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THE ENERGY OF DILUTE BOSE GASES
[after Fournais and Solovej]

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Contents

Introduction

Bosons are quantum particles present in nature either as fundamental particles (such as the photon) or as composite particles (such as the Helium 4 atom). It has been realized since the birth of quantum mechanics that such particles have very peculiar properties: in 1924, Bose and Einstein argued that a system of (identical) such particles must *condensate* at sufficiently low temperatures. Roughly speaking, condensation means that most particles in the system want to occupy the same quantum state. Bose–Einstein condensation has been observed experimentally for the first time in 1995 and led to the 2001 Nobel Prize in Physics.

A major insufficiency of the theoretical argument of Bose and Einstein is the fact that it applies only to *non-interacting* particles. When particles start to interact, their individual behaviour is much harder to understand due to correlations appearing between particles. A milestone in the study of systems of a large number of identical bosons (such a system is often referred to as a *Bose gas*) was the article of Bogoliubov (1947) who proposed an approximate model that is the basis of many subsequent works on the Bose gas. Putting Bogoliubov's approximation on solid mathematical grounds has been the topic of many works, starting from Ginibre (1968) and much later Lieb, Seiringer, and Yngvason (2005). Despite these advances, the mathematical understanding of Bose–Einstein condensation remains incomplete.

Another famous result concerning the Bose gas was a formula proposed by Lee, Huang, and Yang (1957) in a regime where Bogoliubov's method is expected to fail. Their formula is a two-term asymptotics of the energy per unit volume of a Bose gas, in the small density limit. The rigorous proof of this formula has been done in several steps. First, Dyson (1957) proved an upper bound for the first term in the asymptotics, and had a lower bound off by a factor $1/14$. The correct lower bound was proved by Lieb and Yngvason (1998) forty years later! The second term in the asymptotics was proved as an upper bound by Yau and Yin (2009). Finally, a matching lower bound for the second term in the asymptotics was proved by Fournais and Solovej (2020), achieving the full justification of the Lee–Huang–Yang formula. It is this last work that we review here.

The mathematical analysis of the Bose gas has a long history with many names attached to it. A good first reference is the book of Lieb, Seiringer, Solovej, and Yngvason (2005) about the works until 2005. After this date, there were many key results about the Bose gas, sometimes in regimes different from the one that we consider here (for instance, the *mean-field* regime of the *Gross–Pitaevskii* regime). We cannot give all the references attached to these very important results, but let us at least point out the review articles (Lewin, 2015; Schlein, 2022; Nam, 2023).

The result of Fournais and Solovej that we review here is partly the continuation of these previous results, and also has some new key insights that we will try to emphasize here. After their article, several works appeared that both generalize the scope of their result and simplify some of the arguments. In a subsequent article, Fournais and Solovej (2023b) extend their work to include more general interactions between particles (in particular the *hard-core* one). Then, Fournais, Girardot, Junge, Morin, and Olivieri (2024b) extended these results to two-dimensional bosons (while the result of Fournais and Solovej covered the three-dimensional case). The physically important case of *positive* temperature (the work of Fournais and Solovej deals with the zero-temperature case) was then tackled by Haberberger, Hainzl, Nam, Seiringer, and Triay (2023), with a simplified proof and extended result in Fournais, Girardot, Junge, Morin, Olivieri, and Triay (2026). A review of the work of Fournais and Solovej can be found in (Fournais and Solovej, 2023a), and a review video can be found in (Solovej, 2025).

This review is aimed towards general mathematicians. For those who are already familiar with the topic, we refer to the articles mentioned above. Here, we will focus on the basic formalism, problems, and tools around the Bose gas with a particular emphasis on the lower bound of the energy per unit volume in the dilute regime, which is the setting of Fournais and Solovej. As we already said, their result is a two-term asymptotics in this dilute regime. In this review, we will (partly) prove the first term in the asymptotics using their method, to illustrate the techniques and motivate the new ideas that they needed to go to the next order. Originally, the proof of the first term in the asymptotics is due to Lieb and Yngvason (1998), with a different method. The main sources for our presentation were of course the original article of Fournais and Solovej, but also, importantly, Fournais, Girardot, Junge, Morin, and Olivieri (2024a) and Fournais, Girardot, Junge, Morin, Olivieri, and Triay (2026).

This survey is structured as follows. In Section 1, we introduce the formalism of Bose gases, the definition of condensation, as well as the statement of the Lee–Huang–Yang asymptotics. In Section 2, we compute the ground state energy of *two* interacting particles which is the main building block to understand the dilute regime. Interestingly, this exercise already illustrates several key techniques such as the use of *spectral gaps*. In Section 3, we explain the Bogoliubov approximation, a key tool to understand interacting Bose gases. In Section 4, we state a technical but crucial tool in the analysis, which shows that in order to get asymptotics in a large box, it is enough to prove asymptotics in a much smaller box. In Section 5, we explain how to rigorously justify the Bogoliubov approximation to obtain first-order lower bounds to the ground state energy, with a major defect that even if the order of this lower bound is correct, the pre-factor is not. In Section 6, we explain another key idea which allows correcting this pre-factor: the renormalization of the interaction potential. Finally, in Section 7 we explain the broad ideas that allow to go to the next order and obtain the full Lee–Huang–Yang lower bound.

1. The model

We begin with the basic definitions that allow to state the main result that is the topic of this survey, the Lee–Huang–Yang formula of Theorem 1.12.

1.1. States and condensation

Definition 1.1 (Bosonic quantum states). — *Let $\Omega \subset \mathbb{R}^3$ be a measurable set and $N \in \mathbb{N}^*$. A* bosonic N-body quantum state *in Ω is a function in $L^2(\Omega^N, \mathbb{C})$ which is symmetric with respect to the exchange of the N variables in Ω^N (the space of such functions is denoted by $L^2_{\mathrm{sym}}(\Omega^N)$) and such that $\int_{\Omega^N} |\Psi|^2 = 1$.*

Example 1.2. — Let $\varphi \in L^2(\Omega, \mathbb{C})$ be such that $\int_\Omega |\varphi|^2 = 1$ (in other words, φ is a 1-body quantum state). Then, the function

$$\Psi(x_1, \dots, x_N) \coloneqq \varphi(x_1) \cdots \varphi(x_N)$$

is a bosonic N-body quantum state in Ω. Such a state is called a *product state*, denoted by $\varphi^{\otimes N}$.

Let $(\Psi_N)_{N \in \mathbb{N}^*}$ be a sequence of bosonic N-body quantum states in some set Ω. A natural definition of Bose–Einstein condensation would be that there exists φ such that

$$\lim_{N \to +\infty} \|\Psi_N - \varphi^{\otimes N}\|_{L^2(\Omega^N)} = 0.$$

This definition is too restrictive: it does not allow for some particles not to be in the state φ. For instance, if $\psi \in L^2(\Omega)$ is normalized and orthogonal to φ, the (symmetrized version of the) state $\Psi_N := \varphi(x_1)\cdots\varphi(x_{N-1})\psi(x_N)$ satisfies $\|\Psi_N - \varphi^{\otimes N}\|_{L^2} = 1$ while only 1 among the N particles is not in the state φ.

Definition 1.3. — *Let Ψ be a bosonic N-body quantum state in Ω and let φ be a 1-body quantum state. The* average number of particles in Ψ which are in the state φ*, denoted by $n_\varphi(\Psi)$, is*

$$n_\varphi(\Psi) := N \int_{\Omega^{N-1}} \left| \int_\Omega \overline{\varphi(x_1)}\Psi(x_1, x_2, \ldots, x_N)\, dx_1 \right|^2 dx_2 \cdots dx_N \in [0, N].$$

Remark 1.4. — For a product state, one of course has that $n_\varphi(\varphi^{\otimes N}) = N$.

Definition 1.5 (Bose–Einstein condensation). — *Let $(\Psi_N)_{N\in\mathbb{N}^*}$ be a sequence of bosonic N-body quantum states in Ω, and let φ be a 1-body quantum state. We say that $(\Psi_N)_{N\in\mathbb{N}^*}$* condenses in the state φ *if*

$$\liminf_{N\to+\infty} \frac{n_\varphi(\Psi_N)}{N} > 0.$$

When the above liminf is 1, we say that we have full condensation in φ. Otherwise, we may have partial/fractional condensation.

In the remaining, Ω will be a large box

$$C_L := [-\tfrac{L}{2}, \tfrac{L}{2}]^3,$$

for some large $L > 0$.

1.2. Energy

Let $v \in L^\infty(\mathbb{R}^3, \mathbb{R})$ be an even function such that $v \geqslant 0$ a.e. and such that v is compactly supported. This function is called the *interaction potential* which describes how two particles interact. In classical mechanics, a particle situated at $x \in \mathbb{R}^3$ and a particle situated at $y \in \mathbb{R}^3$ have an interaction energy $v(x - y)$. The generalization to quantum mechanics is given in the following definition.

Definition 1.6 (Energy of a quantum state). — *Let $N \in \mathbb{N}^*$ and $L > 0$. Let Ψ be a bosonic N-body quantum state in C_L such that $\Psi \in H^1((C_L)^N)$. The* energy *of Ψ is*

$$\mathcal{E}(\Psi) := \int_{(C_L)^N} \Big(\sum_{j=1}^N |\nabla_{x_j}\Psi(x)|^2 + \tfrac{1}{2} \sum_{\substack{j,k=1 \\ j\neq k}}^N v(x_j - x_k)|\Psi(x)|^2 \Big)\, dx.$$

The infimum of $\mathcal{E}(\Psi)$ over all such Ψ is called the ground state energy in C_L*, denoted by $E(N, L)$. Any Ψ such that $\mathcal{E}(\Psi) = E(N, L)$ is called a* ground state in C_L.

Remark 1.7. — The energy of Ψ may be formally written as $\mathcal{E}(\Psi) = \langle \Psi, H_{N,L}\Psi \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2((C_L)^N)$, and $H_{N,L}$ is the formal Hamiltonian

$$H_{N,L} = \sum_{j=1}^{N} (-\Delta)_{x_j} + \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} v(x_j - x_k).$$

We use the word "formal" because we don't want to enter into the details over which domain this Hamiltonian is defined as an unbounded operator. The term involving the Laplacian describes the kinetic energy of the particles, and the term involving v describes the interaction energy of the particles.

Remark 1.8. — For non-interacting particles, that is when $v = 0$, we have $E(N, L) = 0$ and the unique ground state is given by $\Psi(x_1, \ldots, x_N) = \varphi_0(x_1) \cdots \varphi_0(x_N)$ with

$$\varphi_0(x) = L^{-3/2},$$

i.e., $\Psi \equiv L^{-3N/2}$. In other words, when $v = 0$, the ground state is a product state, and we have full condensation in φ_0.

The following result is due to Ruelle (1969, Thm. 3.5.11).

Proposition 1.9 (Existence of the thermodynamic limit)
Let $\rho > 0$. Then, the following limit exists:

$$e(\rho) := \lim_{\substack{N,L \to +\infty \\ N/L^3 \to \rho}} \frac{E(N, L)}{L^3}.$$

Remark 1.10. — The number N/L^3 is the number of particles per unit volume, which is called the *density* of particles.

A major open problem in the field is the following.

Conjecture 1.11 (Bose–Einstein condensation in the thermodynamic limit)
Let $\rho > 0$. For any $N \in \mathbb{N}^*$, let Ψ_N be a ground state in C_L for $L = \rho^{-1/3} N^{1/3}$. Then, $(\Psi_N)_{N \in \mathbb{N}^*}$ condenses in the state φ_0.

Condensation in the ground state is known in other regimes than the thermodynamic limit, for instance in the mean-field limit or in the Gross–Pitaevskii limit (see the reviews mentioned above).

1.3. The Lee–Huang–Yang formula

A simpler problem than knowing the ground state, and in particular if it condenses, is knowing its energy. In the thermodynamic limit and in the *dilute* regime of small densities $\rho \to 0$, we have the following very precise result:

THEOREM 1.12 (Lee–Huang–Yang formula). — *As $\rho \to 0$, we have*

$$e(\rho) = 4\pi a\rho^2\left(1 + \frac{128}{15\sqrt{\pi}}(\rho a^3)^{1/2}\right) + o(\rho^{5/2}), \tag{1}$$

where $a \geqslant 0$ is given by

$$a = a(v) := \frac{1}{8\pi} \inf_{\omega \in \dot{H}^1(\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(2|\nabla\omega(r)|^2 + v(r)|1-\omega(r)|^2\right) dr. \tag{2}$$

As we mentioned in the introduction, Eq. (1) was first proposed by Lee, Huang, and Yang (1957) by some formal arguments replacing the two-body potential v by a "pseudopotential." The first order upper bound $e(\rho) \leqslant 4\pi a\rho^2 + o(\rho^2)$ was proved by Dyson (1957). The matching lower bound $e(\rho) \geqslant 4\pi a\rho^2 + o(\rho^2)$ was proved by Lieb and Yngvason (1998). The second order upper bound was then proved by Yau and Yin (2009), and finally the second order lower bound by Fournais and Solovej (2020).

The fact that upper and lower bounds are often proved separately is related to different strategies to obtain them: given the definition of $e(\rho)$, obtaining an upper bound amounts to find an adequate Ψ such that $\mathcal{E}(\Psi)$ has the right asymptotics, while obtaining a lower bounds amounts to bound $\mathcal{E}(\Psi)$ from below *for any* Ψ. We will illustrate this strategy on a simple example: the two-body problem.

2. The two-body ground state energy

The following result motivates the appearance of the number a in the statement of Theorem 1.12. It is stated without proof in Lieb (1963, p. 326).

PROPOSITION 2.1. — *We have*

$$\lim_{L\to+\infty} L^3 E(2, L) = 8\pi a,$$

where $E(2, L)$ was defined in Definition 1.6 and a was defined in (2) and satisfies

$$8\pi a = \inf_{\omega \in \dot{H}^1(\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(2|\nabla\omega(r)|^2 + v(r)|1-\omega(r)|^2\right) dr.$$

From Proposition 2.1, the main term $4\pi a\rho^2$ of Theorem 1.12 can be easily interpreted: among the N particles, there are $N(N-1)/2 \sim N^2/2$ pairs. Since the ground state energy of a single pair is $\sim 8\pi a L^{-3}$ by Proposition 2.1, it makes sense that the ground state energy per unit volume of N particles is

$$\sim \frac{1}{L^3} \times \frac{N^2}{2} \times 8\pi a L^{-3} = 4\pi a\rho^2,$$

since we recall that $N/L^3 \sim \rho$. It remains to justify this formal argument.

Proposition 2.1 has another important consequence: from it, we already see that the two-body problem for interacting particles ($v \neq 0$) is very different from the non-interacting case ($v = 0$). Indeed, when $v = 0$ we saw in Remark 1.8 that $\Psi_0 \equiv L^{-3} \in H^1((C_L)^2)$ is the unique ground state. When $v \neq 0$, this product state has energy

$$\mathcal{E}(\Psi_0) \sim_{L\to+\infty} L^{-3}\int_{\mathbb{R}^3} v,$$

which is *not* the behaviour given by Proposition 2.1. Indeed, we always have

$$\inf_{\omega\in \dot{H}^1(\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(2|\nabla\omega(r)|^2 + v(r)|1-\omega(r)|^2\right) dr \leqslant \int_{\mathbb{R}^3} v(r)\, dr$$

by taking $\omega \equiv 0$ as a test function. If $v \neq 0$, the above inequality is actually always strict, taking $\varepsilon\omega$ as a test function and using that

$$\int_{\mathbb{R}^3} \left(2|\nabla\varepsilon\omega(r)|^2 + v(r)|1-\varepsilon\omega(r)|^2\right) dr$$
$$= \int_{\mathbb{R}^3} v(r)\, dr - 2\varepsilon\int_{\mathbb{R}^3} v(r)\omega(r)\, dr + \varepsilon^2\int_{\mathbb{R}^3} (2|\nabla\omega(r)|^2 + v(r)|\omega(r)|^2)\, dr,$$

which is $< \int v$ for ε small enough, if ω is such that $\int v\omega \neq 0$.

Hence, already for two particles we see that ground states are not product states and that two-body correlations appear.

Remark 2.2. — From standard compactness methods, it can be shown that the problem

$$8\pi a(v) = \inf_{\omega\in \dot{H}^1(\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(2|\nabla\omega(r)|^2 + v(r)|1-\omega(r)|^2\right) dr$$

has a minimizer whose Euler–Lagrange equation is

$$-2\Delta\omega = v(1-\omega).$$

Furthermore, one can assume that $0 \leqslant \omega \leqslant 1$ since replacing ω by $\min(1, |\omega|)$ lowers the energy. This will be useful in Section 6.

Below, we provide the full proof of Proposition 2.1 since we could not find it anywhere, and since it illustrates several important aspects of the overall strategy: upper bound using a good test function, lower bound estimating the energy of any state from below, use of spectral gaps through Poincaré inequalities.

Proof of Proposition 2.1. — Recall that

$$E(2, L) = \inf_{\substack{\Psi\in H^1(C_L^2) \\ \int|\Psi|^2=1}} \mathcal{E}(\Psi),$$

where

$$\mathcal{E}(\Psi) = \int_{C_L^2} (|\nabla_x\Psi|^2 + |\nabla_y\Psi|^2 + v(x-y)|\Psi|^2)\, dx\, dy.$$

For any $\Psi \in H^1((C_L)^2)$ such that $\int|\Psi|^2 = 1$, define $F(c, r) = \Psi(c + r/2, c - r/2)$, i.e. $\Psi(x, y) = F\left(\frac{1}{2}(x + y), x - y\right)$. Then, we have $\int|F(c, r)|^2 = 1$ where F is defined on the set of (c,r) in $\mathbb{R}^3\times\mathbb{R}^3$ such that $c\pm r/2\in C_L$. In these variables, the energy has the form

$$\mathcal{E}(\Psi)=\widetilde{\mathcal{E}}(F):=\int\left(\tfrac{1}{2}|\nabla_c F|^2+2|\nabla_r F|^2+v(r)|F|^2\right)dc\,dr.$$

For each fixed $r\in C_{2L}$, $c\pm r/2\in C_L$ if and only if $|c_j|\leqslant(L-|r_j|)/2$ for all $j\in\{1,2,3\}$. We call this cube $\mathcal{C}(r)$, which has volume

$$|\mathcal{C}(r)|=\prod_{j=1}^{3}(L-|r_j|).$$

For the upper bound, let $\omega\in\dot{H}^1(\mathbb{R}^3)$ and consider $F(c,r)=L^{-3}(1-\omega(r))$ so that $\int|\Psi|^2\sim_{L\to+\infty}1$ as well as

$$\mathcal{E}(\Psi)=L^{-3}\int_{C_{2L}}\left(2|\nabla_r\omega|^2+v(r)|1-\omega(r)|^2\right)\prod_j\left(1-\tfrac{|r_j|}{L}\right)dr,$$

so that

$$\limsup_{L\to+\infty}L^3E(2,L)\leqslant\int_{\mathbb{R}^3}\left(2|\nabla_r\omega|^2+v(r)|1-\omega(r)|^2\right)dr,$$

for any $\omega\in\dot{H}^1(\mathbb{R}^3)$, proving the desired upper bound.

To prove the lower bound, let $\Psi\in H^1((C_L)^2)$ with $\int|\Psi|^2=1$. By the upper bound that we just proved, we may also assume that $\mathcal{E}(\Psi)\lesssim L^{-3}$. Define

$$\begin{aligned}f(r)&:=\left(\frac{1}{|\mathcal{C}(r)|}\int_{\mathcal{C}(r)}|F(c,r)|^2\,dc\right)^{1/2}\\&=\left(\int_{C_1}|F((L-|r_1|)c_1,(L-|r_2|)c_2,(L-|r_3|)c_3,r)|^2\,dc\right)^{1/2},\end{aligned}$$

which satisfies

$$\int_{C_{2L}}f(r)^2\prod_j(L-|r_j|)\,dr=1,$$

as well as

$$\int_{C_{2L}}f(r)^2v(r)\prod_j(L-|r_j|)\,dr=\int|F(c,r)|^2v(r)\,dc\,dr.$$

For any $j\in\{1,2,3\}$, we have by Cauchy–Schwarz

$$\begin{aligned}|\partial_{r_j}f(r)|^2\leqslant\int_{C_1}&|(\partial_{r_j}F)((L-|r_1|)c_1,(L-|r_2|)c_2,(L-|r_3|)c_3,r)\\&-\operatorname{sgn}(r_j)c_j(\partial_{c_j}F)((L-|r_1|)c_1,(L-|r_2|)c_2,(L-|r_3|)c_3,r)|^2\,dc\end{aligned}$$

so that

$$|\partial_{r_j}f(r)|^2|\mathcal{C}(r)|\leqslant\int_{\mathcal{C}(r)}\left(2|\partial_{r_j}F(c,r)|^2+\tfrac{1}{2}|\partial_{c_j}F(c,r)|^2\right)dc,$$

and hence $\widetilde{\mathcal{E}}(F)\geqslant\widetilde{\mathcal{E}}(f)$. Define the measure $d\mu_L(r):=\prod_j(1-|r_j|/L)\,dr$ on C_{2L}. It satisfies $\mu_L(C_{2L})=L^3$, as well as a *Poincaré inequality*: for any $u\in H^1(C_{2L})$, one has

$$\int_{C_{2L}}|u-\overline{u}|^2\,d\mu_L\leqslant CL^2\int_{C_{2L}}|\nabla u|^2\,d\mu_L,$$

where $\overline{u} := L^{-3}\int_{C_{2L}} u\, d\mu_L$, for some $C > 0$ independent of L and u. The prefactor L^2 follows from a scaling argument: once the inequality is known for $L = 1$, it follows for any L by considering $u(L\cdot)$ which is defined on C_2 if u is defined on C_{2L}. The fact that $d\mu_1$ satisfies a Poincaré inequality on C_2 follows from the fact that $d\mu_1$ is factorized and the fact that the measure $m(t)\, dt$ satisfies a Poincaré inequality on $[-1, 1]$, with $m(t) := 1 - |t|$. Indeed, the criterion for such a measure to satisfy a Poincaré inequality is (Muckenhoupt, 1972), see also Roustant, Barthe, and Iooss (2017, Thm. 1):

$$\sup_{x\in[0,1)} \int_x^1 m(t)\, dt \int_0^x \frac{dt}{m(t)} < +\infty,$$

which holds here since as $x \to 1$ the second factor behaves as $\log(1 - x)$ while the first behaves as $(1 - x)^2$. Another important consequence of the Poincaré inequality is the fact that for any $u \in H^1(C_{2L})$ one has

$$\|u - \overline{u}\|^2_{L^6(C_L)} \leqslant C \int_{C_{2L}} |\nabla u|^2\, d\mu_L$$

for some $C > 0$ independent of L. Indeed, by scaling it is again enough to prove it for $L = 1$. Then, by the Sobolev embedding $H^1(C_1) \hookrightarrow L^6(C_1)$ we have

$$\|u - \overline{u}\|_{L^6(C_1)} \lesssim \|u - \overline{u}\|_{L^2(C_1)} + \|\nabla u\|_{L^2(C_1)}.$$

Using that $d\mu_1 \geqslant \frac{1}{2} dx$ on C_1, we deduce that

$$\int_{C_1} |\nabla u|^2\, dx \leqslant 2 \int_{C_1} |\nabla u|^2\, d\mu_1 \leqslant 2 \int_{C_2} |\nabla u|^2 d\mu_1,$$

and similarly that

$$\int_{C_1} |u - \overline{u}|^2 dx \leqslant 2 \int_{C_2} |u - \overline{u}|^2\, d\mu_1 \lesssim \int_{C_2} |\nabla u|^2 d\mu_1,$$

where in the last inequality we used the Poincaré inequality. This indeed proves that

$$\|u - \overline{u}\|^2_{L^6(C_1)} \lesssim \int_{C_2} |\nabla u|^2\, d\mu_1,$$

which will be useful later on. Defining $c_f = L^{-3}\int_{C_{2L}} f\, d\mu_L$ and $g = f - c_f$, we deduce by the Poincaré inequality that

$$L^{-3} \gtrsim \widetilde{\mathcal{E}}(f) \geqslant \int_{C_{2L}} |\nabla f|^2\, d\mu_L \gtrsim L^{-2} \int_{C_{2L}} g^2\, d\mu_L,$$

so that $\int g^2\, d\mu_L \lesssim L^{-1}$. Since $\int f^2\, d\mu_L = c_f^2 L^3 + \int g^2\, d\mu_L$, we deduce that $(L^{3/2}c_f)^2 = 1 + O(L^{-1})$ and hence $c_f = L^{-3/2} + O(L^{-5/2})$. Now since $g^2 = (f - c_f)^2 \leqslant 2f^2 + 2c_f^2$, we deduce that $\int vg^2 \leqslant 2\int vf^2 + 2c_f^2 \int v \leqslant 2\widetilde{\mathcal{E}}(f) + 2c_f^2 \int v$ and hence $\int vg^2 \lesssim L^{-3}$. The potential energy may be split as

$$\begin{aligned}\int vf^2 = \int v(c_f + g)^2 = \int v(L^{-3/2} + g + O(L^{-5/2}))^2 \\ = \int v(L^{-3/2} + g)^2 + O(L^{-5}) + O(L^{-5/2}) \int v(L^{-3/2} + g),\end{aligned}$$

and since $|\int vg| \lesssim (\int vg^2)^{1/2} \lesssim L^{-3/2}$, we deduce that $\int vf^2 = \int v(L^{-3/2}+g)^2 + O(L^{-4})$ so that

$$\widetilde{\mathcal{E}}(f) \geqslant \widetilde{\mathcal{E}}(L^{-3/2}+g) + O(L^{-4}).$$

Writing $g = -L^{-3/2}\omega$ we finally deduce that

$$L^{-3} \gtrsim \mathcal{E}(\Psi) \geqslant L^{-3} \int_{C_{2L}} (2|\nabla\omega|^2 + v(1-\omega)^2) \prod_j (1-|r_j|/L)\, dr + O(L^{-4}).$$

Now let $L_n \to +\infty$ be such that $\liminf_{L\to+\infty} L^3 E(2,L) = \lim_{n\to+\infty} L_n^3 E(2,L_n)$. Let $\Psi_n \in H^1((C_{L_n})^2)$ with $\int |\Psi_n|^2 = 1$ and $\mathcal{E}(\Psi_n) \leqslant E(2,L_n) + L_n^{-4}$. From Ψ_n we build ω_n as above. Then, for any $R>0$ the sequence (ω_n) is bounded in $H^1(B(0,R))$ since

$$1 \gtrsim \int_{C_{2L_n}} |\nabla\omega_n|^2 \prod_j (1-|r_j|/L_n)\, dr \geqslant (1-R/L_n) \int_{B(0,R)} |\nabla\omega_n|^2,$$

and since

$$\int_{B(0,R)} |\omega_n|^6 \leqslant \int_{C_{L_n}} |\omega_n|^6 \lesssim \int_{C_{2L_n}} |\nabla\omega_n|^2\, d\mu_{L_n} \lesssim 1$$

as we saw above. Hence, by a diagonal argument we may assume that (ω_n) converges weakly in $H^1(B(0,R))$ for all $R>0$ (and strongly in $L^2(B(0,R))$) towards some $\omega \in H^1_{\text{loc}}(\mathbb{R}^3)$. Since $\int_{C_{2L_n}} |\nabla\omega_n|^2 \prod_j (1-|r_j|/L_n)\, dr \lesssim 1$ we deduce that $\int_{\mathbb{R}^3} |\nabla\omega|^2 \lesssim 1$. Similarly, we have $\int_{\mathbb{R}^3} |\omega|^6 \lesssim 1$ and thus $\omega \in \dot{H}^1(\mathbb{R}^3)$. By strong local convergence and the fact that v is compactly supported, we have

$$\liminf_{n\to+\infty} \int_{C_{2L_n}} (2|\nabla\omega_n|^2 + v(1-\omega_n)^2) \prod_j (1-|r_j|/L_n)\, dr \geqslant \int_{\mathbb{R}^3} (2|\nabla\omega|^2 + v(1-\omega)^2).$$

This proves that

$$\liminf_{L\to+\infty} L^3 E(2,L) \geqslant \int_{\mathbb{R}^3} (2|\nabla\omega|^2 + v(1-\omega)^2)\, dr$$

and the desired lower bound. □

Remark 2.3. — In the above proof, notice that a good trial state is given by $\Psi(x,y) = L^{-3}(1-\omega(x-y))$, which has to be compared to the constant product state $\Psi_0(x,y) = L^{-3}$. Notice that Ψ and Ψ_0 are very close in L^2-norm, since the L^2-norm of $L^{-3}\omega(x-y)$ has size $\sim L^{-3/2}$. However, the energy of Ψ and Ψ_0 are very different.

Remark 2.4. — Another important point of the above proof is the use of Poincaré inequalities: on a box of size length L, the *spectral gap* is of size L^{-2}, much larger than the ground state energy of order L^{-3}. This allows to show that most of the ground state wants to be in the constant state Ψ_0.

3. The Bogoliubov approximation

From the last section, we understand where the main term in the Lee–Huang–Yang expansion (1) comes from. We now give some motivation for the next term, which may be understood formally from Bogoliubov's method (Bogoliubov, 1947). This method not only aims at describing the ground state energy of the Bose gas but also all the low-lying eigenvalues, which are key to understand another important property of the Bose gas: superfluidity. Concerning the ground state energy $E(N, L)$ defined in Definition 1.6, the Bogoliubov method consists in claiming that it is well approximated by a number $E_{\mathrm{Bog}}(v)$ (that we explicitly identify below), which satisfies the following asymptotics:

Proposition 3.1 (Bogoliubov ground state energy for dilute systems)
We have

$$\liminf_{L\to+\infty} \frac{E_{\mathrm{Bog}}(v)}{L^3} \geqslant 4\pi(a_0+a_1)\rho^2 + 4\pi a_0\rho^2 \times \frac{128}{15\sqrt{\pi}}(\rho a_0^3)^{1/2} + o_{\rho\to 0}(\rho^{5/2}), \tag{3}$$

where

$$a_0 = a_0(v) := \frac{\widehat{v}(0)}{8\pi}, \quad a_1 = a_1(v) := -\frac{1}{128\pi^4}\int_{\mathbb{R}^3} \frac{\widehat{v}(k)^2}{|k|^2}\,dk.$$

Compared to the Lee–Huang–Yang asymptotics (1), we see two discrepancies: in the main term, a is replaced by $a_0 + a_1$ while in the next term, a is replaced by a_0. Notice that a_0 is linear in v while a_1 is quadratic in v. It turns out that a_0 and a_1 are the first two terms in the expansion of a for small v. Hence, it seems that Bogoliubov's method only agrees with the Lee–Huang–Yang formula for small v, and this is why it is not expected that it works for v which is not small.

Let us now explain the nature of Bogoliubov's approximation and the origin of the number $E_{\mathrm{Bog}}(v)$. To explain it, we follow the presentation of Lieb, Seiringer, Solovej, and Yngvason (2005, App. A).

3.1. Fock space, creation and annihilation operators

Bogoliubov's method is better explained using the formalism of creation and annihilation operators. This formalism is very useful in a context where the number of particles is fluctuating, as in quantum field theory. In our context, it seems that the number of particles is fixed. However, concerning the question of condensation, we said that the number of particles that actually condense was unclear a priori. Hence, the fluctuating number of particles in our case may be thought as the number of particles in the condensate, of equivalently the number of particles outside the condensate (often called *excitations*).

Definition 3.2 (Bosonic Fock space). — *Let $\mathfrak{H}$ be a complex Hilbert space. For any $n \in \mathbb{N}^*$, we denote by $\mathfrak{H}^{\otimes_{\mathrm{sym}} n}$ the n-fold symmetric tensor product of $\mathfrak{H}$ with itself. By* *convention, we also denote* $\mathfrak{H}^{\otimes_{\rm sym} 0} := \mathbb{C}$*. Then, the* bosonic Fock space *associated to* $\mathfrak{H}$ *is the Hilbert space*

$$\mathcal{F}(\mathfrak{H}) := \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes_{\rm sym} n}.$$

Remark 3.3. — When $\mathfrak{H} = L^2(C_L)$ and $n \in \mathbb{N}^*$, $\mathfrak{H}^{\otimes_{\rm sym} n}$ can be identified with $L^2_{\rm sym}((C_L)^n)$.

Definition 3.4 (Creation and annihilation operators). — *Let* $f \in L^2(C_L)$*.*

1. *For any* $n \geqslant 2$*, we define the* annihilation operator $a(f)\colon L^2_{\rm sym}((C_L)^n) \to L^2_{\rm sym}((C_L)^{n-1})$ *by: for any* $\Psi \in L^2((C_L)^n)$ *and for a.e.* $x_1, \dots, x_{n-1} \in C_L$*,*

$$(a(f)\Psi)(x_1, \dots, x_{n-1}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{C_L} \overline{f(x)} \Psi(x_1, \dots, x, \dots, x_{n-1})\, dx,$$

   where the x *in* $\Psi(x_1, \dots, x, \dots, x_{n-1})$ *is in position* j*. For* $n = 1$*, we just define* $a(f)\Psi = \int_{C_L} \overline{f}\Psi \in \mathbb{C}$*.*
2. *For any* $n \geqslant 1$*, we define the* creation operator $a^*(f)\colon L^2((C_L)^n) \to L^2((C_L)^{n+1})$ *by: for any* $\Psi \in L^2((C_L)^n)$ *and for a.e.* $x_1, \dots, x_{n+1} \in C_L$*,*

$$(a^*(f)\Psi)(x_1, \dots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} f(x_j)\Psi(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}).$$

The creation and annihilation operators can be extended to $\mathcal{F}(L^2(C_L))$ *by setting* $a(f)_{|\mathbb{C}} = 0$ *and for all* $z \in \mathbb{C}$*,* $a^*(f)z = zf$*. On* $\mathcal{F}(L^2(C_L))$*, they satisfy the* canonical commutation relations *(CCR): for any* $f, g \in L^2(C_L)$*,*

$$[a(f), a(g)] = 0 = [a^*(f), a^*(g)], \qquad [a(f), a^*(g)] = \langle f, g\rangle \mathrm{Id}_{\mathcal{F}(L^2(C_L))},$$

where $\langle \cdot, \cdot \rangle$ *denotes the inner product on* $L^2(C_L)$*.*

For any $k \in (2\pi/L)\mathbb{Z}^3$ and $x \in C_L$, we define $e_k(x) := L^{-3/2}e^{ik\cdot x}$ so that $e_k \in L^2(C_L)$ and we use the shortcut notation

$$a_k := a(e_k), \qquad a_k^* := a^*(e_k).$$

Since k is interpreted as a momentum, a_k annihilates a particle of momentum k while a_k^* creates a particle of momentum k. The operator $a_k^* a_k$ counts the number of particles of momentum k. Notice that for $k = 0$, $e_0 = \varphi_0$ is the constant function which represents the condensate. Hence, $a_0^* a_0$ counts the number of particles in the condensate. Indeed, one can show that the operator n_φ given in Definition 1.3 satisfies for all Ψ

$$n_{\varphi_0}(\Psi) = \langle \Psi, a_0^* a_0 \Psi \rangle.$$

For any $f \in L^2(C_L)$ and any $k \in (2\pi/L)\mathbb{Z}^3$, we denote its Fourier coefficients by

$$\widehat{f}(k) = \int_{C_L} f(x) e^{-ik\cdot x}\, dx.$$

Proposition 3.5. — *Let $N \geqslant 2$. Assume that $v\colon \mathbb{R}^3 \to \mathbb{R}$ is measurable, non-negative, bounded, and $L\mathbb{Z}^3$-periodic. Then, for all $\Psi \in H^1((C_L)^N)$ $L\mathbb{Z}^3$-periodic we have*

$$\mathcal{E}(\Psi) = \sum_{k\in(2\pi/L)\mathbb{Z}^3} |k|^2 \langle \Psi, a_k^* a_k \Psi\rangle + \frac{1}{2L^3} \sum_{k,p,q\in(2\pi/L)\mathbb{Z}^3} \widehat{v}(k) \langle \Psi, a_{p+k}^* a_{q-k}^* a_p a_q \Psi\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2((C_L)^n)$.

Remark 3.6. — The above expression of the energy in terms of creation and annihilation may be formally written as

$$H_{N,L} = \sum_{k\in(2\pi/L)\mathbb{Z}^3} |k|^2 a_k^* a_k + \frac{1}{2L^3} \sum_{k,p,q\in(2\pi/L)\mathbb{Z}^3} \widehat{v}(k) a_{p+k}^* a_{q-k}^* a_p a_q. \tag{4}$$

3.2. c-number substitution

Bogoliubov's method consists in claiming that the ground state energy is the same as if, in the above sum (4), among the terms $a_{p+k}^* a_{q-k}^* a_p a_q$, we only keep the ones where either all momenta are 0 or exactly two of them are 0. In the resulting Hamiltonian, all a_0 and a_0^* are replaced by $\sqrt{N}$. This last step is called *c-number substitution* because we replace the operators a_0 and a_0^* by a number $c = \sqrt{N}$. This is motivated by the fact that condensation in the ground state means that $\langle\Psi, a_0^* a_0\Psi\rangle \sim N$ so that $1 = \langle\Psi, [a_0, a_0^*]\Psi\rangle \ll \langle\Psi, a_0^* a_0 \Psi\rangle$ in the ground state, so that a_0 and a_0^* almost commute and may thus be approximated by numbers. The resulting *Bogoliubov Hamiltonian* is

$$H_{\text{Bog}}(v) \coloneqq \frac{1}{2}\widehat{v}(0)\rho N + \sum_{k\in(2\pi/L)\mathbb{Z}^3\setminus\{0\}} (|k|^2 + \rho\widehat{v}(k)) a_k^* a_k + \frac{1}{2}\rho\widehat{v}(k)(a_k a_{-k} + a_{-k}^* a_k^*)$$

Note also that $H_{\text{Bog}}(v)$ does not preserve $L^2((C_L)^N)$ contrary to $H_{N,L}$, so that it acts on the full Fock space $\mathcal{F}(L^2(C_L))$. This should not come as a surprise, since Bogoliubov's method somehow already includes the condensate and thus $H_{\text{Bog}}(v)$ should be interpreted as a Hamiltonian describing the excitations outside the ground state, which may not have a fixed number of particles.

The nice thing about $H_{\text{Bog}}(v)$ is that its ground state can be easily computed (Lieb and Solovej, 2001, Thm. 6.3, Seiringer, 2011, Sec. 3) since it is a quadratic expression in a_k and a_k^*, and hence we may "complete the square" as in the following result:

Lemma 3.7. — *Let $A, B \in \mathbb{R}$ be such that $|B| < A$. Define*

$$\alpha \coloneqq \frac{1}{B}(A - \sqrt{A^2 - B^2}) \in [0, 1).$$

Then, if b, c are two elements of a $*$-algebra such that $[b, c] = 0$ we have

$$\begin{aligned} A(b^*b + c^*c) + B(b^*c^* + bc) = \frac{\sqrt{A^2 - B^2}}{1-\alpha^2}\Big((b^* + \alpha c)(b + \alpha c^*) + (c^* + \alpha b)(c + \alpha b^*)\Big) \\ - \frac{1}{2}(A - \sqrt{A^2 - B^2})([b, b^*] + [c, c^*]) \end{aligned}$$

Proof. — Expanding and using that b and c commute, we get

$$(b^*+\alpha c)(b+\alpha c^*)+(c^*+\alpha b)(c+\alpha b^*) = (1+\alpha^2)(b^*b+c^*c)+2\alpha(b^*c^*+bc)+\alpha^2([b,b^*]+[c,c^*]).$$

It remains to use the relations

$$\sqrt{A^2-B^2}\frac{1+\alpha^2}{1-\alpha^2} = A, \quad \sqrt{A^2-B^2}\frac{2\alpha}{1-\alpha^2} = B, \quad \sqrt{A^2-B^2}\frac{\alpha^2}{1-\alpha^2} = \frac{1}{2}(A-\sqrt{A^2-B^2}).$$

□

Applying the above lemma to $b = a_k$, $c = a_{-k}$, $A = \frac{1}{2}(|k|^2 + \rho\widehat{v}(k))$, $B = \frac{1}{2}\rho\widehat{v}(k)$ for $k \in (2\pi/L)\mathbb{Z}^3 \setminus \{0\}$, and using that v is real and even so that $\widehat{v}(k) = \widehat{v}(-k)$, we obtain that $H_{\mathrm{Bog}}(v) \geqslant E_{\mathrm{Bog}}(v)$ where

$$E_{\mathrm{Bog}}(v) = \frac{1}{2}\widehat{v}(0)\rho N - \frac{1}{2}\sum_{k\in(2\pi/L)\mathbb{Z}^3\setminus\{0\}}\left(|k|^2 + \rho\widehat{v}(k) - |k|\sqrt{|k|^2+2\rho\widehat{v}(k)}\right).$$

By Riemann sums, we get for the thermodynamic limit

$$\liminf_{L\to+\infty}\frac{E_{\mathrm{Bog}}(v)}{L^3} \geqslant \frac{1}{2}\widehat{v}(0)\rho^2 - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(|k|^2 + \rho\widehat{v}(k) - |k|\sqrt{|k|^2+2\rho\widehat{v}(k)}\right)dk.$$

The behaviour of the right side as $\rho \to 0$ is given by the following result:

Lemma 3.8. — *We have*

$$-\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(|k|^2 + \rho\widehat{v}(k) - |k|\sqrt{|k|^2+2\rho\widehat{v}(k)}\right)dk$$
$$= -\frac{\rho^2}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{\widehat{v}(k)^2}{2|k|^2}\,dk + 4\pi a_0\rho^2 \times \frac{128}{15\sqrt{\pi}}(\rho a_0^3)^{1/2} + o_{\rho\to 0}(\rho^{5/2}),$$

where $a_0 \coloneqq (8\pi)^{-1}\widehat{v}(0)$.

Proof. — We first extract the main term in the expansion

$$-\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(|k|^2 + \rho\widehat{v}(k) - |k|\sqrt{|k|^2+2\rho\widehat{v}(k)}\right)dk =$$
$$-\frac{\rho^2}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{\widehat{v}(k)^2}{2|k|^2}\,dk - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(|k|^2 + \rho\widehat{v}(k) - |k|\sqrt{|k|^2+2\rho\widehat{v}(k)} - \rho^2\frac{\widehat{v}(k)^2}{2|k|^2}\right)dk.$$

The last integral is computed by changing variables $k = \sqrt{\rho}k'$,

$$\int_{\mathbb{R}^3}\left(|k|^2 + \rho\widehat{v}(k) - |k|\sqrt{|k|^2+2\rho\widehat{v}(k)} - \rho^2\frac{\widehat{v}(k)^2}{2|k|^2}\right)dk$$
$$= \rho^{5/2}\int_{\mathbb{R}^3}\left(|k|^2 + \widehat{v}(\sqrt{\rho}k) - |k|\sqrt{|k|^2+2\widehat{v}(\sqrt{\rho}k)} - \frac{\widehat{v}(\sqrt{\rho}k)^2}{2|k|^2}\right)dk,$$

and by dominated convergence we have

$$\int_{\mathbb{R}^3}\left(|k|^2+\widehat{v}(\sqrt{\rho}k)-|k|\sqrt{|k|^2+2\widehat{v}(\sqrt{\rho}k)}-\frac{\widehat{v}(\sqrt{\rho}k)^2}{2|k|^2}\right)dk$$
$$\sim_{\rho\to 0}\int_{\mathbb{R}^3}\left(|k|^2+\widehat{v}(0)-|k|\sqrt{|k|^2+2\widehat{v}(0)}-\frac{\widehat{v}(0)^2}{2|k|^2}\right)dk.$$

Again by scaling $k=\sqrt{\widehat{v}(0)}k'$ we have

$$\int_{\mathbb{R}^3}\left(|k|^2+\widehat{v}(0)-|k|\sqrt{|k|^2+2\widehat{v}(0)}-\frac{\widehat{v}(0)^2}{2|k|^2}\right)dk=$$
$$\widehat{v}(0)^{5/2}\int_{\mathbb{R}^3}\left(|k|^2+1-|k|\sqrt{|k|^2+2}-\frac{1}{2|k|^2}\right)dk.$$

Going to radial coordinates, the last integral is equal to

$$\int_{\mathbb{R}^3}\left(|k|^2+1-|k|\sqrt{|k|^2+2}-\frac{1}{2|k|^2}\right)dk=4\pi\int_0^\infty(r^2+1-r\sqrt{r^2+2}-\frac{1}{2r^2})r^2\,dr=-\frac{32\sqrt{2}\pi}{15}.$$

□

Remark 3.9. — The above proof can be made more quantitative to infer that

$$E_{\mathrm{Bog}}(v)\geqslant 4\pi(a_0(v)+a_1(v))\rho^2\ell^3-C\rho^2\ell^2, \tag{5}$$

for small values of ρ. To get this bound, we need some quantitative comparison between Riemann sums and the associated integral, which requires some decay on the derivatives of $\widehat{v}(k)$ that we will ignore. This bound will be useful when we prove rigorous bounds for Bogoliubov's method in Section 5.

In the following sections, we will explain how to implement rigorously Bogoliubov's method (i.e., how to justify the c-number substitution), and how to solve the problem that the numbers a_0 and a_1 appearing in (3) are not the correct ones. We will proceed in two steps: first, in Section 5, we will justify the lower bound (3) at first order for the full ground state energy $E(N,L)$, even though the wrong numbers a_0 and a_1 appear in it. Then, in Section 6, we will explain how to replace these wrong numbers by the correct ones.

4. Reduction to small boxes

Before going to the actual justification of Bogoliubov's method, we explain a crucial argument in order to prove Theorem 1.12: the fact that it is enough to obtain the asymptotics on smaller boxes. The main result that we will prove is the following:

Proposition 4.1. — *Let $\alpha_1 > 1$. Then, there exists $K > 0$ such that the following holds. If, for some $\alpha_2, \beta \in \mathbb{R}$ such that $\alpha_1 < \alpha_2 < \beta$, for some $c_1 > 0$, $c_2 \in \mathbb{R}$, $\delta > 0$, and $C > 0$ one has that for all $\rho \in (0, \delta)$, there exists $\ell \geqslant \rho^{-1/3}$ such that for all $n \in \mathbb{N}^*$ with $n\ell^{-3} \leqslant K\rho$, one has*

$$\frac{E(n,\ell)}{\ell^3} \geqslant c_1 \rho_{n,\ell}^{\alpha_1} + c_2 \rho_{n,\ell}^{\alpha_2} - C\rho^\beta,$$

where $\rho_{n,\ell} := n\ell^{-3}$, then there exist $C' > 0$ and $\beta' > \alpha_2$ such that for all $\rho \in (0, \delta)$ one has

$$e(\rho) \geqslant c_1 \rho^{\alpha_1} + c_2 \rho^{\alpha_2} - C' \rho^{\beta'}.$$

The key point of Proposition 4.1 is that it somehow allows bringing the double limit $N, L \to +\infty$ with $N/L^3 \to \rho$ and then $\rho \to 0$ into a single limit $\rho \to 0$, with some ℓ that depends on ρ (in applications, we will typically take $\ell = \rho^{-\eta}$ for some $\eta > 1/3$). While technical, this statement is crucial because it allows to have a single small parameter ρ, and then we are in a position to prove error bounds only with respect to this parameter.

Proposition 4.1 has its origin in the seminal work (Lieb and Yngvason, 1998) and was extended by Fournais and Solovej. We follow the presentation of Haberberger, Hainzl, Nam, Seiringer, and Triay (2023, Sec. 9) as well as Fournais, Girardot, Junge, Morin, Olivieri, and Triay (2026, App. B).

Lemma 4.2. — *For any $\rho > 0$, any $\ell > 0$, and any $\mu \geqslant 0$ we have*

$$e(\rho) \geqslant \frac{1}{\ell^3} \inf_{n \in \mathbb{N}^*} (E(n,\ell) - \mu n) + \mu\rho.$$

Proof. — Let $M \in \mathbb{N}^*$ and define $L := M\ell$, $J := M^3$, and $N := \lfloor \rho L^3 \rfloor + 1$. Decompose the big cube C_L of side length L into disjoint cubes $C_\ell^{(j)}$ of side length ℓ, for $j = 1, \ldots, J$. Let $\Psi \in H^1((C_L)^N)$ be such that $\int_{(C_L)^N} |\Psi|^2 = 1$. For all $x \in (C_L)^N$ denote by

$$e_\Psi(x) := \sum_{j=1}^N |\nabla_{x_j} \Psi(x)|^2 + \tfrac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^N v(x_j - x_k) |\Psi(x)|^2,$$

so that $\mathcal{E}(\Psi) = \int_{(C_L)^N} e_\Psi(x)\, dx$. Given that $C_L = \cup_{j=1}^J C_\ell^{(j)}$, we deduce that

$$\mathcal{E}(\Psi) = \sum_{j \in \{1,\ldots,J\}^N} \int_{C_\ell^{(j_1)} \times \cdots \times C_\ell^{(j_N)}} e_\Psi(x)\, dx.$$

For any $j' \in \{1, \ldots, J\}$ and any $j \in \{1, \ldots, J\}^N$, denote by

$$n_{j'}(j) := |\{k \in \{1, \ldots, N\},\ j_k = j'\}|.$$

By non-negativity of v, we have

$$\begin{aligned} \int_{C_\ell^{(j_1)} \times \cdots \times C_\ell^{(j_N)}} e_\Psi(x)\, dx &\geqslant \sum_{j'=1}^J E(n_{j'}(j), \ell) \int_{C_\ell^{(j_1)} \times \cdots \times C_\ell^{(j_N)}} |\Psi(x)|^2\, dx \\ &= \sum_{n=0}^N c_n(j) E(n,\ell), \end{aligned}$$

where

$$c_n(j) := |\{j' \in \{1,\dots,J\},\ n_{j'}(j) = n\}| \int_{C_\ell^{(j_1)}\times\cdots\times C_\ell^{(j_N)}} |\Psi(x)|^2\,dx.$$

Finally, defining $c_n := \ell^3 L^{-3} \sum_{j\in\{1,\dots,J\}^N} c_n(j)$, we have

$$L^{-3}\mathcal{E}(\Psi) \geqslant \frac{1}{\ell^3}\sum_{n=0}^N c_n E(n,\ell).$$

The numbers c_n have the following properties. Clearly, $c_n \geqslant 0$. Furthermore, $\sum_{n=0}^N |\{j' \in \{1,\dots,J\},\ n_{j'}(j) = n\}| = J = L^3/\ell^3$ so that

$$\sum_{n=0}^N c_n = \sum_{j\in\{1,\dots,J\}^N} \int_{C_\ell^{(j_1)}\times\cdots\times C_\ell^{(j_N)}} |\Psi(x)|^2\,dx = \int_{(C_L)^N} |\Psi|^2 = 1.$$

Finally, we have

$$\begin{aligned}\sum_{n=0}^N n|\{j' \in \{1,\dots,J\},\ n_{j'}(j) = n\}| &= \sum_{n=0}^N n\sum_{j'=1}^J \mathbb{1}(n_{j'}(j) = n)\\ &= \sum_{j'=1}^J n_{j'}(j)\\ &= \sum_{j'=1}^J\sum_{k=1}^N \mathbb{1}(j_k = j') = N,\end{aligned}$$

so that $\sum_{n=0}^N nc_n = N\ell^3L^{-3} \geqslant \rho\ell^3$. We deduce that

$$\begin{aligned}L^{-3}\mathcal{E}(\Psi) &\geqslant \frac{1}{\ell^3}\sum_{n=0}^N c_n E(n,\ell)\\ &= \frac{1}{\ell^3}\sum_{n=0}^N c_n(E(n,\ell) - \mu n) + \mu\ell^{-3}\sum_{n=0}^N nc_n\\ &\geqslant \frac{1}{\ell^3}\inf_{n\in\mathbb{N}^*}(E(n,\ell) - \mu n) + \mu\rho,\end{aligned}$$

so that $L^{-3}E(N,L) \geqslant \ell^{-3}\inf_{n\in\mathbb{N}^*}(E(n,\ell) - \mu n) + \mu\rho$. We conclude by taking the limit $M \to +\infty$ so that $N, L \to +\infty$ with $N/L^3 \to \rho$. □

Proof of Proposition 4.1. — Let $n_0 := \lfloor K\rho\ell^3\rfloor$. Since $\rho\ell^3 \geqslant 1$, we have $(K-1)\rho \leqslant \rho_{n_0,\ell} \leqslant K\rho$. For $n \leqslant n_0$, we have by assumption that for all $\mu\in\mathbb{R}$,

$$\frac{E(n,\ell)}{\ell^3} - \mu(\rho_{n,\ell} - \rho) \geqslant c_1\rho^{\alpha_1} + c_2\rho^{\alpha_2} - C\rho^\beta + c_1(\rho_{n,\ell}^{\alpha_1} - \rho^{\alpha_1}) - \mu(\rho_{n,\ell} - \rho) + c_2(\rho_{n,\ell}^{\alpha_2} - \rho^{\alpha_2}).$$

We may write with $x := \rho_{n,\ell}/\rho \in [0,K]$,

$$c_1(\rho_{n,\ell}^{\alpha_1} - \rho^{\alpha_1}) - \mu(\rho_{n,\ell} - \rho) = c_1\rho^{\alpha_1}\Big[x^{\alpha_1} - 1 - c_1^{-1}\rho^{1-\alpha_1}\mu(x-1)\Big],$$

which motivates the choice $\mu := c_1\alpha_1\rho^{\alpha_1-1}$, so that by convexity of $x\mapsto x^{\alpha_1}$ one has

$$c_1(\rho_{n,\ell}^{\alpha_1} - \rho^{\alpha_1}) - \mu(\rho_{n,\ell} - \rho) \geqslant 0.$$

By strict convexity of $x \mapsto x^{\alpha_1}$, let $c > 0$ be such that for all $x > 0$ one has

$$x^{\alpha_1} - 1 - \alpha_1(x-1) \geqslant c((x-1)^2 \mathbb{1}_{|x-1|\leqslant 1} + (x-1)^{\alpha_1}\mathbb{1}_{|x-1|\geqslant 1}).$$

Then, recalling that $0 \leqslant x \leqslant K$ we can bound

$$|c_2(\rho_{n,\ell}^{\alpha_2} - \rho^{\alpha_2})| = |c_2|\rho^{\alpha_2}|x^{\alpha_2} - 1| \leqslant |c_2|\rho^{\alpha_2}\alpha_2 K^{\alpha_2-1}|x-1|.$$

For $|x-1| \leqslant 1$ we use that

$$\rho^{\alpha_2}|x-1| \leqslant \varepsilon\rho^{\alpha_1}|x-1|^2 + C_\varepsilon \rho^{2\alpha_2-\alpha_1},$$

(where $2\alpha_2 - \alpha_1 > \alpha_2$) while for $|x-1| \geqslant 1$ we use that

$$\rho^{\alpha_2}|x-1| \leqslant \varepsilon\rho^{\alpha_1}|x-1|^{\alpha_1} + C_\varepsilon \rho^{\frac{\alpha_1(\alpha_2-1)}{\alpha_1-1}},$$

(where $\frac{\alpha_1(\alpha_2-1)}{\alpha_1-1} > \alpha_2$) which proves that (choosing ε so that $\varepsilon|c_2|\alpha_2 K^{\alpha_2-1} \leqslant cc_1/2$)

$$\begin{aligned}
c_1(\rho_{n,\ell}^{\alpha_1} - \rho^{\alpha_1}) - \mu(\rho_{n,\ell} - \rho) + c_2(\rho_{n,\ell}^{\alpha_2} - \rho^{\alpha_2}) \\
\geqslant \frac{1}{2}cc_1\rho^{\alpha_1}((x-1)^2\mathbb{1}_{|x-1|\leqslant 1} + (x-1)^{\alpha_1}\mathbb{1}_{|x-1|\geqslant 1}) - C_K\rho^{\beta'},
\end{aligned}$$

where $\beta' = \min(2\alpha_2 - \alpha_1, \frac{\alpha_1(\alpha_2-1)}{\alpha_1-1}) > \alpha_2$. To sum up, we have proved that for all $n \leqslant n_0$ we have

$$\begin{aligned}
\frac{E(n,\ell)}{\ell^3} - \mu(\rho_{n,\ell} - \rho) \geqslant c_1\rho^{\alpha_1} + c_2\rho^{\alpha_2} - C''\rho^{\beta''} \\
+ \frac{1}{2}cc_1\rho^{\alpha_1}((x-1)^2\mathbb{1}_{|x-1|\leqslant 1} + (x-1)^{\alpha_1}\mathbb{1}_{|x-1|\geqslant 1}),
\end{aligned}$$

where $\beta'' = \min'(\beta, \beta') > \alpha_2$. Now let $n > n_0$. Due to the non-negativity of v, the map $m \mapsto E(m,\ell)$ is *superadditive*, that is $E(m+m',\ell) \geqslant E(m,\ell) + E(m',\ell)$. This implies that

$$E(n,\ell) \geqslant \left\lfloor \frac{n}{n_0} \right\rfloor E(n_0,\ell),$$

and hence

$$E(n,\ell) - \mu n \geqslant \left\lfloor \frac{n}{n_0} \right\rfloor (E(n_0,\ell) - \mu n_0) - \mu r,$$

where $r = n - \lfloor n/n_0 \rfloor n_0 \leqslant n_0$. Recall that $K - 1 \leqslant \rho_{n_0,\ell}/\rho \leqslant K$ so that choosing $K \geqslant 2$ we have seen above that

$$\begin{aligned}
\frac{E(n_0,\ell)}{\ell^3} - \mu\rho_{n_0,\ell} &\geqslant -\mu\rho + c_1\rho^{\alpha_1} + c_2\rho^{\alpha_2} - C''\rho^{\beta''} + \frac{1}{2}cc_1\rho^{\alpha_1}(K-1)^{\alpha_1} \\
&= c_1\rho^{\alpha_1} + c_2\rho^{\alpha_2} - C''\rho^{\beta''} + c_1\rho^{\alpha_1}(\frac{1}{2}c(K-1)^{\alpha_1} - \alpha_1),
\end{aligned}$$

where we used that $\mu\rho = c_1\rho^{\alpha_1}\alpha_1$. Choosing K large enough such that $\frac{1}{2}c(K-1)^{\alpha_1} - \alpha_1 \geqslant 0$, we deduce that for δ small enough we have $E(n_0,\ell) - \mu n_0 \geqslant 0$ and thus (using that $\lfloor n/n_0 \rfloor \geqslant 1$ and that $\mu r \leqslant \mu n_0$)

$$\frac{E(n,\ell)}{\ell^3} - \mu(\rho_{n,\ell} - \rho) \geqslant c_1\rho^{\alpha_1} + c_2\rho^{\alpha_2} - C''\rho^{\beta''} + \frac{1}{2}cc_1\rho^{\alpha_1}(K-1)^{\alpha_1} - \mu\rho_{n_0,\ell}.$$

Recalling again that $\mu = c_1\alpha_1\rho^{\alpha_1-1}$ and that $\rho_{n_0,\ell} \leqslant K\rho$, we deduce that if we choose K large enough so that $\frac{1}{2}c(K-1)^{\alpha_1} - \alpha_1 K \geqslant 0$, we have

$$\frac{E(n,\ell)}{\ell^3} - \mu(\rho_{n,\ell} - \rho) \geqslant c_1\rho^{\alpha_1} + c_2\rho^{\alpha_2} - C''\rho^{\beta''}$$

for all $n > n_0$. Since the same inequality was shown to hold for $n \leqslant n_0$, this proves the result by Lemma 4.2. □

In the remaining sections, our goal will be to bound from below $E(n,\ell)$ for $n \leqslant K\ell^3\rho$ and $\ell = \ell(\rho)$ chosen adequately.

5. Rigorous implementation of Bogoliubov's method

The main result of this section is the following rigorous version of the Bogoliubov approximation introduced in Section 3, at first order in the dilute regime $\rho \to 0$.

Theorem 5.1. — *We have*

$$e(\rho) \geqslant 4\pi(a_0(v) + a_1(v))\rho^2 + o_{\rho\to 0}(\rho^2),$$

where the numbers $a_0(v)$ and $a_1(v)$ are defined in Proposition 3.1.

While rigorous, this result does not give the correct lower bound at first order compared to (1) since we have to replace $a_0(v) + a_1(v)$ by $a(v)$. This will be done in the next section.

5.1. Decomposition of the interaction

Let $P := |\ell^{-3/2}\rangle\langle\ell^{-3/2}|$ be the orthogonal projection on the condensate in $L^2(C_\ell)$ and $Q = 1 - P$. For any $j \in \{1,\dots,n\}$ we denote accordingly P_j and Q_j the projections on $L^2(C_\ell^n)$ which act on the j-th variable. For any $j \neq k$, we use the following identity:

$$\mathrm{Id}_{L^2(C_\ell^n)} = (P_j + Q_j)(P_k + Q_k) = P_jP_k + P_jQ_k + P_kQ_j + Q_jQ_k,$$

from which we deduce the decomposition of the potential:

$$\frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N} v(x_j - x_k) = V_0 + V_2 + V_3 + V_4,$$

where

$$V_0 := \frac{1}{2}\sum_{j\neq k} P_jP_k v(x_j - x_k)P_jP_k,$$

$$V_2 := \sum_{j\neq k} P_jQ_k v(x_j - x_k)P_kQ_j + \sum_{j\neq k} P_jQ_k v(x_j - x_k)Q_kP_j + \frac{1}{2}\sum_{j\neq k} P_jP_k v(x_j - x_k)Q_jQ_k + h.c.,$$

$$V_3 = \sum_{j \neq k} P_j Q_k v(x_j - x_k) Q_j Q_k + h.c.,$$

$$V_4 = \tfrac{1}{2} \sum_{j \neq k} Q_j Q_k v(x_j - x_k) Q_j Q_k \geqslant 0,$$

where "h.c." denotes the Hermitian conjugate. The operator V_4 is non-negative and will not be dropped out completely; it will absorb parts of V_3. Note that the operator V_1 that would appear with only one Q_j is zero since the action of the potential on the constant function gives a constant function which is cancelled by the Q_j.

Using creation and annihilation operators (to set the stage for Bogoliubov's method), we have

$$\sum_{j=1}^{n}(-\Delta_{x_j}) + V_0 + V_2 = \sum_{k \in (2\pi/\ell)\mathbb{Z}^3 \setminus \{0\}} |k|^2 a_k^* a_k + \frac{1}{2\ell^3} \widehat{v}(0) a_0^* a_0^* a_0 a_0$$
$$+ \frac{1}{\ell^3} \sum_{k \in (2\pi/\ell)\mathbb{Z}^3 \setminus \{0\}} \Big(\widehat{v}(k) a_0^* a_k^* a_k a_0 + \frac{1}{2} \widehat{v}(k) (a_0^* a_0^* a_k a_{-k} + h.c.) \Big) + \frac{1}{\ell^3} \widehat{v}(0) n_0 n_+,$$

where $n_0 = a_0^* a_0 = \sum_{j=1}^n P_j$, $n_+ = \sum_{k \in (2\pi/\ell)\mathbb{Z}^3 \setminus \{0\}} a_k^* a_k = \sum_{j=1}^n Q_j$. Rigorously speaking, in order to write such a formula as what we stated in Proposition 3.5, we need the interaction potential and Ψ to be $\ell\mathbb{Z}^3$-periodic which is not the case for us: the interaction potential is compactly supported and Ψ does not satisfy periodic boundary conditions on C_ℓ but rather *Neumann* boundary conditions. We will ignore this issue in this survey but let us emphasize that it is not a minor issue at all. In Fournais and Solovej (2020), this issue is solved by working with the family e_k with $k \in \mathbb{R}$ and the sums are replaced by integrals; they then have to deal with the fact that the e_k are not a Hilbert basis anymore so that the formula are less easy to manipulate. A very nice way around this problem was introduced by Haberberger, Hainzl, Nam, Seiringer, and Triay (2023) and consists in working rather with a Hilbert eigenbasis for the *Neumann* Laplacian (rather than the periodic Laplacian), and show that the interaction potential can be replaced by an adequate symmetrization of it so that the interaction has the same nice form as above in the Neumann basis.

5.2. Coherent states

As we discussed in Section 3.2, a first step in Bogoliubov's method is to replace a_0 by a number ("c-number substitution"). One way to implement rigorously this approach was initiated by Ginibre (1968) and developed by Lieb, Seiringer, and Yngvason (2005), and is based on the use of coherent states.

Define $\mathfrak{H} := L^2(C_\ell)$ and $\mathcal{F}(\mathfrak{H})$ the associated bosonic Fock space. Denote by $|\Omega\rangle \in \mathcal{F}(\mathfrak{H})$ the vacuum state $|\Omega\rangle := (1, 0, 0, \dots)$. Let $\mathfrak{H}_0 := \mathbb{C}\varphi_0 \subset \mathfrak{H}$ where $\varphi_0 := \ell^{-3/2}$ is the condensate wavefunction, and let $\mathfrak{H}' := \mathfrak{H}_0^\perp$ (the "excitation" subspace). The spaces $\mathcal{F}(\mathfrak{H})$ and $\mathcal{F}(\mathfrak{H}_0) \otimes \mathcal{F}(\mathfrak{H}')$ are unitarily equivalent (this is called the "exponential law" for Fock spaces, see Dereziński and Gérard, 2013, Sec. 3.3.7), so we identify them. In particular, in this representation we have $a_0(\Psi_0 \otimes \Psi') = (a_0 \Psi_0) \otimes \Psi'$.

For any $z \in \mathbb{C}$, define the coherent state

$$|z\rangle := e^{-|z|^2/2}(1, z\ell^{-3/2}, \dots, (z\ell^{-3/2})^m/\sqrt{m!}, \dots) = e^{-|z|^2/2} e^{z a_0^*}|\Omega\rangle \in \mathcal{F}(\mathfrak{H}_0),$$

which satisfies $a_0|z\rangle = z|z\rangle$. We denote by T_z the map $\mathcal{F}(\mathfrak{H}') \ni \Psi' \mapsto |z\rangle \otimes \Psi' \in \mathcal{F}(\mathfrak{H}_0) \otimes \mathcal{F}(\mathfrak{H}')$. Its adjoint $T_z^* \colon \mathcal{F}(\mathfrak{H}_0) \otimes \mathcal{F}(\mathfrak{H}') \to \mathcal{F}(\mathfrak{H}')$ satisfies on pure tensors $T_z^*(\Psi_0 \otimes \Psi') = \langle z, \Psi_0\rangle \Psi'$ for all $(\Psi_0, \Psi') \in \mathfrak{H}_0 \times \mathfrak{H}'$.

We have the resolution of the identity on $\mathcal{F}(\mathfrak{H}_0) \otimes \mathcal{F}(\mathfrak{H}')$:

$$\mathrm{Id}_{\mathcal{F}(\mathfrak{H}_0)\otimes\mathcal{F}(\mathfrak{H}')} = \frac{1}{\pi}\int_{\mathbb{C}} T_z T_z^* \, dz,$$

which follows from the fact that for all $\Psi_0 \in \mathcal{F}(\mathfrak{H}_0)$ one has

$$\Psi_0 = \frac{1}{\pi}\int_{\mathbb{C}} \langle z, \Psi_0\rangle |z\rangle \, dz,$$

which itself follows from the identity $\int_{\mathbb{C}} z^n \overline{z}^m e^{-|z|^2}\, dz = \pi n! \delta_{n=m}$. An operator A on $\mathcal{F}(\mathfrak{H}_0) \otimes \mathcal{F}(\mathfrak{H}')$ is said to have an *upper symbol* $(K_A(z))_{z\in\mathbb{C}}$ if for all $z \in \mathbb{C}$, $K_A(z)$ is an operator on $\mathcal{F}(\mathfrak{H}')$ and if

$$A = \frac{1}{\pi}\int_{\mathbb{C}} T_z K_A(z) T_z^* \, dz.$$

Upper symbols are very useful to get bounds; indeed if $K_A(z) \geqslant c$ for all z then $A \geqslant c$ as well. Then, it can be checked that $K_{a_0}(z) = z$ and hence $K_{a_0^*}(z) = z^*$. From these and the CCR we deduce that

$$K_{a_0 a_0^*}(z) = |z|^2, \;\; K_{a_0^* a_0}(z) = |z|^2 - 1, \;\; K_{a_0^* a_0^* a_0 a_0}(z) = |z|^4 - 4|z|^2 + 2.$$

Proposition 5.2. — *Let $\rho_z := |z|^2\ell^{-3}$ and $\mu \in \mathbb{R}$. We have*

$$\sum_{j=1}^n (-\Delta_{x_j}) + V_0 + V_2 = \frac{1}{\pi}\int_{\mathbb{C}} T_z \mathcal{K}(z) T_z^* \, dz + \mathcal{R}_0,$$

where

$$\mathcal{K}(z) = H_{\mathrm{Bog}}(\rho_z, v) + (\widehat{v}(0)\rho_z - \mu)n_+ - \mu|z|^2 + \mu n,$$

$$H_{\mathrm{Bog}}(\rho_z, v) = \frac{1}{2}\widehat{v}(0)\rho_z^2\ell^3 + \sum_{k\in(2\pi/\ell)\mathbb{Z}^3\setminus\{0\}} (|k|^2 + \rho_z\widehat{v}(k))a_k^* a_k + \frac{1}{2}\rho_z\widehat{v}(k)(a_k a_{-k} + a_{-k}^* a_k^*),$$

and $\mathcal{R}_0$ is such that

$$\mathcal{R}_0 \geqslant -Cn\ell^{-3} + \mu.$$

Proof of Proposition 5.2. — The term $\mathcal{R}_0$ comes from the terms of lower degree in $K_{a_0^* a_0^* a_0 a_0}$ and $K_{a_0^* a_0}$, hence

$$\mathcal{R}_0 = -\frac{2}{\ell^3}\widehat{v}(0)a_0 a_0^* + \frac{1}{\ell^3}\widehat{v}(0) - \frac{1}{\ell^3}\sum_{k\in(2\pi/\ell)\mathbb{Z}^3\setminus\{0\}} \widehat{v}(k)a_k^* a_k - \frac{1}{\ell^3}\widehat{v}(0)n_+$$

On $L^2_{\mathrm{sym}}(C_\ell^n)$, we have $a_0 a_0^* \leqslant n+1$, $n_+ = \sum_{k\neq 0} a_k^* a_k \leqslant n$ which proves the bound on $\mathcal{R}_0$. $\square$

Proposition 5.2 relates explicitly the full Hamiltonian with a Bogoliubov Hamiltonian, with a major drawback that the density ρ_z in the Bogoliubov Hamiltonian can be different from the desired density ρ. Let us now see how to solve this problem.

5.3. Error bounds for the Bogoliubov approximation

Proposition 5.3. — *There exists $C > 0$ such that for all $\rho \in (0, \delta)$, all $\ell \geqslant \rho^{-1/3}$, and all $n \leqslant K\rho\ell^3$ we have*

$$\ell^{-3}E(n, \ell) \geqslant 4\pi(a_0(v) + a_1(v))(n\ell^{-3})^2 - C\rho^2\ell^{-1/3} - C\rho^3\ell^2 - C\rho\ell^{-3}$$

Proof. — We start by bounding from below $V_3 + V_4$. Using the positivity of v, we have

$$\pm(P_jQ_kv(x_j - x_k)Q_jQ_k + h.c.) \leqslant C_\varepsilon P_jQ_kv(x_j - x_k)P_jQ_k + \varepsilon Q_jQ_kv(x_j - x_k)Q_jQ_k$$

and using furthermore that

$$P_jv(x_j - x_k)P_j = \ell^{-3}(\int v)P_j,$$

we deduce that

$$V_3 + V_4 \geqslant -C\ell^{-3}n_0n_+ \geqslant -C\rho n_+.$$

Let us now estimate $\mathcal{K}(z)$ from below. To do so, we choose $\mu = 2c_1n\ell^{-3} \lesssim \rho$ with $c_1 := 4\pi(a_0 + a_1)$. Let $A > 0$ be a large parameter to be specified later on. When $|z|^2 \geqslant An$, we drop all the non-negative terms in $\mathcal{K}(z)$ and infer that

$$\mathcal{K}(z) \geqslant -\mu|z|^2 - c(\rho_z + \rho)n_+.$$

We then bound

$$\int_{|z|^2\geqslant An} |z|^2T_zT_z^*\,dz \leqslant (An)^{-1}\int_{\mathbb{C}} |z|^4T_zT_z^*\,dz \lesssim (An)^{-1}(n_0 + 1)^2 \lesssim A^{-1}n \lesssim A^{-1}\rho\ell^3,$$

$$\int_{|z|^2\geqslant An} \rho_zT_zT_z^*\,dz \leqslant \int_{\mathbb{C}} \rho_zT_zT_z^*\,dz \lesssim \rho,$$

to infer than

$$\frac{1}{\pi}\int_{|z|^2\geqslant An} T_z\mathcal{K}(z)T_z^*\,dz \gtrsim -A^{-1}\rho^2\ell^3 - \rho n_+.$$

For $|z|^2 \leqslant An$ we have $\rho_z \leqslant AK\rho$ which is small if $A \ll \rho^{-1}$, hence we may use the lower bound (5) to infer that

$$\mathcal{K}(z) \geqslant c_1\rho_z^2\ell^3 - C\rho_z^2\ell^2 + (\widehat{v}(0)\rho_z - \mu)n_+ - \mu|z|^2 + \mu n,$$

with $c_1 := 4\pi(a_0(v) + a_1(v))$. We first have $\rho_z^2 \leqslant A^2\rho^2$. Next, $(\widehat{v}(0)\rho_z - \mu)n_+ \geqslant -\mu n_+ \gtrsim -\rho n_+$. This proves that

$$\mathcal{K}(z) \geqslant (c_1\rho_z^2 - \mu\rho_z + \mu n\ell^{-3})\ell^3 - CA^2\rho^2\ell^2 - C\rho n_+,$$

which motivates the choice $\mu = 2c_1n\ell^{-3}$ in order to have $c_1\rho_z^2 - \mu\rho_z + \mu\rho = c_1(n\ell^{-3})^2 + c_1(\rho_z - n\ell^{-3})^2$ which is minimal at $\rho_z = n\ell^{-3}$. Hence, for all $|z|^2 \leqslant An$ we have

$$\mathcal{K}(z) \geqslant 4\pi(a_0(v) + a_1(v))(n\ell^{-3})^2\ell^3 - CA^2\rho^2\ell^2 - C\rho n_+.$$

Integrating this bound over z we deduce that

$$\frac{1}{\pi}\int_{|z|^2\leqslant An} T_z\mathcal{K}(z)T_z^*\,dz \geqslant 4\pi(a_0(v)+a_1(v))(n\ell^{-3})^2\ell^3 \\ - c\rho^2\ell^3\int_{|z|^2\geqslant An} T_zT_z^*\,dz - CA^2\rho^2\ell^2 - C\rho n_+,$$

where we used that $\int_{|z|^2\leqslant An} = \int_{\mathbb{C}} - \int_{|z|^2\geqslant An}$. Now multiplying and dividing by $|z|^2$ under the integral we get the bound

$$\int_{|z|^2\geqslant An} T_zT_z^*\,dz \lesssim (An)^{-1}(n_0+1) \lesssim A^{-1}.$$

To control the remaining n_+ term, we use the spectral gap $H_{n,\ell} \gtrsim \ell^{-2}n_+$ and the fact that $E(n,\ell) \lesssim n^2\ell^{-3} \lesssim \rho^2\ell^3$ (which we obtain by computing the energy of the constant function), to infer that for any Ψ such that $\mathcal{E}(\Psi) \lesssim \rho^2\ell^3$, we have $\langle\Psi, n_+\Psi\rangle \lesssim \rho^2\ell^5$. Putting all of these bounds together, we deduce that

$$\ell^{-3}E(n,\ell) \geqslant 4\pi(a_0(v)+a_1(v))(n\ell^{-3})^2 - CA^{-1}\rho^2 - CA^2\rho^2\ell^{-1} - C\rho^3\ell^2 - C\rho\ell^{-3},$$

where the term $\rho^3\ell^2$ comes from the n_+ and the term $\rho\ell^{-3}$ comes from the term $\mathcal{R}_0$. We want the remainders involving A to be of the same order, i.e. $A^{-1}\rho^2 = A^2\rho^2\ell^{-1}$ which is equivalent to $A = \ell^{1/3}$. □

Proof of Theorem 5.1. — We combine Proposition 5.3 together with Proposition 4.1. To do so, we just have to choose ℓ such that $\rho^3\ell^2 = o(\rho^2)$, meaning that $\ell \ll \rho^{-1/2}$, and such that $\rho\ell^{-3} = o(\rho^2)$, meaning that $\ell \gg \rho^{-1/3}$. One can optimize in ℓ by choosing $\rho^3\ell^2 = \rho^2\ell^{-1}$, which leads to $\ell = \rho^{-3/7}$ and a final error in $e(\rho)$ of order $\rho^{2+1/7}$. □

Remark 5.4. — The condition $\ell \ll \rho^{-1/2}$ that we found in the proof above comes from the term n_+ which counts the number of excitations. Hence, $\rho^{-1/2}$ is the length scale under which the contribution of the excitations to the energy is sub-leading. This very important length scale is called the *healing length*.

Theorem 5.1 proves that the lower bound coming from the Bogoliubov approximation of Proposition 3.1 is a true lower bound to the full ground state energy of the Bose gas, at least to first order. There remains the problem that this lower bound has the right order ρ^2 but is not the correct one, namely that we need to replace $a_0(v) + a_1(v)$ by $a(v)$.

6. Renormalization of the interaction potential

In this section, we explain how to adapt the strategy of Section 5 to obtain the correct lower bound for $e(\rho)$ at first order. The main result of this section is the following:

Theorem 6.1. — *For any* $\omega\colon \mathbb{R}^3 \to [0,1]$*, we have*

$$e(\rho) \geqslant \Big(4\pi(a_0(g)+a_1(g)) + \tfrac12 \widehat{g\omega}(0)\Big)\rho^2 + o_{\rho\to 0}(\rho^2),$$

where $g := v(1-\omega)$*.*

There remains to optimize the ρ^2-coefficient:

Lemma 6.2. — *There exists* $\omega\colon \mathbb{R}^3 \to [0,1]$ *such that*

$$4\pi(a_0(g)+a_1(g)) + \tfrac12 \widehat{g\omega}(0) = 4\pi a(v),$$

where we recall that $g = v(1-\omega)$*.*

Proof. — By Remark 2.2, let ω be a minimizer for $8\pi a$ such that $0\leqslant\omega\leqslant 1$. Then, it satisfies the equation $-2\Delta\omega = v(1-\omega)$ which by integration against ω gives $\int 2|\nabla\omega|^2 = \int v(1-\omega)\omega$ and hence

$$8\pi a(v) = \int (2|\nabla\omega|^2 + v(1-\omega)^2) = \int v(1-\omega) = \int g = 8\pi a_0(g),$$

so that $a_0(g) = a(v)$. Using again that $-2\Delta\omega = v(1-\omega) = g$, we have

$$8\pi a_1(g) = \langle g, (2\Delta)^{-1} g\rangle = -\langle g,\omega\rangle = -\int g\omega = -\widehat{g\omega}(0),$$

and hence $4\pi a_1(g) + \frac12\widehat{g\omega}(0) = 0$. □

Remark 6.3. — Given the upper bound (that we did not prove) $e(\rho)\leqslant 4\pi a\rho^2 + o_{\rho\to 0}(\rho^2)$, this shows that this choice of ω is indeed the best. This can be understood in the following way: we want to compute

$$\sup_{0\leqslant\omega\leqslant 1} 4\pi\Big(a_0(g)+a_1(g)+\frac{\widehat{g\omega}(0)}{8\pi}\Big) = \frac12 \sup_{0\leqslant\omega\leqslant 1}(\widehat{g}(0) + \widehat{g\omega}(0) - \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\widehat{g}(k)^2}{2|k|^2}\,dk),$$

and we notice that

$$\begin{aligned}\widehat{g}(0) + \widehat{g\omega}(0) - \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\widehat{g}(k)^2}{2|k|^2}\,dk &= \int_{\mathbb{R}^3} v(1-\omega^2) + \langle 1-\omega, v(2\Delta)^{-1}v(1-\omega)\rangle \\ &= \int v - 2\langle v(2\Delta)^{-1}v,\omega\rangle + \langle\omega,(v(2\Delta)^{-1}v - v)\omega\rangle.\end{aligned}$$

It is not completely clear how to maximize this functional over all $0\leqslant\omega\leqslant 1$, but formally its Euler–Lagrange equation is

$$-v(2\Delta)^{-1}v + (v(2\Delta)^{-1}v - v)\omega = 0,$$

which can be rewritten as

$$-2\Delta\omega = v(1-\omega),$$

which is the same equation satisfied by minimizers for $8\pi a(v)$.

To prove Theorem 6.1, the main idea is to change the decomposition of the interaction that we started with. To do so, let $\omega\colon \mathbb{R}^3 \to \mathbb{R}$ be such that $0 \leqslant \omega \leqslant 1$. Then, we have

$$\mathrm{Id}_{L^2(C_\ell^n)} = \Pi_{jk} + (1-\omega)(x_j - x_k)(1 - Q_jQ_k),$$

where

$$\Pi_{jk} := Q_jQ_k + \omega(x_j - x_k)(1 - Q_jQ_k) = Q_jQ_k + \omega(x_j - x_k)(P_jP_k + P_kQ_j + P_jQ_k).$$

Previously, we used such a decomposition for $\omega = 0$, and we will see that there is a better choice of ω. As before, this induces the following decomposition of the interaction:

$$\tfrac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{n} v(x_j - x_k) = V_0 + V_2 + V_3 + V_4,$$

where

$$V_0 := \frac{1}{2}\sum_{j\neq k} P_jP_k(g + g\omega)(x_j - x_k)P_kP_j,$$

$$V_2 := \sum_{j\neq k} P_jQ_k(g + g\omega)(x_j - x_k)P_kQ_j + \sum_{j\neq k} P_jQ_k(g + g\omega)(x_j - x_k)Q_kP_j + \frac{1}{2}\sum_{j\neq k} P_jP_k g(x_j - x_k)Q_kQ_j + h.c.,$$

$$V_3 = \sum_{j\neq k} P_jQ_k g(x_j - x_k)Q_kQ_j + h.c.,$$

$$V_4 = \frac{1}{2}\sum_{j\neq k} \Pi_{jk}^* v(x_j - x_k)\Pi_{jk} \geqslant 0,$$

where $g := v(1-\omega)$ plays the role of a new interaction potential, the "renormalized" interaction potential. Again as above, we may write the part of the Hamiltonian without V_3 and V_4 using creation and annihilation operators:

$$\sum_{j=1}^{n}(-\Delta_{x_j}) + V_0 + V_2 = \sum_{k\in(2\pi/\ell)\mathbb{Z}^3\setminus\{0\}} |k|^2 a_k^* a_k + \frac{1}{2\ell^3}(\widehat{g}(0) + \widehat{g\omega}(0))a_0^*a_0^*a_0a_0 + \frac{1}{\ell^3}\sum_{k\in(2\pi/\ell)\mathbb{Z}^3\setminus\{0\}}\Big((\widehat{g}(k)+\widehat{g\omega}(k))a_0^*a_k^*a_ka_0 + \frac{1}{2}\widehat{g}(k)(a_0^*a_0^*a_ka_{-k} + h.c.)\Big) + \frac{1}{\ell^3}(\widehat{g}(0)+\widehat{g\omega}(0))n_0n_+.$$

The generalization of Proposition 5.2 is then:

Proposition 6.4. — *Let $\rho_z := |z|^2\ell^{-3}$ and $\mu \in \mathbb{R}$. We have*

$$\sum_{j=1}^{n}(-\Delta_{x_j}) + V_0 + V_2 = \frac{1}{\pi}\int_{\mathbb{C}} T_z\mathcal{K}(z)T_z^*\,dz + \mathcal{R}_0,$$

where

$$\mathcal{K}(z) = \mathcal{Q}(z) + \mathcal{Q}_2(z) + (\widehat{g}(0)\rho_z - \mu)n_+ - \mu|z|^2 + \mu n,$$

$$\mathcal{Q}(z) = H_{\mathrm{Bog}}(\rho_z, g) + \tfrac{1}{2}\rho_z^2\ell^3\widehat{g\omega}(0),$$

$$\mathcal{Q}_2(z) = \rho_z \sum_{k\in(2\pi/\ell)\mathbb{Z}^3\setminus\{0\}} (\widehat{g\omega}(k) + \widehat{g\omega}(0))a_k^* a_k,$$

and $\mathcal{R}_0$ is such that

$$\mathcal{R}_0 \geqslant -Cn\ell^{-3} + \mu.$$

Proof. — The proof follows the one of Theorem 5.1 using Proposition 5.3. We thus only explain what to adapt in the proof of Proposition 5.3. The term $V_3 + V_4$ is bounded using the same idea, controlling part of V_3 using V_4 and the remaining part being of order ρn_+: we write

$$\begin{aligned} P_j Q_k g(x_j - x_k) Q_k Q_j = P_j Q_k g(x_j - x_k)\Pi_{jk} \\ - P_j Q_k g(x_j - x_k)\omega(x_j - x_k)(P_j P_k + P_k Q_j + P_j Q_k) \end{aligned}$$

The term in the first line is again treated by Cauchy–Schwarz, using the important property that $0 \leqslant g \leqslant v$ since $g = v(1-\omega)$ and since $0 \leqslant \omega \leqslant 1$, so that it is controlled by V_4 and a remainder of order ρn_+. The term with $P_j P_k$ on the second line vanishes due to the Q_k on the other side, and the last two terms on the second line are also controlled by ρn_+ by the same argument. We deduce the same bound as before, namely $V_3 + V_4 \gtrsim -\rho n_+$. We now bound $\mathcal{K}(z)$. The term $\mathcal{Q}_2(z)$ satisfies $\mathcal{Q}_2(z) \gtrsim -\rho_z n_+$ and since $\int_{\mathbb{C}} \rho_z T_z T_z^*\,dz \lesssim \rho$, we deduce that

$$\frac{1}{\pi}\int_{\mathbb{C}} T_z \mathcal{Q}_2(z) T_z^*\,dz \gtrsim -\rho n_+.$$

When $|z|^2 \leqslant An$, we get a new lower bound due to the new interaction potential g and the term $\frac{1}{2}\rho_z^2\ell^3\widehat{g\omega}(0)$:

$$\mathcal{K}(z) \geqslant (c_1\rho_z^2 - \mu\rho_z + \mu n\ell^{-3})\ell^3 - CA^{5/2}\rho^{5/2}\ell^3 - C\rho n_+,$$

where $c_1 = 4\pi(a_0(g) + a_1(g)) + \frac{1}{2}\widehat{g\omega}(0)$. With this new c_1, the rest of the proof is identical. □

7. The Lee–Huang–Yang lower bound

Theorem 6.1 provides the first order term in the Lee–Huang–Yang expansion (1). To obtain the second order, a lot of additional work is needed so we only mention some of the key additional points that are needed.

The first point to notice is that the Bogoliubov lower bound of Proposition 3.1

$$\ell^{-3}E_{\mathrm{Bog}}(g) \geqslant 4\pi(a_0(g) + a_1(g))\rho^2 + 4\pi a_0(g)\rho^2 \times \frac{128}{15\sqrt{\pi}}(\rho a_0(g)^3)^{1/2} + o_{\rho\to 0}(\rho^{5/2}) \tag{6}$$

indeed provides the correct next order term when applied to the renormalized interaction potential g chosen from Lemma 6.2 since its proof shows that $a_0(g) = a(v)$. Hence, it seems that we only need to use this bound to get the Lee–Huang–Yang expansion (1). This will be indeed the case, except that we now need to show that all error terms are lower order than this second order term of size $\rho^{5/2}$ (or $\rho^{5/2}\ell^3$ if we look at the energy and not at the energy per unit volume), and this is where things become hard.

Going back to the proof of Theorem 6.1, we see that various error terms arise:

1. The first error term comes from $\mathcal{R}_0$ in Proposition 6.4, which has size ρ. Then, we have $\rho \ll \rho^{5/2}\ell^3$ if $\ell \gg \rho^{-1/2}$, which is already a major difference compared to the proof of Proposition 5.3 where we had $\ell \ll \rho^{-1/2}$. Going to boxes of size ℓ much larger than the healing length $\rho^{-1/2}$ is one of the challenging parts of the proof.
2. Many other error terms were bounded by ρn_+, which, due to the excitation bounds, has size $\rho^3\ell^5$. We have $\rho^3\ell^5 \ll \rho^{5/2}\ell^3$ if $\ell \ll \rho^{-1/4}$, which is incompatible with the condition $\ell \gg \rho^{-1/2}$ that we found in the previous item. This is the point where the proof has to be substantially changed: one has to look more precisely at the terms that were bounded by n_+ and control them better. Also, one needs to improve significantly the excitation bounds on n_+.

Let us now say a few words on how these problems are solved.

The error term ρn_+ arises when we bound $V_3 + V_4 \geqslant -\rho n_+$, as well as when we bounded $\mathcal{Q}_2$. Fournais and Solovej crucially realized that these terms can be controlled thanks to the positive part that was thrown away in the Bogoliubov diagonalization: when we "completed the square" in Lemma 3.7 in Section 3, we actually proved that $H_{\mathrm{Bog}}(v) = E_{\mathrm{Bog}}(v) + D$ for some operator $D \geqslant 0$ that we threw away as a lower bound. An important insight of Fournais and Solovej is that this D operator can help to control V_3 and $\mathcal{Q}_2$ in the following way: they show that

$$D + \mathcal{Q}_2 + V_3^{\mathrm{soft}} \geqslant -C\mathcal{G},$$

where V_3^{soft} includes only "soft" pairs (roughly speaking, V_3 involves 3 excitations; "soft" means that two of them have high momenta and one has low momentum), and $\mathcal{G}$ is a "spectral gap" operator that includes the excitation number n_+ but also a "high momentum" excitation number n_+^H. The part of V_3 which is not coming from V_3^{soft} is essentially controlled by V_4 and n_+^H, as the argument we presented in the proof of Theorem 6.1.

The precise control of these excitations numbers n_+ and n_+^H is also one of the main challenges, in particular because we work on a large box of size $\ell \gg \rho^{-1/2}$. To obtain a good control on these quantities, Fournais and Solovej use a "sliding" localization technique which allows controlling them from a control on smaller boxes of size $\ll \rho^{-1/2}$. Let us mention that in Haberberger, Hainzl, Nam, Seiringer, and Triay (2023) and Fournais, Girardot, Junge, Morin, Olivieri, and Triay (2026), the Neumann symmetrization approach allows for a simpler control of the excitations.

To conclude the argument, one also needs a good control of the Bogoliubov energy $E_{\mathrm{Bog}}(\rho_z, g)$ when replacing sums by integrals, as well as a good splitting in the z-integral of Proposition 5.2 to again show that ρ_z wants to be close to ρ. We refer to the actual articles for the details on these matters.

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