INTRODUCTION TO THE P=W CONJECTURE

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1. **Symmetries of the cohomology of algebraic varieties.** Let X be a complex smooth algebraic variety, endowed with a holomorphic symplectic form. P=W phenomena provide a unified explanation for symmetries, of very different origin, of the cohomology ring $H^*(X, \mathbb{Q})$. These symmetries are recalled below.

(i). If (X, α) is a polarized projective manifold of dimension $2n$, then the hard Lefschetz theorem gives

$$\alpha^i \colon H^{2n-i}(X, \mathbb{Q}) \xrightarrow{\simeq} H^{2n+i}(X, \mathbb{Q}).$$

(ii). If further X admits a symplectic form, i.e. $\sigma \in H^0(X, \Omega^2_X)$ with $\sigma^n \not\equiv 0$, then the non-degeneracy of σ implies

$$\sigma^i \colon \Omega^{n-i}_X \xrightarrow{\simeq} \Omega^{n+i}_X.$$

Taking cohomology, we obtain the symplectic hard Lefschetz theorem

$$\sigma^i \colon H^q(X, \Omega^{n-i}_X) \xrightarrow{\simeq} H^q(X, \Omega^{n+i}_X).$$

Equivalently, if $F^\bullet$ is the Hodge filtration on $H^*_{\mathbb{C}} = H^*(X, \mathbb{C})$, then

$$\boxed{\sigma^i \colon \mathrm{Gr}^{n-i}_F H^d_{\mathbb{C}} \xrightarrow{\simeq} \mathrm{Gr}^{n+i}_F H^{d+2i}_{\mathbb{C}}.} \tag{0.1}$$

(iii). If $f \colon X \longrightarrow B$ is a projective smooth morphism of relative dimension n (with X and B not necessarily proper), then the relative version of hard Lefschetz theorem reads

$$\alpha^i \colon R^{n-i} f_* \mathbb{Q} \xrightarrow{\simeq} R^{n+i} f_* \mathbb{Q}.$$

Taking cohomology, we obtain

$$\alpha^i \colon H^p(B, R^{n-i} f_* \mathbb{Q}) \xrightarrow{\simeq} H^p(B, R^{n+i} f_* \mathbb{Q}).$$

Equivalently, if $L^\bullet$ is the Leray filtration associated to f on $H^* = H^*(X, \mathbb{Q})$, then

$$\alpha^i \colon \mathrm{Gr}^{n-i}_L H^d \xrightarrow{\simeq} \mathrm{Gr}^{n+i}_L H^{d+2i}.$$

(iv). If $f \colon X \longrightarrow B$ is not necessarily smooth, then relative hard Lefschetz theorem continues to hold

$$\boxed{\alpha^i \colon \mathrm{Gr}^P_{n-i} H^d \xrightarrow{\simeq} \mathrm{Gr}^P_{n+i} H^{d+2i},} \tag{0.2}$$

provided that the Leray filtration $L^\bullet$ is replaced with the perverse Leray filtration $P_\bullet$ associated to f. If B is affine, then the perverse filtration is the kernel filtration of a general flag by [5, Thm 4.1.1], i.e.

$$P_k H^d = \mathrm{Ker}\{H^d(X,\mathbb{Q}) \longrightarrow H^d(f^{-1}(\Lambda^{d-k-1}),\mathbb{Q})\}, \tag{0.3}$$

where Λ^k is a general k-dimensional linear section of $B \subseteq \mathbb{A}^N$.

All the previous symmetries require some degree of properness. Analogous hard Lefschetz symmetries in the non-proper case are regarded as *curious* phenomena.

(v). If (X,Δ) is a log symplectic pair with only simple normal crossings, i.e. $\sigma \in H^0(X,\Omega^2_X(\log(\Delta)))$ with $\sigma^n \not\equiv 0$, then the non-degeneracy of σ gives that

$$\sigma^i\colon \Omega^{n-i}_X(\log\Delta) \xrightarrow{\simeq} \Omega^{n+i}_X(\log\Delta).$$

Taking cohomology, we obtain

$$\sigma^i\colon H^q(X,\Omega^{n-i}_X(\log(\Delta))) \xrightarrow{\simeq} H^q(X,\Omega^{n+i}_X(\log(\Delta))).$$

Equivalently, if $F^\bullet$ is the Hodge filtration on $H^*_{\mathbb{C}} = H^*(X\setminus\Delta,\mathbb{C})$, then

$$\sigma^i\colon \mathrm{Gr}^{n-i}_F H^d_{\mathbb{C}} \xrightarrow{\simeq} \mathrm{Gr}^{n+i}_F H^{d+2i}_{\mathbb{C}}.$$

Recall that H^* is a mixed Hodge structure with Hodge and weight filtration $F^\bullet$ and $W_\bullet$ respectively. Suppose that H^* is Hodge–Tate, i.e. $\mathrm{Gr}^W_{2k} H^d_{\mathbb{C}} = \mathrm{Gr}^k_F H^d_{\mathbb{C}}$ (roughly in each cohomological degree the two filtrations coincide). Then H^* has the curious hard Lefschetz property, i.e. there exists $\sigma \in \mathrm{Gr}^W_4 H^2$ such that

$$\boxed{\sigma^i\colon \mathrm{Gr}^W_{2(n-i)} H^d \xrightarrow{\simeq} \mathrm{Gr}^W_{2(n+i)} H^{d+2i}.} \tag{0.4}$$

See [11, Thm 1.7].

2. **P=F for compact symplectic variety.** In the proper case, the analogy between the symplectic and relative hard Lefschetz in (0.1) and (0.2) is explained in the work of Shen and Yin [23].

Theorem 2.1 (P=F)**.** [23, Theorem 0.2] *For any projective irreducible symplectic variety M equipped with a Lagrangian fibration $f\colon M \longrightarrow B$, we have*

$$\mathrm{Gr}^P_p H^{p+q}(M,\mathbb{C}) \simeq \mathrm{Gr}^p_F H^{p+q}(M,\mathbb{C}).$$

The perverse filtration on $H^*(M,\mathbb{C})$ can be identified with the weigth filtration of a Hodge–Tate limiting mixed Hodge structure of a degeneration of irreducible symplectic varieties deformation equivalent to M; see [12] and the survey [14]. A categorification of P=F, formulated in terms of quasi-isomorphisms of complexes of coherent $\mathcal{O}_B$-modules have been conjectured in [22, Conj. 1.2], and recently proved in [21, §19].[1] Partial results for singular irreducible symplectic

[1] Remarkably, X and B are no longer required to be proper!

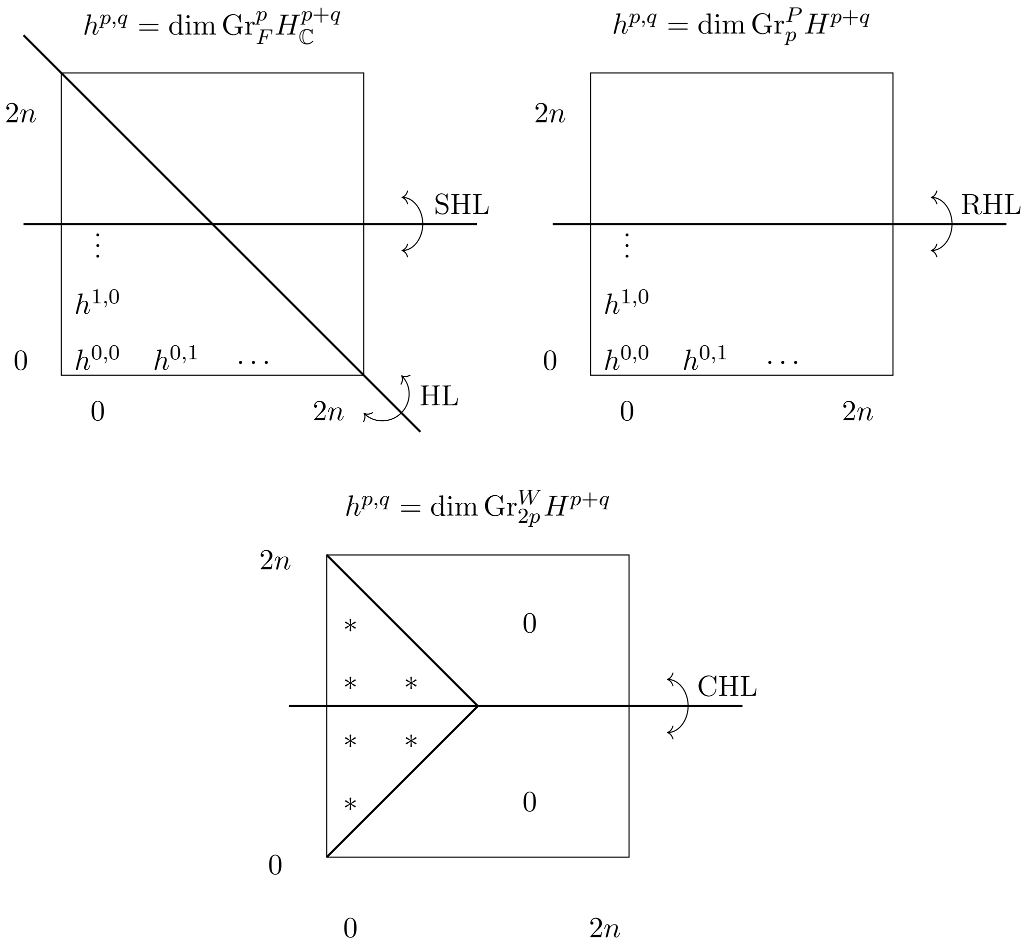

FIGURE 1. Classical, symplectic, relative and curious Hard Lefschetz.

varieties M are due to [9, Thm 0.4] when M admits a symplectic resolution, or [27] when M has only isolated singularities.

3. **P=W phenomena in general.** P=W phenomena have been conjectured in the attempt to interpret curious hard Lefschetz (0.4) as a manifestation of relative hard Lefschetz (0.2). The general structure of a P=W statement should look like the following.

Meta P=W conjecture 3.1. *Let M be a smooth complex manifold of dimension $2n$ endowed with the complex structures I and J (eventually other assumptions may be imposed).*

Suppose that:

- *$f\colon (M, I) \longrightarrow B$ is a proper holomorphic map of relative dimension n (most likely with abelian generic fiber), so that the cohomology of M satisfies the relative hard Lefschetz theorem, and*

- *(M, J) is biholomorphic to an open variety $X \setminus \Delta$ whose cohomology is Hodge–Tate and has the curious hard Lefschetz property.*

Then there exists a homeomorphism $\phi\colon M \longrightarrow M$ such that

$$P_k H^d(M, \mathbb{Q}) = \phi^* W_{2k} H^d(M, \mathbb{Q}),$$

which intertwines relative and curious hard Lefschetz.

It is still unclear what the larger class of varieties satisfying the meta P=W conjecture should be. For this reason, the assumptions of the meta conjecture are deliberately vague; see also [2, §4.4], [10, §4] and the negative result [18, §5.6].

4. **P=W conjecture for surfaces.** The P=W conjecture is essentially settled in dimension two. We follow the short proof of [10, §4], which overrides all previous references in the literature of P=W phenomena for surfaces.[2] Suppose that (X, Δ) is an snc log Calabi–Yau surface, i.e. there exists a no-where vanishing section $\sigma \in H^{2,0}(X, \Omega^2_X(\log(\Delta))) = H^0(X, K_X + \Delta)$.

If $H^*(X \setminus \Delta, \mathbb{Q})$ is Hodge–Tate, then Δ is a cycle of rational curves, and (X, Δ) is birational to a toric pair $(X_\Sigma, \Delta_{\mathrm{tor}})$. The toric moment map $\mu_\Sigma\colon X_\Sigma \longrightarrow \mathbb{D}$ induces an almost toric moment map $\mu\colon X \setminus \Delta \longrightarrow \mathbb{D}$, see [10, Thm 4.1]. The map μ can be endowed with the structure of a holomorphic elliptic fibration $f : M \longrightarrow B$ with at worst nodal fibers, so that a general fiber F of f is exchanged with a general fiber of μ, i.e. a real torus isotopic equivalent to $\mathbb{T} = \{|x| = r, |y| = s\} \subset X_\Sigma \dashrightarrow X \setminus \Delta$, where (x, y) are local coordinates around a node of Δ_{tor}, and $0 < r, s \ll 1$, e.g. $(\mathbb{P}^2, xyz = 0)$, see [10, Prop. 4.3].[3]

$$\begin{array}{ccc} F \subset M & \xrightarrow{\simeq_{\text{diffeo.}}} & X \setminus \Delta \supset \mathbb{T} \\ \Big\downarrow f & & \Big\downarrow \mu \\ B & \xrightarrow{\simeq_{\text{diffeo.}}} & \mathbb{D}. \end{array}$$

Let's compare the graded pieces of the Leray, perverse and weigth filtrations. We denote by b_i the Betti numbers of M.

The Leray filtration is trivial, since all cocycles of M are concentrated on the fibers of f, i.e. $H^d(M, \mathbb{Q}) = H^0(B, R^d f_* \mathbb{Q})$. In particular, if $b_2 \neq 1$, we do not recognize any numerical hard Lefschetz symmetry. This is why it is essential to replace the Leray filtration with the perverse filtration. The smoothness of $X \setminus \Delta$ implies that $\mathrm{Gr}^W_k H^i = 0$ for $k < i$, and so $b_i = 0$ for $i > 2$ by curious hard Lefschetz.[4] Therefore, relative and curious hard Lefschetz imply that

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[2] At the time Harder wrote [10], a similar proof appeared in [25] too, for the case of Painlevé surfaces.

[3] In many places in the literature, the commutative square is a datum of the local setting, but Harder shows that it follows simply from the fact that (X, Δ) is a log Calabi–Yau surface of maximal intersection (which is the case in all known examples).

[4] In general, in any dimension, if the cohomology of a smooth algebraic manifold $X \setminus \Delta$ of dimension $2n$ has the curious Hard Lefschetz property, then $H^d(X \setminus \Delta, \mathbb{Q}) = 0$ for $d > 2n$. Note that the cohomology of any affine

$$h^{p,q} = \dim \mathrm{Gr}^L_p H^{p+q} \qquad h^{p,q} = \dim \mathrm{Gr}^F_p H^{p+q} \qquad h^{p,q} = \dim \mathrm{Gr}^W_{2p} H^{p+q}$$

$$\begin{array}{|ccc|}\hline b_2 & 0 & 0 \\ b_1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline 1 & 0 & 0 \\ b_1 & b_2-1 & 0 \\ 1 & 0 & 0 \\ \hline\end{array} \ \circlearrowleft \text{RHL} \qquad \begin{array}{|ccc|}\hline 1 & 0 & 0 \\ b_1 & b_2-1 & 0 \\ 1 & 0 & 0 \\ \hline\end{array} \ \circlearrowleft \text{CHL}$$

FIGURE 2. Leray, perverse and weigth filtration.

the perverse and weight filtrations are possibly nontrivial only in cohomological degree 2 and in perversity 1 and 2 (weights 2 and 4 respectively). By (0.3), we have

$$P_1 H^2(M, \mathbb{Q}) = \ker\{H^2(M, \mathbb{Q}) \longrightarrow H^2(F, \mathbb{Q})\}.$$

On the other hand, given the natural inclusion $i\colon X \setminus \Delta \hookrightarrow X$, we write

$$W_2 H^2(M, \mathbb{Q}) = \ker\{H^2(M, \mathbb{Q}) \longrightarrow H^0(X, R^2 i_* \mathbb{Q})\} = \ker\{H^2(M, \mathbb{Q}) \longrightarrow H^2(\mathbb{T}, \mathbb{Q})\}.$$

The tori F and $\mathbb{T}$ are isotopic equivalent, and the P=W conjecture holds.

Remark 4.1. It is expected that the isotopy of the tori F and $\mathbb{T}$ is a general geometric incarnation of the cohomological P=W conjecture in top perversity; see [19, Thm A]. This is part of the so-called geometric P=W conjecture; see [15, 19]. It is not clear though what a geometric version of the cohomological P=W conjecture in any degree should be, and a homotopy statement is definitely too week; see [19, Rmk 6.2.11]. Such a geometric statement should explain how to describe the discriminant of $f\colon M \longrightarrow B$ in terms of the topology of the pair (X, Δ). At the end of the introduction of [24], Simpson presents a problematic configuration of discriminant circles in a 4-dimensional example.[5]

5. **P=W conjecture for character varieties.**

Definition 5.1. The Betti moduli space or character variety of a complex smooth projective curve C of genus g is the affine GIT quotient

$$M_B(g, n, d) \coloneqq \left\{ (A_1, B_1, \dots, A_g, B_g) \in \mathrm{GL}_n(\mathbb{C})^{\times 2g} \mid \prod_{j=1}^{g} [A_j, B_j] = e^{2\pi i \frac{n}{d}} 1_n \right\} /\!\!/ \mathrm{GL}_n(\mathbb{C}) \tag{0.5}$$

It parametrises isomorphism classes of semistable representations of the fundamental group of C with prescribed central monodromy around a puncture.

Theorem 5.2. *The P=W conjecture holds for smooth character variety, i.e.* $\gcd(n, d) = 1$.

variety satisfies this vanishing, but not all varieties with curious Hard Lefschetz are affine, e.g. the toric blow-up of $\mathbb{P}^2$ at ≥ 9 points lying in $xyz = 0$ but not nodes.

[5] A proof of the geometric P=W conjecture for this 4-dimensional example follows essentially from [26, §6.2].

Proof. Proved for $n = 2$ in [1], for $g = 2$ in [4], and unconditionally in [16, 13, 17]. □

Remark 5.3 (History of curious hard Lefschetz)**.** Curious hard Lefschetz for Betti moduli space was first observed by Hausel and Rodriguez–Villegas for $(n, d) = (2, 1)$. It was suggested by the symmetries of combinatorial formulas counting representations over finite fields. These representations are in correspondence with points of M_B over finite fields $\mathbb{F}_q$, and their number is a polynomial in q, called E-polynomial, whose coefficients are the Euler characteristic of $\mathrm{Gr}_k^W H_c^*(M_B)$. Hausel and Rodriguez–Villegas observed that the E-polynomial is palindromic and, studying the cup product structure on $H^*(M_B)$, they conjectured curious Hard Lefschetz and proved it in rank 2. In [1], de Cataldo, Hausel and Migliorini first proposed the P=W conjecture as an explanation of the new symmetry.

After that, in [20], Mellit proved curious hard Lefschetz, in any rank and under the smoothness assumption $\gcd(n, d) = 1$ (more generally for smooth character varieties of a curve with finitely many punctures), independently of P=W. In [20], he deforms the Betti moduli space to a character variety with very general monodromy at a new extra puncture. This latter variety is endowed with a vector bundle whose total space admits a stratification in strata $(\mathbb{C}^*)^a \times \mathbb{C}^b$, which all enjoy the curious hard Lefschetz property with respect to the restriction of the same global 2-form. Finally, he pushes back the curious hard Lefschetz property to the original character variety, see [20, §8].[6] The proof of the P=W conjecture in [16] relies on [20], while [13] provides an alternative independent proof of curious hard Lefschetz.

Note that the existence of log symplectic compactifications of character varieties would provide a new geometric explanation of curious hard Lefschetz.

Conjecture 5.4. *Character varieties admit log symplectic compactifications.*

At the moment, the conjecture is known to hold trivially for $n = 1$, $(d, g) = (0, 1)$ by [19, Thm E and D], and $n = 2$ on a punctured sphere with general monodromy by [7, Cor. 1.9].[7]

Remark 5.5. (Multiplicativity of the perverse filtration) The weight filtration is multiplicative under cup product, i.e.

$$\cup\colon W_k H^d \times W_{k'} H^{d'} \longrightarrow W_{k+k'} H^{d+d'},$$

see [6, Corollaire 8.2.11]. Hence, P=W implies that the perverse filtration is compatible with cup product. In fact, by [4, Thm 0.6], P=W for smooth character varieties is equivalent to the multiplicativity of the perverse filtration. In [17], Maulik, Shen and Yin have recently proved the multiplicativity of the perverse filtration for so-called dualizable abelian fibrations with Fourier vanishing, which includes families of compactified Jacobians of integral curves

[6] A similar strategy have been employed in the proof of P=W too both in [16, 13].

[7] In [19, 7], the authors exhibit log Calabi–Yau compactifications of character varieties, i.e. $K_X + \Delta \sim 0$. Note that they are actually log symplectic. Indeed, by curious hard Lefschez [20], there exists $\sigma \in F^2 H^2_{\mathbb{C}} = H^0(X, \Omega_X^{[2]}(\log \Delta))$ with $\sigma^n \neq 0$. Since (X, Δ) is log Calabi–Yau, then $\sigma^n \in H^0(X, \Omega_X^{[2n]}(\log \Delta) = H^0(X, K_X + \Delta) = H^0(X, \mathcal{O})$ is nowhere vanishing.

with planar singularities (with a section, or for special twisted families with no sections). In particular, using the reduction step in [13, §8], they provide a third alternative proof of the P=W conjecture. Note that the multiplicativity of the perverse filtration fails in general, see [28, 29].

Examples 5.6. [29, Thm 1.5] Let $g\colon S \longrightarrow B$ be a proper morphism from a smooth quasi-projective surface and a smooth quasi-projective curve. Then the perverse filtration f is automatically multiplicative, but the induced morphisms of Hilbert scheme $g^{[n]}\colon S^{[n]} \longrightarrow B^{(n)}$ is multiplicative only if f is an elliptic fibration. The converse holds if $n = 2$, or f is a fibration of Hitchin type, i.e. $f\colon S \longrightarrow \mathbb{A}^1$ with $H^*(S, \mathbb{Q}) \simeq H^*(g^{-1}(0), \mathbb{Q})$.

Remark 5.7. The P=W phenomena in the compact case [23, 12, 21], §4, Theorem 5.4 and Theorem 5.6 suggests that symplectic structures should be key ingredients of P=W phenomena. However, the symplectic nature of character varieties does not seem to play an essential role in [16, 13, 17], except possibly [20, §5-7].

Remark 5.8 (Singular character varieties)**.** In the singular case, Poincaré duality and relative hard Lefschetz theorems fail for singular cohomology. These symmetries can be restored taking instead intersection cohomology. In [3, Question 4.1.7], de Cataldo and Maulik[8] proposed the PI=WI conjecture for the intersection cohomology of singular character varieties, proved only for $(d, g) = (0, 1)$ and $(d, g, n) = (0, 2, 2)$ in [8, Main Thm] (leaving aside the smooth cases). See [18, §5] for partial results in rank two, in particular a numerical evidence for curious hard Lefschetz in [18, Cor. 1.5].

At the moment, curious hard Lefschetz is still unknown in arbitrary degree d. The current proofs of the P=W conjectures rely on the multiplicative structure of singular cohomology. However, intersection cohomology does not have a canonical ring structure. This means that the proofs in the smooth context do not naturally extend to the singular case, in particular to the fundamental case $d = 0$, thus challenging the community to understand better P=W phenomena.

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[8] The Dolbeault moduli spaces M_{Dol} parametrises semistable Higgs bundles on the curve C, and can be obtained from the Betti moduli space via hyperkähler rotations. It plays the role of the complex space (M, I) in the meta P=W conjecture. In [3], the authors show that the perverse filtration on $IH^*(M_{\text{Dol}})$ is independent of the complex structure of the underlying curve, exactly as it happens for the weight filtration on the Betti side.

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