is that, if $F\in\widetilde{\mathcal{B}}_n$, then there exists $a\in\mathbb{B}^{n+1}$ such that $\int_M F_a = 0$ by topological degree theory. Thus, by the variational characterization of the first eigenvalue we get

$$\overline{\lambda}_1(M,g) = \lambda_1(M,g)\text{area}(M,g) = \lambda_1(M,g)\int_M |F_a|^2\,dv_g \leq \int_M |dF_a|^2\,dv_g = 2E(F_a),$$

---

(12) To be precise, one first has to consider a suitable weak notion of metrics, and then prove that the maximizer is smooth away from finitely many conical singularities.

(13) Note that we use the tilde here, i.e., $\widetilde{\mathcal{E}}_n$, because this is not the final definition of the energy $\mathcal{E}_n$, which will be given in Definition 5.7.

exactly as in (12), which is the core inequality in the proof of Theorem 3.3. Taking the infimum over all $F \in \widetilde{\mathcal{B}}_n$ and the supremum over all $\tilde{g} \in [g]$, we then get

$$\Lambda_1(M,[g]) \leq 2\widetilde{\mathcal{E}}_n(M,[g]). \tag{13}$$

Remark 5.5. — Note that the maps F_a, for $F \in \widetilde{\mathcal{B}}_n$ and $a \in \overline{\mathbb{B}}^{n+1}$, are not necessarily conformal. Conformality of $G_a \circ \phi$ was needed in the Li–Yau construction to have $\operatorname{area}(G_a \circ \phi) = E(G_a \circ \phi)$ (see (8)).

It is well known that the Dirichlet energy E presents a lack of compactness due to its conformal invariance. Therefore, it is often convenient to perturb the Dirichlet functional to a functional E_ε that approaches E as $\varepsilon \to 0$ and has some compactness property for $\varepsilon > 0$ (e.g. the Palais–Smale condition, see e.g. Ambrosetti and Malchiodi, 2007, Section 7.4), which allows one to find critical points easily. This idea was first introduced by Sacks and Uhlenbeck (1981) and the second ingredient in (Karpukhin and Stern, 2024c) is indeed a similar perturbation. In particular, they consider the Ginzburg–Landau approximation of the Dirichlet energy introduced by Chen (1989) and Chen and Struwe (1989):

$$E_\varepsilon(u) \coloneqq \int_M \frac{1}{2}|du|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2}\, dv_g \geq E(u),$$

defined for functions $u \in W^{1,2}(M,\mathbb{R}^{n+1})$. The second term in the integrand is called the *potential term.*

Remark 5.6. — The choice of the Ginzburg–Landau energy is not essential; the Sacks–Uhlenbeck perturbation could potentially work too.

Note that $E_\varepsilon(u) = E(u)$ if u takes values in $\mathbb{S}^n$. Moreover, E_ε is nondecreasing as $\varepsilon \searrow 0$. Therefore, the following energies are well-defined.

Definition 5.7 (cf. Karpukhin and Stern, 2024c, Section 3)
For every $\varepsilon > 0$, we define the perturbed min-max energy

$$\mathcal{E}_{n,\varepsilon}(M,g) \coloneqq \inf_{F\in\mathcal{B}_n} \max_{a\in\mathbb{B}^{n+1}} E_\varepsilon(F_a),$$

where[14] $\mathcal{B}_n = \{F \in C^0(\overline{\mathbb{B}}^{n+1}, W^{1,2}(M,\mathbb{R}^{n+1})) : F_a \equiv a$ *for* $a \in \mathbb{S}^n\}$. *As* $\varepsilon \searrow 0$, $\mathcal{E}_{n,\varepsilon}(M,g)$ *is nondecreasing. Thus, we can define*

$$\mathcal{E}_n(M,g) \coloneqq \sup_{\varepsilon>0} \mathcal{E}_{n,\varepsilon}(M,g) = \lim_{\varepsilon\searrow 0} \mathcal{E}_{n,\varepsilon}(M,g).$$

As mentioned above, the definition of $\mathcal{E}_n$ is devised to be a relaxation of the conformal volume such that Theorem 3.3 still holds. We make this precise in the following proposition.

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(14) The difference with $\widetilde{\mathcal{B}}_n$ is that we now allow the maps to take values in $\mathbb{R}^{n+1}$ and not only in $\mathbb{S}^n$.

PROPOSITION 5.8 (Karpukhin and Stern, 2024c, Section 3.1)

The energy $\mathcal{E}_n(M,g)$ is a conformal invariant, and we write $\mathcal{E}_n(M,[g]) \coloneqq \mathcal{E}_n(M,g)$. Moreover, for every $n \geq 2$, we have

$$\Lambda_1(M,[g]) \leq 2\mathcal{E}_n(M,[g]) \leq 2V_c(n,M,[g]) < \infty.$$

Idea of the proof. — Note that the energies $\mathcal{E}_{n,\varepsilon}(M,g)$ are not conformally invariant because of the potential term in the definition of E_ε. However, for every $\tilde{g} \in [g]$, there exists $C = C(g,\tilde{g}) > 0$ such that $\mathcal{E}_{n,C\varepsilon}(M,g) \leq \mathcal{E}_{n,\varepsilon}(M,\tilde{g}) \leq \mathcal{E}_{n,\varepsilon/C}(M,g)$. Taking the limit as $\varepsilon \to 0$, we get $\mathcal{E}_n(M,g) = \mathcal{E}_n(M,\tilde{g}) = \mathcal{E}_n(M,[g])$.

The inequality $\mathcal{E}_n(M,[g]) \leq V_c(n,M,[g])$ follows from the fact that every map $\mathbb{B}^{n+1} \ni a \mapsto G_a \circ \phi \in W^{1,2}(M,\mathbb{S}^n)$, in the definition of $V_c(n,M,[g])$, can be approximated by maps in $\mathcal{B}_n$ by mollification. We refer to (Karpukhin and Stern, 2024c, Proposition 3.3) for the details of the proof.

We are left with the proof of the inequality $\Lambda_1(M,[g]) \leq 2\mathcal{E}_n(M,[g])$. The idea is the same as for the proof of (13). Here, the difference is that $\mathcal{E}_n(M,[g])$ is obtained as an approximation of the energies $\mathcal{E}_{n,\varepsilon}(M,[g])$. Therefore, one first proves that

$$\frac{1-\varepsilon}{2+\varepsilon}\overline{\lambda}_1(M,g) \leq \mathcal{E}^{\Gamma}_{n,\varepsilon}(M,g),$$

by using a similar argument but taking care of the potential term (see Karpukhin and Stern, 2024c, Proposition 3.4 and KKMS, Proposition 3.4). Then, one gets the desired inequality by taking the limit as $\varepsilon \to 0$. □

The third and final ingredient to adapt the energy $\mathcal{E}_n(M,[g])$ to the setting of KKMS is to consider its equivariant adaptation. Let (M,g) be a closed, smooth, Riemannian surface and let Γ be a finite group of isometries of M (in particular, this applies to the case when (M,τ) is a Γ-equivariant basic reflection surface and $g \in \mathrm{Met}_\Gamma(M)$). Then, (KKMS, Section 3) replaces the set $\mathcal{B}_n$ in the definition of $\mathcal{E}_{n,\varepsilon}(M,g)$ with a suitable subset(15) $\mathcal{B}^{\Gamma}_m$ of $C^0(\overline{\mathbb{B}}^m, W^{1,2}_{\Gamma}(M,\mathbb{R}^{m|\Gamma|}))$, where $W^{1,2}_{\Gamma}(M,\mathbb{R}^{m|\Gamma|})$ is the set of Γ-equivariant $W^{1,2}$ Sobolev maps with respect to a natural action of Γ on $\mathbb{R}^{m|\Gamma|}$.

Then, again, one can define

$$\mathcal{E}^{\Gamma}_{m,\varepsilon}(M,g) \coloneqq \inf_{F\in\mathcal{B}^{\Gamma}_m} \max_{a\in\mathbb{B}^m} E_\varepsilon(F_a)$$

and

$$\mathcal{E}^{\Gamma}_m(M,[g]) = \mathcal{E}^{\Gamma}_m(M,g) \coloneqq \sup_{\varepsilon>0} \mathcal{E}^{\Gamma}_{m,\varepsilon}(M,g) = \lim_{\varepsilon\searrow 0} \mathcal{E}^{\Gamma}_{m,\varepsilon}(M,g).$$

By similar arguments as in the proof of Proposition 5.8, we get the inequality $\Lambda^{\Gamma}_1(M,[g]) \leq 2\mathcal{E}^{\Gamma}_m(M,[g]) < \infty$, for every m sufficiently large (cf. KKMS, Lemma 3.3 and Proposition 3.4).

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(15) Note that we use the index m instead of n for the equivariant setting, because we use $n+1$ for the dimension of the target space $\mathbb{R}^{n+1}$ (see e.g. Theorem 5.17 below).

5.2. Min-max characterization of the first conformal eigenvalue

We now show that the inequality $\Lambda_1^\Gamma(M,[g]) \leq 2\mathcal{E}_m^\Gamma(M,[g])$ is actually an equality for every m sufficiently large, obtaining the following min-max characterization for the first conformal eigenvalue.

THEOREM 5.9 (Karpukhin and Stern, 2024c, Theorem 1.3, KKMS, Theorem 3.11)
For every $m \in \mathbb{N}$ sufficiently large, if $\Lambda_1^\Gamma(M,[g]) > 8\pi$, then we have

$$\Lambda_1^\Gamma(M,[g]) = 2\mathcal{E}_m^\Gamma(M,[g]).$$

Remark 5.10. — Karpukhin and Stern (2024c, Theorem 1.13) find a min-max characterization for the second conformal eigenvalue, but it is still not known if there is an analogous result for $\Lambda_k(M,[g])$ with $k > 2$.

We keep discussing the Γ-equivariant case, but the proofs are similar to the case without equivariance and the reader interested in the nonequivariant setting can ignore the group Γ for the rest of the section. Note that in (KKMS, Section 3), the authors adapt the proofs in (Karpukhin and Stern, 2024c) to the equivariant setting, simplifying some of the arguments. Figure 8 contains a schematic representation of the main elements in the proof presented in this section.

$$\begin{tikzcd}
\begin{array}{c}\textbf{Perturbed energy}\\ \text{(Lemma 5.11)}\end{array} \arrow[r, "\text{(Proposition 5.15)}"'] & \begin{array}{c}\textbf{Min-max energy}\\ \text{(Theorem 5.17)}\end{array} \\
\mathcal{E}^\Gamma_{m,\varepsilon}(M,g) = \inf\limits_{F\in\mathcal{B}^\Gamma_m}\max\limits_{a\in\mathbb{B}^m} E_\varepsilon(F_a) \arrow[r, "\varepsilon\to 0"] \arrow[d, equal] & \mathcal{E}^\Gamma_m(M,[g]) \arrow[d, equal] \\
E_\varepsilon(u_{m,\varepsilon}) & E(u_m) \\
u_{m,\varepsilon}\colon M \to \mathbb{R}^{m|\Gamma|} \arrow[r, "\text{smoothly}"] & u_m\colon M \to \mathbb{S}^n \subset \mathbb{R}^{m|\Gamma|-1} \text{ harmonic} \\
g_{m,\varepsilon} := \dfrac{1-|u_{m,\varepsilon}|^2}{2\varepsilon^2} g \arrow[r, "\text{smoothly}"] & g_m := \dfrac{1}{2}|du_m|^2 g \\
\overline{\lambda}_1(M,g_{m,\varepsilon}) \sim \Lambda_1^\Gamma(M,[g]) & 2\mathcal{E}^\Gamma_m(M,[g]) = 2E(u_m) = \overline{\lambda}_1(M,g_m) = \Lambda_1^\Gamma(M,[g])
\end{tikzcd}$$

FIGURE 8. Schematic representation of the proof of the min-max characterization of the first conformal eigenvalue. This scheme holds for every m sufficiently large, namely the energies $\mathcal{E}_m^\Gamma(M,[g])$ stabilize equal to $\Lambda_1^\Gamma(M,[g])$ as $m \to \infty$.

As a byproduct of the proof of Theorem 5.9, we get a proof of Theorem 5.1. Indeed, we aim to obtain a harmonic map u_m whose associated metric $g_m := \frac{1}{2}|du_m|^2 g$ maximizes $\overline{\lambda}_1$ in its conformal class (for every m sufficiently large) and such that $\Lambda_1^\Gamma(M,[g]) = \overline{\lambda}_1(M,g_m) = 2E(u_m) = 2\mathcal{E}_m^\Gamma(M,[g])$ (see Theorem 5.17). To do this, we first study the perturbed problem, we get maps $u_{m,\varepsilon}$ and metrics $g_{m,\varepsilon}$, and then we prove smooth convergence as $\varepsilon \to 0$.

The first step is to fix $m \in \mathbb{N}$, and consider critical points $u_{m,\varepsilon} \in W^{1,2}_\Gamma(M, \mathbb{R}^{m|\Gamma|})$ for $\varepsilon > 0$. The existence of such critical points follows from standard arguments in the calculus of variations, given that E_ε satisfies a Palais–Smale condition (see KKMS, Proposition 3.1(iii)):

Lemma 5.11 (KKMS, Lemma 3.2 and Propositions 3.1 and 3.6)

For every $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists a smooth Γ-equivariant critical point $u_{m,\varepsilon} \in W^{1,2}_\Gamma(M, \mathbb{R}^{m|\Gamma|})$ for E_ε that realizes the perturbed min-max energy $\mathcal{E}^\Gamma_{m,\varepsilon}(M,g)$, i.e.,

$$E_\varepsilon(u_{m,\varepsilon}) = \mathcal{E}^\Gamma_{m,\varepsilon}(M,g).$$

Moreover, we have that:

(i) *$u_{m,\varepsilon}$ solves the Euler–Lagrange equation $\Delta_g u_{m,\varepsilon} + \frac{1-|u_{m,\varepsilon}|^2}{\varepsilon^2} u_{m,\varepsilon} = 0$ for E_ε;*
(ii) *$|u_{m,\varepsilon}| < 1$, and in particular the metric $g_{m,\varepsilon} := \frac{1-|u_{m,\varepsilon}|^2}{2\varepsilon^2} g \in [g]$ is well defined;*
(iii) *the Γ-equivariant Morse index $\operatorname{ind}^\Gamma_{E_\varepsilon}(u_{m,\varepsilon})$ of $u_{m,\varepsilon}$ is at most m;*
(iv) *there is a universal constant $C > 0$ such that, for every sufficiently large m,*

$$\lambda_1(M, g_{m,\varepsilon}) \geq 2 - \frac{C}{m}.$$

Remark 5.12. — Note that our definition of $g_{m,\varepsilon}$ differs from that in (KKMS, Proposition 3.6) by a factor of $\frac{1}{2}$. As a consequence, we have a different constant in the estimate in item (iv).

Proof. — We give a brief outline of the proof. The existence of a smooth critical point $u_{m,\varepsilon}$ satisfying the desired index bound follows from standard min-max methods (see e.g. Ambrosetti and Malchiodi, 2007, Chapter 8), since E_ε satisfies the Palais–Smale condition, and from elliptic regularity (cf. KKMS, Proposition 3.1 and Lemma 3.2). The index bound holds because the maximum in the definition of $\mathcal{E}^\Gamma_{m,\varepsilon}(M,g)$ is over an m-dimensional parameter space. The bound $|u_{m,\varepsilon}| < 1$ is deduced using the inequality $\Delta_g(1-|u_{m,\varepsilon}|^2) \geq -2\frac{1-|u_{m,\varepsilon}|^2}{\varepsilon^2}|u_{m,\varepsilon}|^2$ and the elliptic maximum principle.

We are left with (iv), which is the most delicate point, and is derived from the index bound $\operatorname{ind}^\Gamma_{E_\varepsilon}(u_{m,\varepsilon}) \leq m$. The idea is to use the constant vector fields in $\mathbb{R}^{m|\Gamma|}$ and the first eigenfunction of $\Delta_{g_{m,\varepsilon}}$ to construct a number of negative directions for the second derivative of E_ε at $u_{m,\varepsilon}$, whose existence violates the index bound $\operatorname{ind}^\Gamma_{E_\varepsilon}(u_{m,\varepsilon}) \leq m$ unless $\lambda_1(M, g_{m,\varepsilon}) \geq 2 - \frac{C}{m}$. See (KKMS, Proposition 3.6) for the details. □

A consequence of the previous lemma is that the metric $g_{m,\varepsilon}$ from the statement is close to being a maximizer of the first renormalized eigenvalue $\overline{\lambda}_1$ in the conformal class $[g]$, when m is sufficiently large and ε is sufficiently small.

Corollary 5.13. — *Given a metric $g_{m,\varepsilon}$ as in Lemma 5.11, we have that*

$$\left(2 - \frac{C}{m}\right) \mathcal{E}^\Gamma_{m,\varepsilon}(M,g) \leq \overline{\lambda}_1(M, g_{m,\varepsilon}) \leq \Lambda^\Gamma_1(M,[g]).$$

Remark 5.14. — Since $\lim_{\varepsilon\to 0} 2\mathcal{E}^{\Gamma}_{m,\varepsilon}(M,g) = 2\mathcal{E}^{\Gamma}_{m}(M,[g]) \geq \Lambda_1^{\Gamma}(M,[g])$, we get

$$\left(1-\frac{C}{2m}\right)\Lambda_1^{\Gamma}(M,[g]) \leq \lim_{\varepsilon\to 0}\overline{\lambda}_1(M,g_{m,\varepsilon}) \leq \Lambda_1^{\Gamma}(M,[g]). \tag{14}$$

In this sense, we say that $g_{m,\varepsilon}$ is close to be a maximizer for $\overline{\lambda}_1$ in $[g]$ if m is large and ε is small enough.

Proof. — Observe that

$$\begin{aligned}
&\mathcal{E}^{\Gamma}_{m,\varepsilon}(M,g) = E_\varepsilon(u_{m,\varepsilon}) = \int_M \frac{1}{2}|du_{m,\varepsilon}|^2 + \frac{(1-|u_{m,\varepsilon}|^2)^2}{4\varepsilon^2}\,dv_g\\
&= \int_M \frac{(1-|u_{m,\varepsilon}|^2)}{2\varepsilon^2}|u_{m,\varepsilon}|^2 + \frac{(1-|u_{m,\varepsilon}|^2)^2}{4\varepsilon^2}\,dv_g = \int_M \frac{(1-|u_{m,\varepsilon}|^2)}{2\varepsilon^2}\cdot\frac{(1+|u_{m,\varepsilon}|^2)}{2}\,dv_g\\
&\leq \int_M \frac{(1-|u_{m,\varepsilon}|^2)}{2\varepsilon^2}\,dv_g = \mathrm{area}(M,g_{m,\varepsilon}),
\end{aligned}$$

where we integrated by parts and we used (i) and (ii) from Lemma 5.11. As a result, using (iv), we conclude the desired inequality

$$\overline{\lambda}_1(M,g_{m,\varepsilon}) = \lambda_1(M,g_{m,\varepsilon})\mathrm{area}(M,g_{m,\varepsilon}) \geq \left(2-\frac{C}{m}\right)\mathcal{E}^{\Gamma}_{m,\varepsilon}(M,g_{m,\varepsilon}). \qquad \square$$

The next step is to show that, fixed $m\in\mathbb{N}$ sufficiently large, the maps $u_{m,\varepsilon}\colon M\to\mathbb{R}^{m|\Gamma|}$ converge smoothly as $\varepsilon\to 0$ to a harmonic map $u_m\colon M\to\mathbb{S}^{m|\Gamma|-1}\subset\mathbb{R}^{m|\Gamma|}$, whose Dirichlet energy realizes the energy $\mathcal{E}^{\Gamma}_{m}(M,[g])$.

Proposition 5.15 (KKMS, Propositions 3.6 and 3.9). — *For any $m\in\mathbb{N}$ and $\varepsilon>0$, let $u_{m,\varepsilon}$ be a map satisfying the properties in Lemma 5.11. Then, for every sufficiently large $m\in\mathbb{N}$, up to extracting a subsequence the maps $u_{m,\varepsilon}$ converge smoothly as $\varepsilon\to 0$ to a smooth Γ-equivariant harmonic map $u_m\colon M\to\mathbb{S}^{m|\Gamma|-1}$ such that*

$$E(u_m) = \mathcal{E}^{\Gamma}_{m}(M,[g]), \quad \mathrm{ind}^{\Gamma}_{E}(u_m)\leq m, \quad |du_m|^2_g \leq C,$$

where $C>0$ is a constant independent of m. In particular, the associated metrics $g_{m,\varepsilon}$ from Lemma 5.11 converge smoothly as $\varepsilon\to 0$ to the (well-defined, smooth with possibly finitely many conical singularities) metric $g_m := \frac{1}{2}|du_m|^2_g g\in[g]$, as $\varepsilon\to 0$.

Remark 5.16. — As a consequence of (14), the metric g_m in the statement satisfies

$$\left(1-\frac{C}{2m}\right)\Lambda_1^{\Gamma}(M,[g]) \leq \overline{\lambda}_1(M,g_m) \leq \Lambda_1^{\Gamma}(M,[g]). \tag{15}$$

Idea of the proof. — We give a rough idea of the proof, which requires some careful analysis. Let $u_{m,\varepsilon}\in W^{1,2}_{\Gamma}(M,\mathbb{R}^{m|\Gamma|})$ be the critical points of E_ε from Lemma 5.11. The idea is to show that bubbling cannot appear in the limit of $u_{m,\varepsilon}$ as $\varepsilon\to 0$. This follows from the bound $\lambda_1(g_{m,\varepsilon})\geq 2-C/m$ from Lemma 5.11(iv) and the assumption

$\Lambda_1^\Gamma(M,[g]) > 8\pi$ (see KKMS, Lemma 3.8 and Proposition 3.9). The Euler–Lagrange equation for $u_{m,\varepsilon}$ in Lemma 5.11(i) is also used. In particular, one shows that

$$\frac{1-|u_{m,\varepsilon}|^2}{\varepsilon^2} \le C, \tag{16}$$

for some $C = C(M,g)$, which together with the Euler–Lagrange equation implies smooth convergence to a harmonic map $u\colon M \to \mathbb{S}^{m|\Gamma|-1}$ (cf. KKMS, Proposition 3.9).

Note that the potential terms $\int_M \frac{(1-|u_{m,\varepsilon}|^2)^2}{\varepsilon^2}\, dv_g$ converge to 0 as $\varepsilon \to 0$ by (16). Thus

$$E(u_m) = \lim_{\varepsilon\to 0} E_\varepsilon(u_{m,\varepsilon}) = \lim_{\varepsilon\to 0} \mathcal{E}^\Gamma_{m,\varepsilon}(M,g) = \mathcal{E}^\Gamma_m(M,[g]).$$

Finally, by the smooth convergence $u_{m,\varepsilon} \to u_m$ as $\varepsilon \to 0$, using the Euler–Lagrange equation $-\Delta_g u_m = |du_m|_g^2 u_m$ for harmonic maps, we have

$$\begin{aligned}|du_m|_g^2 &= \langle -\Delta_g u_m, u_m\rangle = \lim_{\varepsilon\to 0}\langle -\Delta_g u_{m,\varepsilon}, u_{m,\varepsilon}\rangle = \lim_{\varepsilon\to 0} \frac{(1-|u_{m,\varepsilon}|^2)|u_{m,\varepsilon}|^2}{\varepsilon^2}\\ &= \lim_{\varepsilon\to 0}\frac{1-|u_{m,\varepsilon}|^2}{\varepsilon^2}.\end{aligned}$$

Recall that the terms $\frac{1-|u_{m,\varepsilon}|^2}{\varepsilon^2}$ are uniformly bounded by (16). Therefore, $|du_m|_g^2$ is also uniformly bounded independently of m, and we obtain that $g_{m,\varepsilon} = \frac{1-|u_{m,\varepsilon}|^2}{2\varepsilon^2} g$ converges smoothly to $g_m = \frac{1}{2}|du_m|_g^2 g$ as $\varepsilon \to 0$. ☐

The final step consists in proving that the metrics $g_m = \frac{1}{2}|du_m|^2 g$ are not only close to maximizing the first Γ-equivariant renormalized eigenvalue in $[g]$ (cf. (15)), but actually maximize it for every sufficiently large m. In particular, the following results prove Theorems 5.1 and 5.9.

Theorem 5.17 (KKMS, Lemma 3.10 and Theorem 3.11). — *Let $u_m\colon M \to \mathbb{S}^{m|\Gamma|-1}$, $m \in \mathbb{N}$, be any family of Γ-equivariant harmonic maps satisfying the properties of Proposition 5.15. Then, for every m sufficiently large, the metric $g_m := \frac{1}{2}|du_m|^2 g$ maximizes $\overline{\lambda}_1$ among Γ-equivariant metrics in $[g]$. In particular, we obtain*

$$\Lambda_1^\Gamma(M,[g]) = \overline{\lambda}_1(M,g_m) = 2E(u_m) = 2\mathcal{E}^\Gamma_m(M,[g]).$$

Moreover, there exists a number $n = n(M,[g])$ such that u_m takes values in a totally geodesic sphere of dimension n, i.e., $u_m(M) \subset \mathbb{S}^n \subset \mathbb{S}^{m|\Gamma|-1}$.

Sketch of the proof. — One first bounds the dimension of the image of u_m. Assuming it is not bounded, the space spanned by the coordinate functions of u_m has dimension going to infinity as $m \to \infty$. However, the coordinate functions of u_m are eigenfunctions with eigenvalue 2 of Δ_{g_m}. As a result, we get $\lim_{m\to\infty} \lambda_k(g_m) \le 2$ for all $k \in \mathbb{N}$. By the Bochner identity for sphere-valued harmonic maps (see Karpukhin and Stern, 2024a, Section 3), we have that $|du_m|_g^2$ is uniformly bounded in $W^{1,2}(M)$ and thus, up to passing to a subsequence, we find $\rho \in W^{1,2}(M)$ such that $\frac{1}{2}|du_m|_g^2 \to \rho$ in L^p for all $p \in [1,\infty)$.

In particular, $\lambda_k(M, \rho g) = \lim_{m\to\infty} \lambda_k(g_m) \in [0, 2]$. However, this contradicts(16) the fact that $\lambda_k(M, \rho g) \to \infty$ as $k \to \infty$.

Next, one shows that $\lambda_1(M, g_m) = 2$. The proof is similar to Lemma 5.11(iv). Namely, one uses the constant vectors on $\mathbb{R}^{m|\Gamma|}$ as test functions for the second derivative of the energy E at u_m, and the index bound $\operatorname{ind}_E^\Gamma(u_m) \leq m$. However, we can now use as test functions the constant vectors orthogonal to the image $u_m(M)$, which is contained in a sphere $\mathbb{S}^n$ of bounded dimension independent of m. This helps improve the estimate to $\lambda_1(M, g_m) \geq 2$. The inequality $\lambda_1(M, g_m) \leq 2$ follows from the fact that the coordinates of u_m are eigenfunctions with eigenvalue 2, therefore we get the desired equality $\lambda_1(M, g_m) = 2$. This concludes the proof, since

$$\begin{aligned}\Lambda_1^\Gamma(M, [g]) \geq \overline{\lambda}_1(M, g_m) = \lambda_1(M, g_m)\operatorname{area}(M, g_m) = \int_M |du_m|_g^2 \, dv_g \\ = 2E(u_m) = 2\mathcal{E}_m^\Gamma(M, [g]) \geq \Lambda_1^\Gamma(M, [g]).\end{aligned}$$

□

Let us conclude this section with a corollary of the min-max characterization of $\Lambda_1^\Gamma(M, [g])$, which will be useful in the next section.

Corollary 5.18 (cf. Karpukhin, Nahon, Polterovich, and Stern, 2025, Proposition 3.1, Karpukhin, Petrides, and Stern, 2025, Proposition 2.2)

Under the assumptions of Theorem 5.1, there exists $n = n(M, [g]) \in \mathbb{N}$ such that, for every continuous linear functional $T\colon W^{1,2}(M, \mathbb{R}^{n+1}) \to \mathbb{R}$, there exists a map $f \in W^{1,2}(M, \mathbb{S}^n)$ such that $T(f) = 0 \in \mathbb{R}^{n+1}$ and $\int_M |df|_g^2 \, dv_g \leq \Lambda_1^\Gamma(M, [g])$.

We do not give the proof of this result, but the key idea is that, for every $F \in \mathcal{B}_m^\Gamma$, we can choose $a \in \mathbb{B}^{m+1}$ such that $T(F_a) = 0$. Using the min-max characterization of $\Lambda_1^\Gamma(M, [g])$, this leads to the desired result. Note that this is the same principle that we used above with $T(f) = \int_M f \, dv_g$, generalized to any linear operator T.

6. Existence of an eigenvalue maximizer

The following theorem asserts the existence of a metric maximizing the first renormalized Laplace eigenvalue among the symmetric metrics on any equivariant basic reflection surface (see Section 4.3). This section is devoted to its proof.

Theorem 6.1 (KKMS, Theorem 8.7). — *Let (M, τ) be a closed Γ-equivariant basic reflection surface. There then exists a metric $g_0 \in \mathrm{Met}_\Gamma(M)$ (smooth away from finitely many conical singularities) maximizing the first renormalized Laplace eigenvalue on M among Γ-equivariant metrics.*

---

(16) Although ρg is not smooth, its spectrum is still well defined, is discrete, and converges to infinity, see e.g. (Kokarev, 2014, Proposition 1.3).

Remark 6.2. — A version of this theorem works for the first renormalized Steklov eigenvalue on surfaces with boundary, but does not cover all possible equivariant basic reflection surfaces with boundary (see KKMS, Theorem 9.15). As mentioned in Section 4.4, the case when $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, which is sufficient to get all possible topologies and prove Theorem B together with Theorem 4.12, is covered by KKMS.

6.1. Degenerations at the boundary of the moduli space of conformal classes

Let g_n be a maximizing sequence of metrics in $\mathrm{Met}_\Gamma(M)$ for $\Lambda_1^\Gamma(M)$, that is to say $\overline{\lambda}_1(M, g_n)$ goes to $\Lambda_1^\Gamma(M)$ as $n \to \infty$. Thanks to Theorem 5.17, without loss of generality, we can assume that g_n maximizes the first renormalized Laplace eigenvalue in its conformal class, i.e., $\overline{\lambda}_1(M, g_n) = \Lambda_1^\Gamma(M, [g_n])$. If g_n converges (in some good sense) to a metric g_0 on M, then we conclude that g_0 is the desired maximizing metric by arguing that $\overline{\lambda}_1(M, g_0) = \lim_{n\to\infty} \overline{\lambda}_1(M, g_n) = \lim_{n\to\infty} \Lambda_1^\Gamma(M, [g_n]) = \Lambda_1^\Gamma(M)$.

However, a priori, the conformal classes $[g_n]$ could converge to the boundary of the space of (equivariant) conformal classes on M, as we will see in the proof of Theorem 6.7. In this case, one can show that the sequence of Riemannian surfaces (M, g_n) converges to a Riemannian manifold (M', g'), which is a *topological degeneration* of M, as defined below.

Definition 6.3 (cf. KKMS, Definition 6.1). — *Let M, M' be two closed*(17) *Riemannian surfaces and let Γ be a finite group acting on M and M' by isometries. We say M' is a* topological Γ-degeneration *of M, and write $M' \overset{\Gamma}{\curlyeqprec} M$, if M' is obtained from M via a Γ-equivariant surgery. That is, if M' is obtained from M by removing a Γ-equivariant set $\mathfrak{U}$ consisting of a disjoint union of annuli that are not homotopically trivial and gluing a disk to each boundary component of $M \setminus \mathfrak{U}$ (two disks for each annulus removed).*

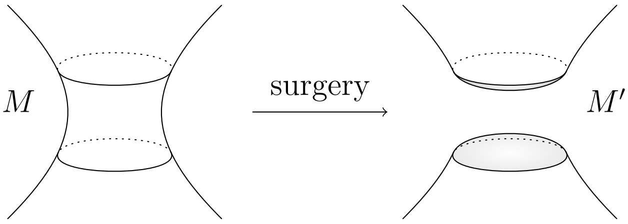

Figure 9. Representation of one of the surgeries described in Definition 6.3, consisting of removing an annulus and gluing in two disks. Note that a Γ-equivariant surgery consists of multiple surgeries in such a way that the procedure is equivariant.

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(17) The definition in KKMS requires the surfaces to be connected. We remove this assumption to simplify the exposition.

Remark 6.4. — Note that these degenerations are the same ones that occur when one tries to find a surface M' minimizing the area in the isotopy class of a given surface M, as described by Meeks, Simon, and Yau (1982). Moreover, these are the same pathologies that one encounters when applying a Simon–Smith min-max procedure to obtain minimal surfaces, cf. (De Lellis and Pellandini, 2010, Section 2.2).

Remark 6.5. — For the entirety of this exposition, we discuss only the case of orientable surfaces. In KKMS, nonorientable surfaces are also considered. In that case, the definition of surgery allows not only for annuli to be removed but also Möbius strips.

Remark 6.6. — It is possible to describe all Γ-equivariant surgeries of a Γ-equivariant basic reflection surface, see (KKMS, Propositions 6.4, 6.5, 6.6). In particular, if (M, τ) is a Γ-equivariant basic reflection surface and M' is a topological Γ-degeneration of M, then (M', τ) is also a Γ-equivariant basic reflection surface.

The following theorem provides a condition which precludes the convergence of maximizing sequence g_n for $\Lambda_1^\Gamma(M)$ (as introduced above) to the boundary of the moduli space of conformal classes on M. Namely, it is sufficient to check that the value of Λ_1^Γ on M is greater than its value on all topological Γ-degenerations.

THEOREM 6.7 (Petrides, 2014, Theorem 2, KKMS, Theorems 7.6 and 7.7)
Let (M, τ) be a Γ-equivariant basic reflection surface. If

$$\Lambda_1^\Gamma(M) > \max\{\Lambda_1^\Gamma(M') \ :\ M' \overset{\Gamma}{\precsim} M,\ M' \neq M\}, \tag{17}$$

then there exists a Γ-equivariant metric g_0 on M realizing $\Lambda_1^\Gamma(M)$. This metric is smooth up to finitely many conical singularities.

Ideas of the proof. — First observe that, if the statement holds and M is not a topological sphere, then $\Lambda_1^\Gamma(M) > 8\pi$. Indeed, by the characterization of all possible topological Γ-degenerations in (KKMS, Proposition 6.4, 6.5 and 6.6) (see also the discussion at the beginning of the proof of KKMS, Theorem 8.7), then starting from M and performing a succession of Γ-equivariant surgeries, it is possible to obtain a topological sphere, for which we know that $\Lambda_1^\Gamma(\mathbb{S}^2) = 8\pi$ (see Example 4.9). In particular, we get that $\Lambda_1^\Gamma(M) > \Lambda_1^\Gamma(\mathbb{S}^2) = 8\pi$ and we can apply Theorem 5.1.

Let g_n be a sequence of metrics such that $\Lambda_1^\Gamma(M, [g_n]) \to \Lambda_1^\Gamma(M)$. By Theorem 5.1, we can assume $\overline{\lambda}_1(M, g_n) = \Lambda_1^\Gamma(M, [g_n])$. The idea is that, up to a subsequence, (M, g_n) converges (in a suitable sense) to a topological Γ-degeneration (M', g') of M, such that

$$\Lambda_1^\Gamma(M') \geq \overline{\lambda}_1(M', g') \geq \limsup_{n\to\infty} \overline{\lambda}_1(M, g_n) = \limsup_{n\to\infty} \Lambda_1^\Gamma(M, [g_n]) = \Lambda_1^\Gamma(M).$$

By the assumption (17), this implies that the degeneration M' has to be trivial equal to M and $g_0 = g'$ is the desired maximizing metric.

The proof of what we just stated is based on two steps:

- First, one needs to appeal to a compactness result to show that, up to a subsequence, (M, g_n) converges to a topological Γ-degeneration (M', g'). The convergence is roughly defined as local smooth convergence away from the surgery regions, up to composing with diffeomorphisms of M. We refer to (KKMS, Theorem 7.4) for a precise statement.
- The second step is to prove $\overline{\lambda}_1(M', g') \geq \limsup_{n\to\infty} \overline{\lambda}_1(M, g_n)$, the upper semicontinuity along the sequence. This, a priori, is not always true and follows from a careful analysis of g_n as $n \to \infty$, which was first performed in (Petrides, 2014, Section 4). We refer to (KKMS, Theorem 7.6) for the proof, but remark here that the inequality $\Lambda_1^\Gamma(M) > 8\pi$ is used to prevent the formation of bubbles and concentration of energy in the surgery regions via Hersch-type arguments. □

6.2. New technical ingredients in KKMS

Thanks to the previous theorem, the proof of Theorem 6.1 reduces to showing the strict inequality (17). In order to do this, Karpukhin, Kusner, McGrath, and Stern (KKMS) introduced a new method. Indeed, proving this strict inequality turns out to be very delicate. In the nonequivariant case, there had been many failed attempts before KKMS. Those attempts tried to prove the inequality directly, by taking a metric g' on M' and explicitly constructing a metric g on M, where M' is obtained by M through surgeries along some necks, such that $\overline{\lambda}_1(M, g) > \overline{\lambda}_1(M', g')$. Unfortunately, this approach turned out not to be fruitful. We now explain the new techniques introduced in KKMS. We present a sketch of the proof, with simplifications by Karpukhin, Petrides, and Stern (2025) and personal communication with the authors.

Sketch of the proof of Theorem 6.1. — Assume by contradiction that there exists a Γ-equivariant basic reflection surface (M, τ) and a topological Γ-degeneration M' of M such that $\Lambda_1^\Gamma(M) \leq \Lambda_1^\Gamma(M')$. To simplify the discussion, let us only discuss the case when M' is obtained from M via surgery on a single neck. In other words, the surface M' is obtained from M by removing an annulus (which is not homotopically trivial) and gluing in two disks. Since $\overset{\Gamma}{\prec}$ is a well-defined partial ordering, arguing by induction, we can assume that M' admits a maximizing metric g', i.e., $\overline{\lambda}_1(M', g') = \Lambda_1^\Gamma(M')$.

Since M is obtained by M' via surgery on one neck, M can be recovered from M' by removing two disks around two points $p, q \in M'$ and gluing in an annulus. More precisely, we can consider two disks $D_\varepsilon(p)$, $D_\varepsilon(q)$ in M and consider the metric g_ε on M (for $\varepsilon > 0$ sufficiently small) obtained by removing from M' the two disks $D_\varepsilon(p), D_\varepsilon(q)$ and gluing in the flat cylinder $C_\varepsilon = \mathbb{S}^1_\varepsilon \times [0, 2\varepsilon]$ to $\partial D_\varepsilon(p)$ and $\partial D_\varepsilon(q)$ (see Figure 10). Note that g_ε is not necessarily smooth, however, it is smooth up to a conformal factor and this is sufficient for our purposes.

Thanks to Corollary 5.18, there exists a Γ-equivariant map $f_\varepsilon \in W^{1,2}(M, \mathbb{S}^n)$, for some n independent of ε (a priori n depends on $[g_\varepsilon]$, but the dependence on ε can be

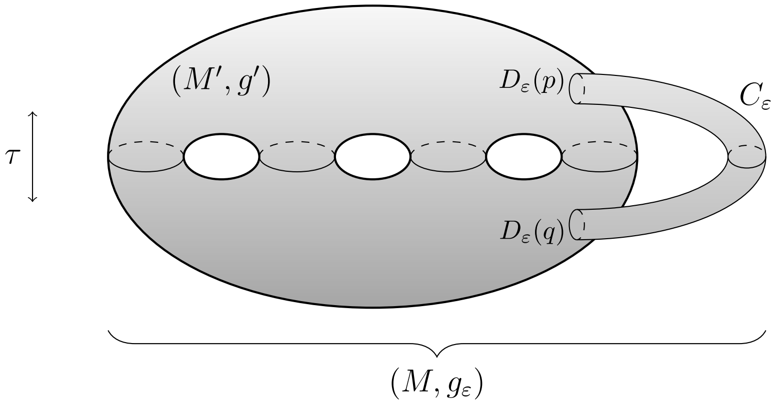

FIGURE 10. The surface (M, g_ε) is obtained from (M', g') by gluing in a flat cylinder. In the equivariant case, this operation has to be done equivariantly and may involve gluing multiple necks. For simplicity, we discuss only the case in which one neck is added.

removed, cf. Karpukhin, Petrides, and Stern, 2025), such that

$$\int_M |df_\varepsilon|^2 \, dv_{g_\varepsilon} \le \Lambda_1^\Gamma(M, [g_\varepsilon]) \le \Lambda_1^\Gamma(M') = \overline{\lambda}_1(M', g') \tag{18}$$

and

$$\int_{M'} \hat{f}_\varepsilon \, dv_{g'} = 0, \tag{19}$$

where $\hat{f}_\varepsilon \colon M' \to \mathbb{R}^{n+1}$ is obtained by $f_\varepsilon \colon M \to \mathbb{S}^n$ by harmonic extension on $D_\varepsilon(p), D_\varepsilon(q)$. More precisely, the map $\hat{f}_\varepsilon$ is defined as

$$\hat{f}_\varepsilon \coloneqq \begin{cases} f_\varepsilon & \text{on } M' \setminus D_\varepsilon(p) \cup D_\varepsilon(q) \\ \text{harmonic extension of } f_\varepsilon|_{\partial D_\varepsilon(p)} & \text{on } D_\varepsilon(p) \\ \text{harmonic extension of } f_\varepsilon|_{\partial D_\varepsilon(q)} & \text{on } D_\varepsilon(q). \end{cases}$$

Note that we applied Corollary 5.18 on (M, g_ε) with $T(f) = \int_{M'} \hat{f} \, dv_{g'}$.

Step 1 : In the regime when $\varepsilon \to 0$, we have(18)

- $\displaystyle\int_{M'} |d\hat{f}_\varepsilon|^2 \, dv_{g'} = \overline{\lambda}_1(M', g') + o(1);$
- $\displaystyle\int_{M'} |1 - |\hat{f}_\varepsilon|^2| \, dv_{g'} = o(1);$
- $\displaystyle\fint_{\partial D_\varepsilon(p)} \hat{f}_\varepsilon \, dv_{g'} - \fint_{\partial D_\varepsilon(q)} \hat{f}_\varepsilon \, dv_{g'} = o(1).$

---

(18) Here, the symbol $\fint$ denotes the average integral, that is the integral divided by the measure of the domain of integration.

Let us now sketch the proof of these estimates. First note that we decided to impose (19) in order to have a test function for $\lambda_1(M',g')$. Indeed, we get

$$\int_{M'}|d\hat{f}_\varepsilon|^2\,dv_{g'} \geq \lambda_1(M',g')\int_{M'}|\hat{f}_\varepsilon|^2\,dv_{g'}.$$

Moreover, since $|f_\varepsilon|\equiv 1$ and $\hat{f}_\varepsilon$ is either equal to f_ε or to its harmonic extension inside $D_\varepsilon(p)\cup D_\varepsilon(q)$, we have $|\hat{f}_\varepsilon|\leq 1$ and hence

$$\int_{M'}|1-|\hat{f}_\varepsilon|^2|\,dv_{g'} = \int_{D_\varepsilon(p)\cup D_\varepsilon(q)}|1-|\hat{f}_\varepsilon|^2|\,dv_{g'} = O(\varepsilon^2) = o(1).$$

Combining this with the previous equation gives

$$\int_{M'}|d\hat{f}_\varepsilon|^2\,dv_{g'} \geq \overline{\lambda}_1(M',g') + o(1).$$

Now observe that

$$\int_{M'}|d\hat{f}_\varepsilon|^2\,dv_{g'} = \int_M|df_\varepsilon|^2\,dv_{g_\varepsilon} + \|d\hat{f}_\varepsilon\|^2_{L^2(D_\varepsilon(p)\cup D_\varepsilon(q))} - \|df_\varepsilon\|^2_{L^2(C_\varepsilon)}. \tag{20}$$

The choice of $\hat{f}_\varepsilon$ as the harmonic extension of f_ε in the removed disks has been done in order to being able to estimate it well in terms of f_ε. Thanks to (KKMS, Lemma 8.11, applied with $L=2>\frac{3}{2}\ln(2)$), we indeed get

$$\|d\hat{f}_\varepsilon\|^2_{L^2(D_\varepsilon(p)\cup D_\varepsilon(q))} - \|df_\varepsilon\|^2_{L^2(C_\varepsilon)} \leq C\|df_\varepsilon\|^2_{L^2(C_\varepsilon)}, \tag{21}$$

for some universal constant $C>0$. Moreover, using the assumption $\Lambda_1^\Gamma(M)\leq\Lambda_1^\Gamma(M')$, it is possible to estimate the Dirichlet energy of f_ε on the neck C_ε as follows.

Lemma 6.8. — *There exists $C>0$ such that*

$$\|df_\varepsilon\|^2_{L^2(C_\varepsilon)} \leq C\varepsilon^2.$$

Proof of the lemma. — Using $|\hat{f}_\varepsilon|\leq 1$ together with (19), we have that

$$A_\varepsilon := \fint_{M'\setminus(D_\varepsilon(p)\cup D_\varepsilon(q))}\hat{f}_\varepsilon\,dv_{g'}$$

is of order ε^2. Therefore, using $\hat{f}_\varepsilon - A_\varepsilon$ as test function in the variational characterization (7) of the first Neumann eigenvalue $\lambda_1^N(M'\setminus(D_\varepsilon(p)\cup D_\varepsilon(q)),g')$, we get

$$\begin{aligned}\int_{M\setminus(D_\varepsilon(p)\cup D_\varepsilon(q))}|df_\varepsilon|^2\,dv_{g_\varepsilon} &= \int_{M'\setminus(D_\varepsilon(p)\cup D_\varepsilon(q))}|d\hat{f}_\varepsilon|^2\,dv_{g'}\\ &\geq \lambda_1^N(M'\setminus(D_\varepsilon(p)\cup D_\varepsilon(q)),g')\int_{M'\setminus(D_\varepsilon(p)\cup D_\varepsilon(q))}|\hat{f}_\varepsilon - A_\varepsilon|^2\,dv_{g'}\\ &\geq \lambda_1^N(M'\setminus(D_\varepsilon(p)\cup D_\varepsilon(q)),g')\ (\mathrm{area}(M'\setminus(D_\varepsilon(p)\cup D_\varepsilon(q)),g') - C\varepsilon^2)\\ &\geq (\lambda_1(M',g') - C\varepsilon^2)(\mathrm{area}(M',g') - C\varepsilon^2) \geq \overline{\lambda}_1(M',g') - C\varepsilon^2,\end{aligned}$$

where we used that $\lambda_1^N(M' \setminus (D_\varepsilon(p) \cup D_\varepsilon(q)), g') \geq \lambda_1(M', g') - C\varepsilon^2$ by (Matthiesen and Siffert, 2019, Theorem 5.1). Combining this inequality with (18) and the fact that $\|df_\varepsilon\|^2_{L^2(C_\varepsilon)} = \|df_\varepsilon\|^2_{L^2(M)} - \|df_\varepsilon\|^2_{L^2(M\setminus(D_\varepsilon(p)\cup D_\varepsilon(q)))}$, we get the desired result. $\square$

Remark 6.9. — The estimate in Lemma 6.8 is stronger than the one used in (KKMS, see Lemma 8.13 therein), and simplifies some of the arguments. It was used in (Karpukhin, Petrides, and Stern, 2025, see (4.9) in their paper), but the proof presented there is different, uses a different definition of $\hat{f}_\varepsilon$, and is not always possible in the equivariant setting. However, as explained by Karpukhin, Petrides, and Stern in personal communication, the proof sketched above does work in the equivariant setting.

Using this better estimate in place of the one in (KKMS, Lemma 8.13), it is possible to fix the length of the neck $L = 2$, instead of having to take it going to ∞ as $\varepsilon \to 0$ (see KKMS, p. 113).

Therefore, Lemma 6.8 together with (20) and (21) gives

$$\int_{M'} |d\hat{f}_\varepsilon|^2 \, dv_{g'} \leq \int_M |df_\varepsilon|^2 \, dv_{g_\varepsilon} + C\varepsilon^2 \leq \overline{\lambda}_1(M', g') + o(1).$$

Combining this with the reverse inequality (18) obtained earlier, we thus get

$$\int_{M'} |d\hat{f}_\varepsilon|^2 \, dv_{g'} = \overline{\lambda}_1(M', g') + o(1).$$

Finally, the last point in the statement of Step 1 follows by

$$\begin{aligned} \left| \fint_{\partial D_\varepsilon(p)} \hat{f}_\varepsilon \, dv_{g'} - \fint_{\partial D_\varepsilon(q)} \hat{f}_\varepsilon \, dv_{g'} \right| &= \left| \fint_{\partial D_\varepsilon(p)} f_\varepsilon \, dv_{g'} - \fint_{\partial D_\varepsilon(q)} f_\varepsilon \, dv_{g'} \right| \\ &\leq \frac{1}{\varepsilon} \|df_\varepsilon\|_{L^2(C_\varepsilon)} \mathrm{area}(C_\varepsilon)^{1/2} \leq C\varepsilon^2 = o(1). \end{aligned}$$

Step 2 : Define a map $v_\varepsilon \colon M' \to \mathbb{R}^{n+1}$ to be the projection of $\hat{f}_\varepsilon$ onto the space of $\lambda_1(M', g')$-eigenfunctions. Then v_ε also has the properties of Step 1, namely

- $\displaystyle\int_{M'} |dv_\varepsilon|^2 \, dv_{g'} = \overline{\lambda}_1(M', g') + o(1);$
- $\displaystyle\int_{M'} |1 - |v_\varepsilon|^2| \, dv_{g'} = o(1);$
- $\displaystyle\fint_{\partial D_\varepsilon(p)} v_\varepsilon \, dv_{g'} - \fint_{\partial D_\varepsilon(q)} v_\varepsilon \, dv_{g'} = o(1).$

This step is useful to have nice convergence properties of the maps v_ε as $\varepsilon \to 0$ and obtain in the limit a map by first eigenfunctions, as described in Step 3 below. The idea to prove this is to write $\hat{f}_\varepsilon = v_\varepsilon + R_\varepsilon$ and prove that $\|R_\varepsilon\|^2_{W^{1,2}(M')} \leq C\varepsilon$. Observe that this estimate is better than the one in (KKMS, (8.16)) thanks to our use of Lemma 6.8 in place of (KKMS, Lemma 8.13).

Step 3 : Since the maps v_ε are uniformly bounded in $W^{1,2}(M',g')$ and their components are first eigenfunctions, we can extract a subsequence converging smoothly to a map $v\colon M' \to \mathbb{S}^n$ by first eigenfunctions. Moreover, we have that

$$v(p) - v(q) = \lim_{\varepsilon \to 0} \left(\fint_{\partial D_\varepsilon(p)} v_\varepsilon \, dv_{g'} - \fint_{\partial D_\varepsilon(q)} v_\varepsilon \, dv_{g'} \right) = 0.$$

Now recall that (M',g') maximizes $\overline{\lambda}_1$ among Γ-equivariant metrics. Therefore, by Theorem 4.8, the multiplicity of $\lambda_1(M',g')$ is 4 and there exists a minimal embedding $u\colon (M',g') \to \mathbb{S}^3$. As a result, if $\operatorname{genus}(M') \neq 1$, applying Proposition 4.10 we get a contradiction to the fact that $v(p) = v(q)$ for $p \neq q$, proving $\Lambda_1^\Gamma(M') < \Lambda_1^\Gamma(M)$. The inequality when M' has genus one is showed separately in (KKMS, p. 103), using that the only genus one minimal surface in $\mathbb{S}^3$ is the Clifford torus. This concludes the proof. ☐

Proof of Theorem A. — Thanks to Theorems 6.1 and 4.8, for every Γ-equivariant basic reflection surface (M,τ) there exists a minimal embedding $u\colon M \to \mathbb{S}^3$ by first Laplace eigenfunctions, with area less than 8π, and which is a doubling of an equatorial sphere. Applying this for different actions of the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ on a surfaces M with genus γ gives $\lfloor \frac{\gamma-1}{4} \rfloor + 1$ such embeddings which are nonisometric (see Section 4.2). For the proof of this final statement see (KKMS, Proposition 8.8). ☐

6.3. Maximizing metric without symmetries

The ideas introduced by KKMS to prove the strict inequality (17) in the equivariant setting have also turned out to be very useful for proving the existence of maximizing metrics on closed surface in the case without symmetries, as stated in Theorem C. However, since Proposition 4.10 is no longer available in the nonequivariant setting, a more careful analysis is necessary. In particular, note that:

– Petrides (2024, Theorem 0.1) proves the result for orientable surfaces. The paper uses some ideas from KKMS but is based on an approach via Ekeland's variational principle from (Petrides, 2025). The idea is to find a maximizing sequence of metrics and associated maps by first eigenfunctions that are almost harmonic and *almost conformal*. Then, the existence of a maximizer follows by performing a careful analysis of the limit of the sequence of metrics and maps. The method is very robust: it works for higher eigenvalues (under suitable assumptions), and its application to the equivariant setting might cover more general group actions than KKMS, e.g. the ones missing in the Steklov case as described in Remark 6.2.

– Karpukhin, Petrides, and Stern (2025, Theorem 1.2) cover both the orientable and nonorientable cases with a proof along the lines of KKMS. The proof works only for the first eigenvalue, but it is shorter and simpler, once one knows the min-max characterization of conformal eigenvalues discussed in Section 5. In order to bypass Proposition 4.10, it is necessary to attach the cylinder C_ε to M' more carefully (with a twist), in order to define a metric g_ε on M. Moreover, one needs to define the map $\hat{f}_\varepsilon$ more carefully.

Remark 6.10. — For the kth eigenvalue with $k \geq 2$, it is not true in general that there is a metric maximizing it which is smooth away from finitely many conical singularities. See (Karpukhin, Nadirashvili, Penskoi, and Polterovich, 2021), which also contains a nice survey of results about higher eigenvalues.

Remark 6.11. — The existence of a maximizing metric for the first renormalized Steklov eigenvalue is still open in general. However, KKMS proves the result for some topologies (see e.g. Corollary 1.11 therein), and it is expected that the methods described in this section will soon lead to a proof the general result.

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Abbreviations

KKMS Mikhail Karpukhin, Robert Kusner, Peter McGrath, and Daniel Stern (2024). "Embedded minimal surfaces in $\mathbb{S}^3$ and $\mathbb{B}^3$ via equivariant eigenvalue optimization", *preprint*. arXiv: `2402.13121 [math.DG]`.

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